J. DIFFERENTIAL GEOMETRY
12 (1977) 425-434

GLOBAL PROPERTIES OF SPHERICAL CURVES

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Let α be a closed curve regularly embedded in Euclidean three-space satisfying suitable differentiability conditions. In addition, suppose α is nonsingular, i.e., free of multiple points. In 1968, B. Segre [4] proved the following about such curves.

Theorem. *If α is nonsingular and lies on a sphere, and* 0 *denotes any point of the convex hull of α with the condition that* 0 *(if lying on α) is not a vertex of α, then there are always at least four points of α whose osculating plane at each of those points passes through* 0. *If* 0 *is a vertex of α then there are at least three points of α whose osculating plane at each of those points passes through* 0.

All terms used in the statement of the theorem are defined later in this paper.

To quote H. W. Guggenheimer [2] who reviewed [4], "The 12-page proof is rather complicated." Here we present a shorter and hopefully more transparent proof of this theorem. In addition, we need only require that the spherical curve α be of class C^2 whereas Segre's proof requires α be of class C^3. Also, we obtain, with no extra effort, a similar theorem which holds if α's only singularity is one double point; in this case, the above mentioned minimums must be reduced by two.

In the last section of this paper we characterize spherical curves with the following property: for every point 0 of the convex hull of α, other than a vertex of α, there exists the same (necessarily even) number of distinct points of α whose osculating plane at each of those points passes through 0.

The proofs of many results in this paper ultimately depend on ideas contained in a paper by W. Fenchel [1].

Throughout this paper we use the following conventions. By a curve we mean a regular C^2 function $\alpha: D \to E^3$, where D is an interval (with or without end points) or a circle, and E^3 is Euclidean three-space. We let α denote both the function and its configuration $\alpha(D)$ in E^3. When D is a circle we say α is closed. If D is a closed interval we may sometimes refer to α as an arc. We say a point P in E^3 is a multiple point of α if it is the image of $k > 1$ points of D. If $k = 2$ then P is called a double point. At a multiple point P we will think of P as k distinct points each traversed once by α as we traverse D once. If α has no multiple points, then we say α is nonsingular.

Received July 16, 1975. This research was supported by NSF Grant GP 43030.

1. Geodesic curvature

Let α be an oriented spherical curve; i.e., α lies on a sphere S in E^3 and has a preferred direction of traversal. Let S be oriented, say, with respect to the outward pointing normal. We denote by k the geodesic curvature of α as a curve in S. It is defined by $k = (d^2\alpha/ds^2)\cdot n$, where s is the arc length parameter of α consistent with its orientation, and n is $d\alpha/ds$ rotated $+90°$ in the tangent plane to S at its point of contact with S. Since α is C^2, k is a continuous function on α.

At each point P of α there is in S a circle tangent to α which best approximates α near P. This circle $\omega(P)$ is the osculating circle to α at P; it is easy to see that $\omega(P)$ is the intersection of the sphere S and the osculating plane $\pi(P)$ to α at P, when α is viewed as a curve in E^3. We have the following obvious lemma.

Lemma 1. *Let α be a spherical curve and $P \in \alpha$. Then $k(P) = 0$ if and only if $\pi(P)$ goes through the center of S.*

We will need some lemmas about spherical curves proved by Fenchel [1]. Actually we state mild generalizations of these lemmas; see [1], [5] for their proofs. In these lemmas we speak of a set on the sphere being to the left of a curve. By this we mean that when the tangent vector to the curve in the preferred direction is rotated $+90°$ it points into the set. Also when we say a point P is between points A and B we mean that either A and B are antipodes or if A and B are not antipodes then P lies on the shorter geodesic arc through A and B.

