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THE DIMENSION OF BASIC SETS

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Let $f: M \to M$ be a C^1 diffeomorphism of a compact connected manifold M. A closed f-invariant set $\Lambda \subset M$ is said to be *hyperbolic* if the tangent bundle of M restricted to Λ is the Whitney sum of two Df-invariant bundles, i.e., if $T_\Lambda M = E^u(\Lambda) \oplus E^s(\Lambda)$, and if there are constants $C > 0$ and $0 < \lambda < 1$ such that

$$|Df^n(V)| \leq C\lambda^n |v| \qquad \text{for } v \in E^s,\ n > 0\ ,$$
$$|Df^{-n}(V)| \leq C\lambda^n |v| \qquad \text{for } v \in E^u,\ n > 0\ .$$

The diffeomorphism f is said to satisfy *Axiom A* if (a) the non-wandering set $\Omega(f) = \{x \in M: U \cap \bigcup_{n>0} f^n(U) \neq \emptyset$ for every neighborhood U of $x\}$ of f is a hyperbolic set, and (b) $\Omega(f)$ equals the closure of the set of periodic points of f. If f satisfies Axiom A, one has the spectral decomposition theorem of Smale [9] which says $\Omega(f) = \Lambda_1 \cup \cdots \cup \Lambda_l$ where Λ_i are pairwise disjoint, f-invariant closed sets and $f|_{\Lambda_i}$ is topologically transitive.

These Λ_i are called the *basic sets* of f, and it is the object of this article to investigate restrictions on their dimensions imposed by the homotopy type of f and the fiber dimensions of the bundles E^s and E^u. In [11] S. Smale showed that any diffeomorphism can be isotoped to a diffeomorphism satisfying Axiom A with all basic sets of dimension zero. This disproved earlier conjectures that some homotopy classes might contain only diffeomorphisms with a basic set of positive dimension. Theorem 1 below shows that if one restricts either the fiber dimensions of the bundles E^u or the total number of basic sets for f, then there are indeed homotopy classes all of whose diffeomorphisms (subject to these restrictions) have basic sets of positive dimension. In Theorem 2 we investigate diffeomrphisms with a single infinite basic set, the others being isolated periodic orbits. It is a pleasure to acknowledge valuable conversations with R. F. Williams.

We consider diffeomorphisms which in addition to Axiom A satisfy the no-cycle property [10] which we now define. If Λ_i is a basic set of f then its stable and unstable manifolds ([5] or [9]) are defined by

$$W^s(\Lambda_i) = \{x \in M \,|\, d(f^n(x), \Lambda_i) \to 0 \text{ as } n \to \infty\}\ ,$$

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$$W^u(\Lambda_i) = \{x \in M \mid d(f^{-n}(x), \Lambda_i) \to 0 \text{ as } n \to \infty\} .$$

One says $\Lambda_i \leq \Lambda_j$ if $W^u(\Lambda_j) \cap W^s(\Lambda_i) \neq \emptyset$. If this extends to a total ordering on the basic sets Λ_i, then f is said to satisfy the *no-cycle property* and we re-index so that $\Lambda_i \leq \Lambda_j$ when $i \leq j$. If Λ_i is a basic set of $f: M \to M$ then we define the *index* u_i of Λ_i with respect to f to be the fiber dimension of $E^u(\Lambda_i)$. All homology and cohomology will be singular with real coefficients unless otherwise stated.

Theorem 1. *If f: $M \to M$ satisfies Axiom A and the no-cycle property and $H^k(M) \neq 0$, then there is a basic set Λ_i satisfying* $\dim \Lambda_i \geq |k - u_i|$ *where u_i is the index of Λ_i.*

Hence, if f has fewer basic sets than nonzero cohomology groups, it must have a basic set of positive dimension, or equivalently:

Corollary 1. *If f has only basic sets of dimension zero, then there is a basic set Λ_i with index $u_i = k$ for each k such that $H^k(M) \neq 0$.*

Theorem 2. *Suppose $f: M \to M$ satisfies Axiom A and the no-cycle property and has one infinite basic set Λ, the others being isolated periodic orbits. If $f^*: H^k(M) \to H^k(M)$ has an eigenvalue which is not a root of unity, then* $\dim \Lambda \geq |n - 2k|$ *where $n =$* $\dim M$. *It Λ is an attractor, then* $\dim \Lambda \geq \max\{(n-k), k\}$.

