J. DIFFERENTIAL GEOMETRY
12 (1977) 377-401

HOLOMORPHIC AND DIFFERENTIABLE TANGENT SPACES TO A COMPLEX ANALYTIC VARIETY

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An important invariant in the study of analytic varieties is the local embedding dimension. To measure this precisely one defines $T(V, \mathcal{O}_p)$, the tangent space to V at p with respect to the analytic functions. Similarly one can define tangent spaces with respect to the infinitely differentiable functions C^∞, and the k times continuous differentiable functions C^k, whose dimension is the local C^k or C^∞ embedding dimension. It is known [6], [18] that $T(V, C_p^\infty) = T(V, \mathcal{O}_p)$. In this paper we strengthen that result as follows: there is a locally bounded function $k: V \to \boldsymbol{Z}^+$ such that $T(V, C_p^{k(p)}) = T(V, \mathcal{O}_p)$.

An outline of the paper is the following. First show that for curves, k can be picked $\leq N$, where N is the exponent of the conductor. Then find a curve C in V such that $T(C, \mathcal{O}_p) = T(V, \mathcal{O}_p)$. The local boundedness of k follows by showing there is an upper bound for the conductor number of all nearby linear one-dimensional sections of V. One finds this upper bound by stratifying V into finitely many "equisingular" varieties so that the conductor number is constant on each one.

For curves, we derive some precise estimates for k, and in § 3 we give examples to show these estimates are in general the best possible. Also for each k we show there exists a variety V so that $T(V, C^{k-1}) \neq T(V, \mathcal{O})$, but $T(V, C^k) = T(V, \mathcal{O})$, that is, k is the precise critical degree of differentiability. This enables us to construct a Stein complex space X with no C^∞ embedding in any $\boldsymbol{C}^m$, but for every k there is a C^k embedding into some $\boldsymbol{C}^n$.

The author would like to thank K. Spallek and the referee for pointing out that Theorem 1 of this paper can be obtained directly via 1.1.5 and the last remark of [16]. The methods employed in [16] are somewhat different and do not seem to yield a proof of the curve selection lemma (Theorem 2) or the slicing lemma (Lemma 2) of this paper.

1. Definitions and preliminaries

From [18] we have all of the following. Let V be a complex analytic variety in $\boldsymbol{C}^n$, $p \in V$, C_p^k the ring of germs at p of k times continuously differentiable

Received July 3, 1975, and, in revised form, March 31, 1976. Research supported by a grant from the Purdue Research Foundation.

complex valued functions on C^n, $k = 1, 2, \cdots, \infty$, and $I(V, C_p^k)$ the ideal of functions in C_p^k vanishing identically on V. Then

$$T(V, C_p^k) = \left\{a \in C^n = R^{2n} : \sum_{j=1}^{n} a_j \frac{\partial f}{\partial z_j}(p) + \bar{a}_j \frac{\partial f}{\partial \bar{z}_j} = 0 \text{ for all } f \in I(V, C_p^k)\right\}$$
$$= \left\{(r_1, \cdots, r_{2n}) \in R^n : \sum_{i=1}^{2n} r_i \frac{\partial f}{\partial x_i} = 0 \text{ for all } f \in I(V, C_p^k)\right\} ,$$

where we identify $C^n = R^{2n}$ by $a_k = r_{2k-1} + ir_{2k}$. This is clearly a vector space over the field of real numbers but not necessarily over the complexes: Write $C^n = R^n \oplus iR^n$, $C = R \oplus iR$. Then

$$\begin{aligned} a &= a_x + ia_y , & f &= f_x + if_y , & df &= df_x + idf_y , \\ ia &= -a_y + ia_x , & if &= -f_y + if_x , & d(if) &= -df_y + idf_x = idf , \end{aligned}$$

$$\begin{aligned} a \in T \Leftrightarrow 0 &= a_x(d_x f_x + id_x f_y) + a_y(d_y f_x + id_y f_y) \\ &= a_x d_x f_x + a_y d_y f_x + i(a_x d_x f_y + a_y d_y f_y) \\ &\Leftrightarrow a_x d_x f_x + a_y d_y f_x = 0 = a_x d_x f_y + a_y d_y f_y . \end{aligned}$$

Hence it is sufficient to consider only the real valued f_x and f_y in computing the tangent space.