Lemma 2. *A nonsingular spherical curve α with $k \geq 0$ and not identically zero connects two points A and B of a great circle γ without otherwise meeting it. Then A and B are not antipodes of one another. In addition the region bounded by the curve and the smaller great circular arc AB of γ and lying in a hemisphere is to the curve's left.*

Lemma 3. *Let α be a nonsingular spherical curve with $k \geq 0$, and let γ be an arbitrary great circle which meets α in at least two points. Then there is a subarc α_γ of α with the following characteristics*:

1. *The end points A and B of α_γ lie on γ.*
2. *α_γ has otherwise no points in common with γ.*
3. *All other points of intersection of α with γ lie between A and B.*

Remark. If α_γ contains a point P for which $k(P) > 0$, then A and B are not antipodal by Lemma 2. In particular, more than a half circle of γ is free of points of intersection with α.

2. Fenchel's theorem

The convex hull of a point set M in Euclidean space is the smallest convex set containing M. Let Ω be the convex hull of a spherical curve α. The next lemma characterizes the points of Ω; for its proof see [1, Satz A].

Lemma 4. *For 0 to be an element of Ω it is necessary and sufficient that there exists a plane λ through 0 such that 0 is in the convex hull of $\alpha \cap \lambda$.*

Throughout this section we take 0 to be the center of the sphere S on which α lies. With this choice for 0, Lemmas 3 and 4 lead immediately to a theorem due to Fenchel [1, Satz II′]. This theorem is restated to include the possibility that 0 is an element of the boundary of Ω as well as the interior of Ω.

Theorem 1 (*Fenchel*). *Suppose α is closed and nonsingular except perhaps for one double point. If $0 \in \Omega$, and α does not contain a great semicircular arc, then the geodesic curvature of α changes sign at least twice.*

The same lemmas can be used to prove the following extension of Theorem 1. This will be shown here.

Theorem 2. *Suppose α is closed and nonsingular. If $0 \in \Omega$, and α does not contain a great semicircular arc, then the geodesic curvature of α changes sign at least four times.*

Remark. It is easy to construct examples of closed nonsingular spherical curves whose geodesic curvature changes sign only twice and which necessarily contain a great semicircular arc. It is a consequence of Lemma 2 that these curves lie in a hemisphere determined by the great semicircular arc.

The remainder of this section is devoted to a proof of Theorem 2. Before we proceed we introduce some notation. If α is a non-closed spherical curve, and P, Q are two points of α, then by $P\alpha Q$ we mean the oriented arc running along α from P to Q. If P, Q are two points of the sphere S which are not antipodal, then PQ denotes the smaller great circular arc through P and Q oriented from P towards Q. To denote the larger great circular arc connecting P and Q, we write PAQ where A is on the great circle through P and Q but $A \notin PQ$. By a Jordan curve we mean a nonsingular continuous image of a circle.

Proof of Theorem 2. Let α be a closed nonsingular curve lying on a sphere S with center 0, and suppose that α contains no great semicircular arc. In particular, α's geodesic curvature k is not identically zero. Also suppose $0 \in \Omega$, the convex hull of α. By Theorem 1 we already know that k changes sign at least twice. We will show that the supposition that k changes sign only twice leads to a contradiction. Therefore suppose k changes sign twice at the points A and B of α. Let α^1 and α^2 be the two curves into which α is separated by A and B, both oriented so that their geodesic curvature is nonnegative (and, of course, not identically zero). Suppose α^1 and α^2 begin at A and end at B.

By Lemma 2 there is a plane λ through 0 such that 0 is in the convex hull of $\lambda \cap \alpha$. Let $\gamma = \lambda \cap S$; it is, of course, a great circle. There are two cases to consider. Either

1. α meets γ in at least three points and these points do not lie in an open half circle of γ, or
2. α meets γ in two points, which are necessarily antipodal.

Case 1. Let C, D, E be distinct points at which $\alpha = \alpha^1 \cup \alpha^2$ meets γ and which do not lie in an open half circle of γ. We may suppose that C and D are points of α^1; in fact, suppose C precedes D in α^1. Since α^1 meets γ in at least two points, Lemma 3 implies that there exists a subarc α^1_γ with the characteristics 1, 2, and 3 of that lemma. Also α^1_γ is not a great semicircular arc. The remark following Lemma 3 implies that E must be a point of α^2. We may assume that C and D are the end points of α^1_γ; if the new C, D, E lie in an open half circle of γ so do the old C, D, E.