We note that M. Shub [8] has shown that whenever $f^*: H^*(M) \to H^*(M)$ has an eigenvalue which is not a root of unity, then f must have at least one infinite basic set.

In case M is the n-dimensional torus T^n we can strengthen Theorem 2 because either $f^*: H^1(T^n) \to H^1(T^n)$ has an eigenvalue which is not a root of unity or $f^*: H^*(T^n) \to H^*(T^n)$ is quasi-unipotent (i.e., has only roots of unity as eigenvalues).

Corollary 2. *If $f^*: T^n \to T^n$ satisfies Axiom A and the no-cycle property and has only one basic set Λ which is infinite, then either $f^*: H^*(T^n) \to H^*(T^n)$ is quasi-unipotent or* $\dim \Lambda \geq n - 2$.

It is not difficult to construct diffeomorphisms on T^n with a single infinite basic set of dimension $n, n-1$, but the author does not know if there is a diffeomorphism of T^3 which is not unipotent on homology and with a single infinite basic set of dimension one (dimensions 2 and 3 can be realized in this case). The hypothesis that f^* not be quasi-unipotent on cohomology is necessary since it is easy to construct $f: T^n \to T^n$ homotopic to the identity with a single infinite basic set of dimension zero.

We review briefly the filtrations of [10] associated with a diffeomorphism which satisfies Axiom A and the no-cycle property. It is possible to find submanifolds (with boundary and of the same dimension as M),

$$M = M_l \supset \cdots \supset M_1 \supset M_0 = \emptyset ,$$

such that

$$M_{i-1} \cup f(M_i) \subset \operatorname{int} M_i \ ,$$
$$\Lambda_i = \bigcap_{m \in Z} f^m(M_i - M_{i-1}) \ ,$$
$$W^u(\Lambda_i) \cup M_{i-1} = M_{i-1} \cup \bigcap_{m \geq 0} f^m(M_i) \ .$$

Henceforth $f: M \to M$ will be a diffeomorphism of a compact manifold satisfying Axiom A and the no-cycle property and $M = M_l \supset M_{l-1} \supset \cdots \supset M_0 = \emptyset$ will be a filtration for f. The proofs of Theorems 1 and 2 use the following proposition which may be of some independent interest.

Proposition 1. *Suppose $f: M \to M$ satisfies Axiom A and the no-cyclic property and $\Lambda_i \subset M_i - M_{i-1}$ is a basic set of f. Let $S = \{k \mid f_k^* : H^k(M_i, M_{i-1}) \to H^k(M_i, M_{i-1})$ has a nonzero eigenvalue$\}$. Then* $\dim \Lambda_i \geq \max S - \min S$.

We procede now with a sequence of lemmas leading to the proofs of the results above. We will use closed local stable and unstable manifolds of a point $x \in \Lambda$, denoted $W_\varepsilon^S(x)$ and $W_\varepsilon^u(x)$ (see [5] or [9]).

Since it is not in general true that $\dim(X \times Y) = \dim X + \dim Y$ it is necessary to use the concept of *cohomological dimension over R* [3] defined as follows: If X is a compact Housdorff space, then $\dim_R X = \sup\{k \mid \check{H}^k(X, A; R) \neq 0\}$ where A runs over all closed subspaces of X and $\check{H}^k$ is Čech cohomology with real coefficients. By a result of [7, p. 152] $\dim_R X \leq \dim X$.

Lemma 1. *Suppose $\Lambda_i \subset M_i - M_{i-1}$ is a basic set for f and M_i, M_{i-1} are the elements of a filtration for f. If $k > \dim_R W_\varepsilon^u(\Lambda_i)$, then the map f_k^*: $H^k(M_i, M_{i-1}) \to H^k(M_i, M_{i-1})$ is nilpotent.*

Proof. This is essentially the same as [4, Lemma 6] which drew heavily on [1]. Let $X = W^u(\Lambda_i) \cup M_{i-1}$ and let $\check{H}^k$ denote Čech cohomology with real coefficients. We use the closed local unstable manifolds of [5]. The inclusion $(W_\varepsilon^u(\Lambda_i), \partial W_\varepsilon^u(\Lambda_i)) \to (X, W)$ is a relative homeomorphism where $W = cl(X - W_\varepsilon^u(\Lambda_i))$. Hence by a standard result [12, p. 266],

$$\check{H}^k(W_\varepsilon^u(\Lambda_i), \partial W_\varepsilon^u(\Lambda_i)) \cong \check{H}^k(X, W) \ .$$

By definition of $\dim_R$,

$$\check{H}^k(W_\varepsilon^u(\Lambda_i), \partial W_\varepsilon^u(\Lambda_i)) = 0 \ ,$$

when $k > \dim_R W_\varepsilon^u(\Lambda_i)$. Since W is compact and $X \subset \{\bigcap_{n \geq 0} f^{-n}(\operatorname{int} M_{i-1})\} \cup \Lambda_i$ it follows that $f^m(W) \subset M_{i-1}$ for some $m > 0$. The diagram

$$\begin{array}{ccc} (X, M_{i-1}) & \xrightarrow{\ i\ } & (X, W) \\ & {\searrow}^{f^m} \quad {\swarrow}_{f^m} & \\ & (X, M_{i-1}) & \end{array}$$

commutes. Thus the map $(f^m)^* : \check{H}^k(X, M_{i-1}) \to \check{H}^k(X, M_{i-1})$ factors through $\check{H}^k(X, W)$ so that $(f^m)^* = (f^*)^m = 0$ when $k > \dim_R W^u(\Lambda_i)$.

Now if $f^* : H^k(M_i, M_{i-1}) \to H^k(M_i, M_{i-1})$ is not nilpotent, there is a subspace $V \neq 0$ with $f^*(V) = V$. By [1, Lemma 1], the map h^* is one-to-one on V where $h^* : H^k(M_i, M_{i-1}) = \check{H}^k(M_i, M_{i-1}) \to \check{H}^k(X, M_{i-1})$ is induced by the inclusion $h : (X, M_{i-1}) \to (M_i, M_{i-1})$. Thus we have a commutative diagram

$$\begin{array}{ccc} H^k(M_i, M_{i-1}) & \xrightarrow{(f^*)^m} & H^k(M_i, M_{i-1}) \\ \downarrow h^* & & \downarrow h^* \\ \check{H}^k(X, M_{i-1}) & \xrightarrow{(f^*)^m} & \check{H}^k(X, M_{i-1}) \end{array} .$$

But, $(f^*)^m h^*(V) = h^*(f^*)^m V = h^*(V) \neq 0$, which is a contradiction if $k > \dim_R W^u(\Lambda_i)$, since $(f^*)^m : \check{H}^k(X, M_{i-1}) \to \check{H}^k(X, M_{i-1})$ is zero in this case. Thus it must be the case that $f^* : H^k(M_i, M_{i-1}) \to H^k(M_i, M_{i-1})$ is nilpotent when $k > \dim_R W^u(\Lambda_i)$. q.e.d.

If Λ is a basic set and $x \in \Lambda$, we let $\hat{W}^s_\varepsilon(x) = W^s_\varepsilon(x) \cap \Lambda$ and $\hat{W}^u_\varepsilon(x) = W^u_\varepsilon(x) \cap \Lambda$. While it is true [9] that $x \in \Lambda$ has a neighborhood homeomorphic to $\hat{W}^s_\varepsilon(x) \times \hat{W}^u_\varepsilon(x)$, it appears to be an open question whether or not $\dim \Lambda = \dim \hat{W}^s_\varepsilon(x) + \dim \hat{W}^u_\varepsilon(x)$. For the cohomological dimension over R however we have the following.