Let H be the closed hemisphere determined by γ and not containing α^1_γ except for the end points C and D. Let L be the region to the left of the oriented Jordan curve $C\alpha^1 D \cup DC$ together with its boundary. Lemma 3 implies that $\alpha^1 \subset H \cup L$. In particular $A, B \in H \cup L$; hence α^2 must begin and end in $H \cup L$. The boundary of $H \cup L$ is the Jordan curve $\alpha^1_\gamma \cup DEC$. Now if α^2 is not contained in $H \cup L$, it must cross the boundary along DEC (excluding the end points D and C). Remember that α^1 and α^2 meet only at A and B. We assume without loss of generality that α^2 crosses DEC. If α^2 did not cross DEC, then it would be tangent to γ at E. We could then rotate λ a bit about the diameter of S through C or D so that α crosses γ at points which we still call C, D, E and which still do not lie in an open half circle of γ. Since α^2 meets γ at least twice, Lemma 3 implies the existence of a subarc α^2_γ. Let α^2_γ begin at F and terminate at G. Characteristic 3 of α^2_γ implies that at least one of the points F and G is not between C and D. At this stage of the argument we suppose that F does not lie between C and D. The argument is similar if we suppose that G does not lie between C and D.

Consider the oriented Jordan curve $A\alpha^1 D \cup DF \cup F\alpha^2 A$. If D and F are antipodal, then here DF is the half great circle not containing G; see Fig. 1.

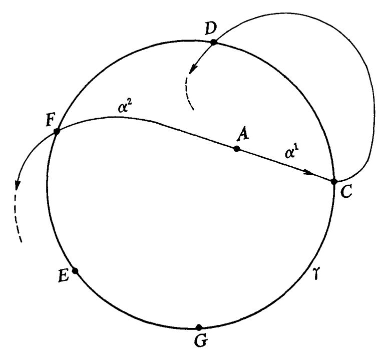

Fig. 1

Note that $D\alpha^1 B$ and $F\alpha^2 B$ cannot cross the Jordan curve. That $F\alpha^2 B$ does not cross DF is the only part of the preceding statement which may not be immediately clear. However $F\alpha^2B$ may only cross γ along FG which is less than a half circle; also DF is at most a half circle. Thus DF meets FG only at F. Thus $F\alpha^2B$ meets DF only at F. Now $D\alpha^1B$ and $F\alpha^2B$ are on opposite sides of the Jordan curve near D and F, respectively. This is clear since α^1 is entering H at D and α^2 is leaving H at F. Thus B is both to the right and the left of the Jordan curve, which is a contradiction.

Case 2. Let C and D be the two points in which α meets γ. As already noted C and D are necessarily antipodal. This case can be reduced to Case 1 since there must be a great circle through C and D which intersects α at a third point E. Clearly C, D, E do not lie in an open half circle.

Remark. We do not use the fact that α^1 and α^2 join at A and B in a C^2 fashion, but only that they begin and end at A and B, respectively.

3. Segre's theorem

Generally, if P is a point of a curve α then at P α passes through the osculating plane to α at P. However if this does not happen we call P a vertex of α. Thus by a vertex of a curve α we mean a point P of α with the property that near P α lies on one side of the osculating plane to α at P.

Theorem 3. *Let α be a closed curve on the sphere S and let $0 \in \Omega$, α's convex hull. Then*

(i) *if α is nonsingular and* 0 *is not a vertex of α, there exist at least four points of α whose osculating plane at each of those points passes through* 0,

(ii) *if α is nonsingular and* 0 *is a vertex of α, there exist at least three points of α whose osculating plane at each of those points passes through* 0,

(iii) *if α's only singularity is one double point and* 0 *is not a vertex of α, there exist at least two points of α whose osculating plane at each of those points passes through* 0.