Lemma 2. *Suppose Λ is a basic set for f, $u =$* fiber dim $E^u(\Lambda)$, *and $s =$* fiber dim $E^s(\Lambda)$. *Then*

(a) $\dim_R W^u_\varepsilon(\Lambda) = \dim_R \hat{W}^s_\varepsilon(x) + u$,

(b) $\dim_R W^s_\varepsilon(\Lambda) = \dim_R \hat{W}^u_\varepsilon(x) + s$,

(c) $\dim_R \Lambda = \dim_R \hat{W}^u_\varepsilon(x) + \dim_R \hat{W}^s_\varepsilon(x)$,

where x is any point of Λ and $\varepsilon > 0$ is sufficiently small.

Proof. We will use the following results from [13, Theorem 2.2 and Lemma 2.1]. If X and Y are compact Hausdorff spaces, then (1) $\dim_R (X \times Y) = \dim_R X + \dim_R Y$, and (2) if $n = \dim_R X$, there exists a point $p \in X$ such that if U is any sufficiently small neighborhood of p in X, then $\check{H}^n(X, X - U) \neq 0$.

Also if Y is a compact subset of X, then consideration of the exact sequence of the triple (X, Y, A), where A is a closed subset of Y,

$$\check{H}^n(X, A) \longrightarrow \check{H}^n(Y, A) \xrightarrow{\delta} \check{H}^{n+1}(X, Y) ,$$

shows that $\dim_R X \geq \dim_R Y$.

We begin the proof of (a) by showing that $\dim_R \hat{W}^s_\varepsilon(x)$ is independent of $x \in \Lambda$. If $y \in \Lambda$, then using the canonical coordinates [9, p. 781] for Λ and the fact that $W^s(\text{orb}(y))$ is dense in Λ it is easy to show that $\hat{W}^s_\varepsilon(x)$ is homeomorphic to a compact subset of $f^m(\hat{W}^s_\varepsilon(y))$ for some m. This implies $\hat{W}^s_\varepsilon(x)$ is homeomorphic to a subset of $\hat{W}^s_\varepsilon(y)$ since f^m is a diffeomorphism. Thus $\dim_R \hat{W}^s_\varepsilon(x) \leq \dim_R \hat{W}^s_\varepsilon(y)$ and the same argument shows $\dim_R \hat{W}^s_\varepsilon(y) \leq \dim_R \hat{W}^s_\varepsilon(x)$.

By results of [6] there is a continuous map $\varphi: \Lambda \to \mathrm{Emb}\,(D, M)$ such that $\varphi(z)(D) = W_\varepsilon^u(z)$ where D is the disk of dimension u. The map $\psi: \hat{W}_\varepsilon^s(x) \times D \to W_\varepsilon^u(\Lambda)$ given by $\psi(y, t) = \varphi(y)(t)$ is a homeomorphism onto a compact neighborhood K_x of x in $W_\varepsilon^u(\Lambda)$. But it is not possible that $\dim_R W_\varepsilon^u(\Lambda) > \dim K_x$ because the sets K_x cover $W_\varepsilon^u(\Lambda)$ and by (2) above together with excision at least one of them must have dimension over R equal to that of $W_\varepsilon^u(\Lambda)$. Thus $\dim_R W_\varepsilon^u(\Lambda) = \dim_R \hat{W}_\varepsilon^s(X) + u$ for all $x \in \Lambda$ and (a) is proven. Applying this result to f^{-1} proves (b).

To prove (c) we consider the canonical coordinate map $\rho: \hat{W}_\varepsilon^s(x) \times \hat{W}_\varepsilon^u(x) \to \Lambda$ which is a homeomorphism onto a compact neighborhood J_x of x in Λ. By (1) above $\dim_R J_x = \dim_R \hat{W}_\varepsilon^s(x) + \dim_R \hat{W}_\varepsilon^u(x)$. Since $J_x \subset \Lambda$, $\dim_R J_x \leq \dim_R \Lambda$ and again using (2) above and excision, it follows that $\dim_R \Lambda = \dim_R J_x$ for some x (and hence for all x since $\dim_R \hat{W}_\varepsilon^s(x)$ and $\dim_R \hat{W}_\varepsilon^u(x)$ are independent of x). Thus (c) is proven. q.e.d.

Lemma 3. *If* $A_3 \xrightarrow{i} A_2 \xrightarrow{j} A_1$ *is a sequence of vector spaces exact at* A_2, $\alpha_i: A_i \to A_i$ *are linear maps commuting with i and j, and* λ *is an eigenvalue of* α_2, *then* λ *is also an eigenvalue ef either* α_3 *or* α_1.

This is [4, Lemma 2]; the proof is not difficult and will not be repeated here.

Lemma 4. *If* λ *is an eigenvalue of* $f_k^*: H^k(M) \to H^k(M)$, *then there is an* M_i *in the filtration for f such that* $f_k^*: H^k(M_i, M_{i-1}) \to H^k(M_i, M_{i-1})$ *has* λ *as an eigenvalue.*

Proof. Consider the exact cohomology sequence of the triple

$$H^k(M, M_j) \to H^k(M, M_{j-1}) \to H^k(M_j, M_{j-1}) .$$

There is a map f^* induced by f on each of these groups, and these maps commute with the maps of the sequence. We now apply Lemma 1 to this sequence when $j = 1$. In this case the sequence is

$$H^k(M, M_1) \to H^k(M) \to H^k(M_1, M_0) ,$$

so either λ is an eigenvalue of f^* on $H^k(M_1, M_0)$ or an eigenvalue of f^* on $H^k(M, M_1)$. If the latter we set $j = 2$ and reapply Lemma 1 to show λ is an eigenvalue of f^* on either $H^k(M_2, M_1)$ or $H^k(M, M_2)$. Continuing this procedure it follows that λ is an eigenvalue of f^* on $H^k(M_i, M_{i-1})$ for some i, since $H^k(M, M_l) = H^k(M, M) = 0$.

Proof of Proposition 1. Let $k_1 = \max S$. Then by Lemma 1, $k_1 \leq \dim_R W_\varepsilon^u(\Lambda_i)$ and by Lemma 2, $\dim_R W_\varepsilon^u(\Lambda_i) = \dim_R \hat{W}_\varepsilon^s(x) + u_i$ where $x \in \Lambda_i$ and $u_i =$ fiber $\dim E^u(\Lambda_i)$, so $k_1 - u_i \leq \dim_R \hat{W}_\varepsilon^s(x)$. Let $k = \min S$ and let $\tilde{M}_j = cl(M - M_j)$. Then since $f_k^*: H^k(M_i, M_{i-1}) \to H^k(M_i, M_{i-1})$ has a nonzero eigenvalue, its adjoint $f_{*k}: H_k(M_i, M_{i-1}) \to H_k(M_i, M_{i-1})$ has the same eigenvalue. Suppose M is orientable and $n = \dim M$. Then [1, Lemma 4] shows that if $g = f^{-1}: M \to M$, $g_{n-k}^*: H^{n-k}(\tilde{M}_{i-1}, \tilde{M}_i) \to H^{n-k}(\tilde{M}_{i-1}, \tilde{M}_i)$ is similar to either $f_{*k}: H_k(M_i, M_{i-1}) \to H_k(M_i, M_{i-1})$ or to $-f_{*k}$. In either case g_{n-k}^*