The idea behind the proof lies in the observation that Theorem 3 follows trivially from Theorems 1 and 2 by means of Lemma 1 if 0 is the center of S. So if 0 is not the center of S we let α^* be the projection of α into a sphere Σ centered at 0 and apply Theorems 1 and 2 to α^* to get the required number of points of α^* whose osculating plane at each of those points passes through 0. If $0 \in \alpha$, then α^* is not a closed curve but one can still show that α^* has the required number of points whose osculating plane at each of those points passes through 0. Finally we observe by Lemma 5 that an osculating plane at a point of α^* passes through 0 if and only if the osculating plane at the corresponding point of α does so.

We now introduce the notation which will be used in the proofs of Lemma 5 and Theorem 3. Let α be a closed curve on S, and Ω the convex hull of α. Suppose that 0 is any element of Ω and Σ is a sphere centered at 0. Let $p: S \to \Sigma$ be the projection of S into Σ through 0. When $0 \in \alpha$, p is understood to be defined only on $S - \{0\}$. Denote the image of $P \in S$ under $p: S \to \Sigma$ by P^*.

If 0 is in interior of S, we let α^* denote the image of α under p. If $0 \in \alpha$, note first that $p(\alpha)$ is contained in a hemisphere H with boundary γ^*, where γ^* is the intersection of the tangent plane to S at 0 with Σ. Assume 0 is not a multiple point of α; then the limits of P^* as P approaches 0 along α first from one side and then the other are two antipodal points on γ^*. We adjoin these points to $p(\alpha)$ and denote the resulting arc by α^*. When 0 is a multiple point of α, we adjoin points of γ^* to $p(\alpha)$ as above to get a collection of arcs denoted by α^*. Then let Ω^* be the convex hull of α^*. Let $\pi(P)$ and $\pi^*(P^*)$ denote the osculating planes to α at P and α^* at P^*, respectively.

Lemma 5. *Suppose* $P \neq 0$. *Then* $\pi(P)$ *passes through* 0 *if and only if* $\pi^*(P^*)$ *goes through* 0. *Moreover, if* $\pi(P)$ *passes through* 0, *then* P *is a vertex of* α *if and only if* P^**is a vertex of* α^*.

Proof. The projection $p: S \to \Sigma$ is a C^∞ diffeomorphism of S onto its image. Thus the order of contact between two curves on S and their images under p on Σ is preserved (except if the contact is at $0 \in \alpha$).

Let $\omega(P)$ and $\omega^*(P^*)$ denote the osculating circles to α at P and α^* at P^*, respectively. Suppose $\pi(P)$ passes through 0. Since $\omega(P)$ lies in $\pi(P)$ which passes through 0, its image under p is a (great) circle on Σ if $0 \notin \alpha$ and is a half (great) circle on Σ if $0 \in \alpha$. Let $\omega(P)^*$ denote the circle in which $p(\omega(P))$ lies on Σ. Since the order of contact is preserved, $\omega(P)^* = \omega^*(P^*)$. Thus both $\pi(P)$ and $\pi^*(P^*)$ contain $\omega(P)^*$. Hence $\pi(P) = \pi^*(P^*)$ passes through 0. The converse is proved in an identifical fashion.

Now suppose $\pi(P)$ passes through 0. Then, by the above, $\pi(P) = \pi^*(P^*)$. If α lies on one side of $\pi(P)$ near P, clearly α^* lies on one side of $\pi^*(P^*)$ near P^* and conversely. That is, P is a vertex of α if and only if P^* is a vertex of α^*.

Proof of Theorem 3. We separate the proof into two cases according as $0 \in \alpha$ or not.