has a nonzero eigenvalue. Since g has the same basic sets as f (with $W^s(f; \Lambda_i) = W^u(g; \Lambda_i)$) and $M = \tilde{M}_0 \supset \tilde{M}_1 \supset \cdots \supset \tilde{M}_l = \emptyset$ is a filtration for g, we can apply to g the argument which showed $k_1 - u_i \leq \dim_R \hat{W}^s_\varepsilon(x)$. We have then that $(n - k) -$ fiber $\dim E^u(g; \Lambda_i) \leq \dim_R \hat{W}^s_\varepsilon(g; x)$ or $(n - k) - s_i \leq \dim_R \hat{W}^u_\varepsilon(f; x)$ where $s_i =$ fiber $\dim E^s(f; \Lambda_i)$. Adding this inequality to the one for k_1 we have

$$k_1 - u_i + (n - k) - s_i \leq \dim_R \hat{W}^s_\varepsilon(x) + \dim_R \hat{W}^u_\varepsilon(x) .$$

Since $n = u_i + s_i$, $k_1 - k \leq \dim_R \Lambda$ by Lemma 2. That is, $\max S - \min S \leq \dim_R \Lambda_i \leq \dim \Lambda_i$.

In case M is not orientable, we let $\pi : \bar{M} \to M$ be an oriented double cover of M and $\bar{f} : \bar{M} \to M$ a lift of f. If $\bar{\Lambda}_i = \pi^{-1}(\Lambda_i)$ and $\bar{M}_i = \pi^{-1}(M_i)$, then the $\bar{\Lambda}_i$ have all the properties of basic sets for $\bar{f}$ except they may not be topologically transitive. But $\bar{f}$ together with the nontrivial covering transformation on $\bar{M}$ will be transitive, and this is sufficient for everything we have done. So exactly as above, we use the filtration $\bar{M}_i$ and prove the result for $\bar{\Lambda}_i$ ($\pi_* : H_j(\bar{M}_i, \bar{M}_{i-1}) \to H_j(M_i, M_{i-1})$ is surjective—see [1, Theorem 1]). Since $\dim \bar{\Lambda}_i = \dim \Lambda_i$, this completes the proof.

Proof of Theorem 1. If $\lambda \neq 0$ is an eigenvalue of $f^* : H^k(M) \to H^k(M)$ then by Lemma 4 there is an i such that λ is an eigenvalue of $f^* : H^k(M_i, M_{i-1}) \to H^k(M_i, M_{i-1})$. Now if $u_i =$ fiber $\dim E^u(\Lambda_i)$, then from the proof of Proposition 1 we have $k - u_i \leq \dim_R \hat{W}^s_\varepsilon(x)$ and $u_i - k = (n - k) - s_i \leq \dim_R \hat{W}^u_\varepsilon(x)$ for $x \in \Lambda_i$. Since

$$\begin{aligned}\dim \Lambda_i \geq \dim_R \Lambda_i &= \dim_R \hat{W}^s_\varepsilon(x) + \dim_R \hat{W}^u_\varepsilon(x) \\ &\geq \max \{(k - u_i), (u_i - k)\} = |k - u_i| ,\end{aligned}$$

the proof is complete.

Proof of Theorem 2. If $\Lambda_i \subset M_i - M_{i-1}$ is a periodic orbit of period p, then f^p fixes each point of Λ_i and Df^{2p} preserves an orientation on $E^u(\Lambda_i)$. Let $g = f^{2p}$. Since $\dim \Lambda_i = 0$, it follows from the proof of Theorem 1 or from [1, Theorem 1] that $g^*_k : H^k(M_i, M_{i-1}) \to H^k(M_i, M_{i-1})$ is nilpotent unless $k =$ fiber $\dim E^u(\Lambda_i)$.