Suppose $0 \notin \alpha$. Then it is clear that $0 \in \Omega^*$ since $0 \in \Omega$. Thus we may apply Theorems 1 and 2 to α^* lying on Σ. If α is nonsingular, so is α^*; thus α^* has at least four points where its geodesic curvature is zero. If α has just one double point, so does α^*; thus α^* has at least two points where its geodesic curvature is zero. By Lemma 1, at each of these points of α^* the osculating plane passes through 0. Hence by Lemma 5 the osculating planes at the corresponding points of α pass through 0. Thus we have proved (i) and (iii) for the case $0 \notin \alpha$.

Suppose $0 \in \alpha$ and 0 is not a multiple point of α. Assume now α is oriented. By means of p we orient α^*. Denote the beginning of α^* by A and the end by B. Let ω be the osculating circle to α at 0. Its image under p including end points, denoted by ω^*, is a half great circular arc of Σ. It is easy to see that ω^* also begins at A and ends at B. Also ω^* and α^* are tangent at A and B. If 0 is not a vertex of α, then α^* is on opposite sides of ω^* in H near A and B; see Fig. 2. If 0 is a vertex of α, then α^* is on the same side of ω^* in H

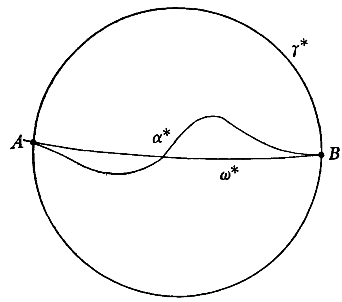

Fig. 2

near A and B. Let k^* be the geodesic curvature of α^*. Then using Lemma 2 and the idea of parity, one can show the following hold:

1. k^* changes sign at least twice if 0 is not a vertex of α and α is nonsingular,
2. k^* changes sign at least twice if 0 is a vertex of α and α is nonsingular,
3. k^* changes sign at least once if 0 is not a vertex of α and α's only singularity is one double point.

Again apply Lemmas 1 and 5, in that order, to prove (i), (ii), and (iii) for the case where $0 \in \alpha$ and 0 not a multiple point of α. If 0 is the double point of α the proof of (iii) is immediate.

Corollary. *Let α be a C^3 closed nonplanar curve in E^3 with no pair of directly parallel tangents. Then α has at least four vertices.*

For the proof of this corollary see Segre [4, p. 263] where the same result is proven for C^4 curves. Our results allow his proof to go through for C^3 curves. Actually the corollary follows immediately from Theorem 2 and the remark following Theoerm 2 since the tangent indicatrix of a nonplanar curve cannot lie in a hemisphere.

4. A characterization

In this section we find a characterization for a (possibly singular) closed curve α lying on the sphere S and having the property that for each point 0 in its convex hull Ω except for vertices of α there exists the same (necessarily even) number of distinct points of α whose osculating plane at each of those points passes through 0.

The next lemma is especially important in this section. It follows by means of stereographic projection from a similar fact for plane curves due to Kneser; see [3, p. 48] for Kneser's theorem and its proof. When we say that the circle ω lies between the (disjoint) circles ω^1 and ω^2 on the sphere S we mean that ω is in the connected component of $S - (\omega^1 \cup \omega^2)$ whose boundary is $\omega^1 \cup \omega^2$.

Lemma 6. *Let α be spherical arc with monotone geodesic curvature k. Let P, Q, and R be three points of α with Q between P and R. Then $\omega(Q)$ is be-*

tween $\omega(P)$ *and* $\omega(R)$ *if it is not equal to* $\omega(P)$ *or* $\omega(R)$. *Moreover,* $\omega(Q) = \omega(P)$ (*respectively,* $\omega(R)$) *only if* $k(Q) = k(P)$ (*respectively,* $k(R)$).

At this point we make some additional assumptions about the closed spherical curve α which will hold throughout the remainder of this section. First, we require that there exists at most a finite number of points of α at which the geodesic curvature k takes on an extreme value. This is equivalent to requiring that α has at most a finite number of vertices since the vertices of α occur at the extremes of k. Secondly, we assume k is strictly monotone between the vertices of α. This second condition rules out the possibility of α having an arc of points with the same osculating plane.

Let B denote the closed ball whose boundary S contains the closed curve α. Clearly $\Omega \subset B$.

Theorem 4. *Suppose α has n vertices. If $0 \in B$, then there exist at most n points of α whose osculating plane at each of those points passes through* 0.

Proof. Let $V_1, V_2, \cdots, V_n$ denote the vertices of α as they occur in making one circuit of α. Using the notation of § 2, we set $\alpha^i = V_i \alpha V_{i+1}$ for $i = 1, 2, \cdots, n$, where $V_{n+1} = V_1$. We will show for each integer i, where $1 \leq i \leq n$, there exists at most one point $P \in \alpha^i$ such that $0 \in \pi(P)$. This immediately implies the theorem.

Suppose, to the contrary, that α^i contains two points P and Q such that $0 \in \pi(P) \cap \pi(Q)$. In particular, $\pi(P) \cap \pi(Q) \neq \emptyset$; hence $\omega(P) \cap \omega(Q) \neq \emptyset$. This is impossible by Lemma 6 since k is strictly monotone on α^i.

Remark. Note that $V_i \in \alpha^{i-1} \cap \alpha^i$ for $i = 1, 2, \cdots, n$, where $\alpha^0 = \alpha^n$. Hence if $0 \in B$ and, in addition, $0 \in \pi(V_i)$, then there exist strictly less than n points of α whose osculating plane at each of those points passes through 0.

Corollary. *Suppose α has n vertices. If $0 \in \Omega$, then there exist at most n points of α whose osculating plane at each of those points passes through* 0.

Let $V_1, V_2, \cdots, V_n$ be the vertices of α. Note that n is necessarily even since it is the number of extreme points of the geodesic curvature of α.

Theorem 5. *Suppose $\omega(V_i) \cap \alpha = \{V_i\}$ for $i = 1, 2, \cdots, n$. Then for every $0 \in \Omega - \{V_1, V_2, \cdots, V_n\}$ there exist exactly n points $P_1, P_2, \cdots, P_n$ of α such that $0 \in \pi(P_i)$ for $i = 1, 2, \cdots, n$, and conversely.*

Proof. Let $B' = B - \bigcup_{i=1}^n \pi(V_i)$. Also let B'_m be the set of points 0 in B' with the property that there exist exactly m points $P_1, P_2, \cdots, P_m$ of α such that $0 \in \pi(P_i)$ for $i = 1, 2, \cdots, m$.

Let $\Omega' = \Omega - \{V_1, V_2, \cdots, V_n\}$. For $i = 1, 2, \cdots, n$, the assumption $\omega(V_i) \cap \alpha = \{V_i\}$ implies $\Omega \cap \pi(V_i) = \{V_i\}$. Thus Ω' is a connected subset of B'. The theorem is proved by showing that for any nonnegative integer m, B'_m is an open and closed subset of B'. This implies $\Omega' \subset B'_m$ for some nonnegative integer m. Then we show $m = n$.

The fact that B'_m is both open and closed in B' follows in three steps:

Step 1. $B'_m \subset$ interior $\bigcup_{m \leq j} B'_j$. Let $0 \in B'$ and suppose there exist m points $P_1, P_2, \cdots, P_m$ of α such that $0 \in \pi(P_i)$ and P_i is not a vertex of α for

$i = 1, 2, \cdots, m$. We will show for each integer i, where $1 \leq i \leq m$, there exists a neighborhood N_i of P_i in α with the property that $U_i = \bigcup_{P \in N_i} \pi(P) \cap B'$ is an open set of B' containing 0. Moreover, we may assume $N_1, N_2, \cdots, N_m$ are mutually disjoint. It is then clear that $U = \bigcap_{i=1}^{m} U_i$ is a neighborhood of 0 in $\bigcup_{m \leq j} B'_j$.