Now let $L(g) = \sum_{k=0}^n (-1)^k \operatorname{tr}(g^*_k) = (-1)^u \operatorname{tr}(g^*_u)$ where $u =$ fiber $\dim E^u(\Lambda_i)$. By Lefschetz fixed point theory (see [4, Lemma 3] and [2, Theorem 4.1]). $L(g) = \sum_{q \in \Lambda_i} I(g; q)$ where $I(g; q)$ denotes the index of q under g, which by a result of [9, p. 767] is $(-1)^u$. Hence $(-1)^u \operatorname{tr}(g^m_u)^* = L(g^m) = (-1)^u p$ for all $m > 0$. That is, $\operatorname{tr}(g^m_u)^* = p$ for all $m > 0$, and it follows that the only nonzero eigenvalue of g^*_u is 1, with multiplicity p. This is because the nonzero eigenvalues with multiplicity of a matrix A are determined by the poles of $\exp(\sum_{m=1}^\infty (\operatorname{tr} A^m) z^m / m)$ (see [1] or [9]) and hence g^*_u has the same nonzero eigenvalues as the $p \times p$ identity matrix. Consequently every nonzero eigenvalue of $f^* : H^*(M_i, M_{i-1}) \to H^*(M_i, M_{i-1})$ is a root of unity when Λ_i is finite. This argument is essentially a reproof of a result of M. Shub [8].

Suppose now that M is orientable. If λ is an eigenvalue of $f_k^*: H^k(M) \to H^k(M)$ which is not a root of unity, then it follows by Poincaré duality (see [1, Lemma 4]) that $f_*: H_{n-k}(M) \to H_{n-k}(M)$ has an eigenvalue $\pm\lambda^{-1}$ and hence $f_{n-k}^*: H^{n-k}(M) \to H^{n-k}(M)$ has an eigenvalue which is not a root of unity. Hence, if $\Lambda \subset M_s - M_{s-1}$ is the infinite basic set, then $f_j^*: H^j(M_s, M_{s-1}) \to H^j(M_s, M_{s-1})$ has an eigenvalue which is not a root of unity when $j = k$ and when $j = n - k$. This follows from Lemma 4 and the fact shown above that $f^*: H^*(M_i, M_{i-1}) \to H^*(M_i, M_{i-1})$ has only roots of unity and zero as eigenvalues when $i \neq s$. Thus by Proposition 1, $\dim \Lambda \geq (n - k) - k$ if $n - k \geq k$ and $\dim \Lambda \geq k - (n - k)$ if $k \geq n - k$ so in any case $\dim \Lambda \geq |n - 2k|$. If Λ is an attractor, then the filtration can be chosen such that $(M_s, M_{s-1}) = (M_1, M_0 = \emptyset)$ so $f^*: H^0(M_s, M_{s-1}) = H^0(M_1) \to H^0(M_1)$ is nontrivial and it follows from Proposition 1 that $\dim \Lambda \geq \max\{(n - k), k\}$. This proves the theorem in the case M is orientable.

If M is not orientable, let $\pi: \bar{M} \to M$ be an oriented two-fold covering of M and let $\bar{f}: \bar{M} \to \bar{M}$ cover f. The map $\pi_*: H_k(\bar{M}) \to H_k(M)$ is surjective (see [1, Theorem 1]) so $\pi^*: H^k(M) \to H^k(\bar{M})$ is injective and it follows that $\bar{f}_k^*: H^k(\bar{M}) \to H^k(\bar{M})$ has an eigenvalue which is not a root of unity. Now if $\bar{\Lambda}_i = \pi^{-1}(\Lambda_i)$ it may be that $\bar{f}: \bar{\Lambda}_i \to \bar{\Lambda}_i$ is not topologically transitive, but the proof for the orientable case applied to $\bar{f}: \bar{M} \to \bar{M}$ (using the filtration $\bar{M}_i = \pi^{-1}(M_i)$) still shows that if $\bar{\Lambda} = \pi^{-1}(\Lambda)$ then $\dim \bar{\Lambda} \geq |n - 2k|$ and that if Λ is an attractor then $\dim \bar{\Lambda} \geq \max\{(n - k), k\}$. Since $\dim \Lambda = \dim \bar{\Lambda}$, the result follows.

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