Consider the point P_i. Since P_i is not a vertex there exists an open neighborhood N_i of P_i in α on which k is strictly monotone. By Lemma 6, N_i does not contain P_j, where $j \neq i$. Let P'_i and P''_i be the boundary points of N_i. It follows from Lemma 6 that $\bigcup_{P \in N_i} \omega(P)$ is an open set of S; it is the component of $S - [\omega(P'_i) \cup \omega(P''_i)]$ containing P_i. Then $U_i = \bigcup_{P \in N_i} \pi(P) \cap B'$ is an open set of B'. In fact U_i is the component of $B' - [\pi(P'_i) \cup \pi(P''_i)]$ containing P_i. Clearly $0 \in U_i$ since $P_i \in N_i$.

Step 2. B'_m is closed in B'. Let $0_i, i = 1, 2, \cdots$, be a sequence of points in B'_m approaching $0 \in B'$. Thus for each $i = 1, 2, \cdots$, there exist exactly m points $P_{i1}, P_{i2}, \cdots, P_{im}$ of α such that $0_i \in \pi(P_{ij})$ for $j = 1, 2, \cdots, m$. By taking subsequences if necessary, we may assume that P_{ij} approaches a point P_j as i approaches infinity for $j = 1, 2, \cdots, m$. By continuity $0 \in \pi(P_j)$ for $j = 1, 2, \cdots, m$. Thus there are at least m points of α whose osculating plane at each of those points passes through 0 unless $P_j = P_k$ for some $j \neq k$. Suppose this; then in any neighborhood of $P_j = P_k$ there exist the distinct points P_{ij}, P_{ik}, for i sufficiently large. Since $0_i \in \pi(P_{ij}) \cap \pi(P_{ik})$, $\omega(P_{ij}) \cap \omega(P_{ik}) \neq \emptyset$. By Lemma 6, $P_j = P_k$ is a vertex of α. But this contradicts the assumption $0 \notin \bigcup_{i=1}^{n} \pi(V_i)$. Thus $P_j \neq P_k$ for all $j \neq k$ between 1 and m inclusive. By Step 1 there exist at most m points $P_1, P_2, \cdots, P_m$ of α with $0 \in \pi(P_j)$.

Step 3. B'_m is open in B'. This step follows immediately from Step 1 and Step 2 since $B'_m = \emptyset$ for $m > n$ by Theorem 4.

We now know that $\Omega' \subset B'_m$ where $m \leq n$. Suppose $m < n$. We will show this leads to a contradiction. Let $0 \in \alpha \cap \Omega'$. Since $0 \in \Omega'$, there exist m points $P_1, P_2, \cdots, P_m$ with $0 \in \pi(P_i)$ for $i = 1, 2, \cdots, m$. In the notation of the proof of Theorem 4, there exists an arc α^i for some integer between 1 and n inclusive with the following property: there exists no point $Q \in \alpha^i$ such that $0 \in \pi(Q)$. Thus $\omega(V_i)$ and $\omega(V_{i+1})$ do not have 0 between them. Hence, say, $\omega(V_i)$ and 0 are separated by $\omega(V_{i+1})$. In particular V_i and 0 are on opposite sides of $\omega(V_{i+1})$. Thus α must meet $\omega(V_{i+1})$ at points other than V_{i+1}.

The converse follows from the remark following the proof of Theorem 4.

q.e.d.

It may still be that for every point 0 of Ω' there exists the same number of points of α whose osculating plane at each of those points passes through 0 even though $\omega(V_i) \cap \alpha \neq \{V_i\}$ for some integer $i, 1 \leq i \leq n$. For this to happen the following must be true: if, say, V_1 is a vertex of α and $\omega(V_1)$ intersects α in more than V_1, then there must be another vertex V_i for some integer i, $2 \leq i \leq n$, such that $\pi(V_i) = \pi(V_1)$. Also, for points P near V_1 and Q near V_i, $\pi(P)$ and $\pi(Q)$ must be on opposite sides of $\pi(V_1) = \pi(V_i)$.

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