By $T(V, \mathcal{O}_p)$, we will mean the usual Zariski tangent space, sixth tangent cone of Whitney $C_6(V, p) = \{a \in C^n : ad_pF = 0 \text{ for all } F \in I(V, \mathcal{O}_p)\}$. Other useful tangent cones are the third, fourth, and fifth of Whitney:

$$\begin{aligned} C_3(V, p) &= \{a \in C^n : \exists \text{ sequences } q_i \in V,\ \lambda_i \in C,\ q_i \to p,\ \lambda_i(p - q_i) \to a\} , \\ C_4(V, p) &= \{a \in C^n : \exists \text{ sequences } q_i \in \operatorname{Reg}(V),\ q_i \to p,\ v_i \in T(V, \mathcal{O}_{q_i}) \\ &\qquad\qquad \text{with } v_i \to a\} , \\ C_5(V, p) &= \{a \in C^n : \exists \text{ sequences } q_i, p_i \in V,\ \lambda_i \in C,\ q_i, p_i \to p , \\ &\qquad\qquad \lambda_i(p_i - q_i) \to a\} . \end{aligned}$$

We have the following sequence of strong inclusions:

$$\begin{aligned} &C_3(V, p) \subset C_4(V, p) \subset C_5(V, p) \subset T(V, C_p^1) \subset \cdots \subset T(V, C_p^k) , \\ &T(V, C_p^{k+1}) \subset \cdots \subset T(V, C_p^\infty) \subset T(V, \mathcal{O}_p) . \end{aligned}$$

In addition, Bloom has shown [5] that if p is an isolated singular point of V, then $T(V, C_p^1)$ is the complex linear span of $C_5(V, p)$.

2. One dimensional case

Throughout this section V will be a one-dimensional complex analytic subvariety of C^n with the origin as a singular point. If V is irreducible, $\phi : C \to V$

will denote its normalization. Unless otherwise stated, V will be assumed to be holomorphically imbedded in its minimal possible dimension, that is, $T(V, \mathcal{O}_0) = C^n$. We begin with some rather technical results, the first similar to paragraph 2.2 of [17].

Lemma 1. *If $f \in I(V, C_0^k)$, there is a holomorphic polynomial $P_k(z) = \sum_{|\alpha|\le k} a_\alpha z^\alpha$, with $D^\alpha f(o) = \alpha! a_\alpha$ such that $P_k(z) = o(|z|^k)$ on V.*

Proof. By appropriate choice of coordinates, the normalization ϕ can be written as $\phi(t) = (t^{q_1}, t^{q_2}u_2(t), \cdots, t^{q_n}u_n(t))$, where $q = q_1 \le q_2 \le \cdots \le q_n$ and the u_i's are units; hence $o(|z|) = o(\sum_{i=1}^n |z_i|) = o(|z_1|)$. There exists a polynomial $A_k(z, \bar{z}) = k$th order Taylor expansion of f about the origin such that $f - A_k = \sum_{|\beta|=k} z^\beta g_\beta(z) = o(|z|^k)$, where the g_β are continuous functions such that $g_\beta(o) = 0$. Let $A_k = P_k + Q_k$ be the sum of polynomials with P_k holomorphic and Q_k having no holomorphic terms. Now composing with the normalization and writing holomorphic polynomial $P(t) = P_k(\varphi(t))$, polynomial $Q(t, \bar{t}) = Q_k(\varphi(t))$ with no holomorphic terms, and $l = qk$, we have

$$P(t) + Q(t, \bar{t}) = t^l g(t) + \bar{t}^l h(t) = o(|t|^l) ,$$

where g and h are continuous functions such that $g(o) = h(o) = 0$. Hence neither P nor Q can have any terms of degree l or less, and we conclude that $P(t) = o(t^l)$. Thus $P_k(z) = o(|z|^k)$ on V. So far V has been assumed to be irreducible; but if V is reducible the argument given is valid on each component, and the lemma as stated clearly holds if it holds for z in each component.

Lemma 2. *There is a biholomorphic change of coordinates in C^n so that the normalization has the form $\phi(t) = (t^{q_1}u_1(t), \cdots, t^{q_n}u_n(t))$ where the u_i are units, $q_1 < q_2 < \cdots < q_n$, are there is no polynomial in $\phi_1(t), \cdots, \phi_{k-1}(t)$ whose order is precisely q_k.*

Proof. By induction on k; first given a normalization $\phi(t)$ rearrange the variables $z_1, \cdots, z_n$ so that the lowest q_i is first—this completes the first step of the induction. Now suppose no polynomial in $\phi_1(t), \cdots, \phi_{k-1}(t)$ has order q_k, $q_1 < \cdots < q_k$, and $q_k \le q_l$ for all $l \ge k$. Then rearrange the variables $z_{k+1}, \cdots, z_n$ so $\phi_{k+1}(t)$ has lowest order. If there is a polynomial $h(z_1, \cdots, z_k)$ such that $h(\phi_1(t), \cdots, \phi_k(t))$ has same leading term as $\phi_{k+1}(t)$, make the change of coordinates: $(z_1, \cdots, z_n) \to (z_1, \cdots, z_k, z_{k+1} - h(z_1, \cdots, z_k), z_{k+2}, \cdots, z_n)$ eliminating the leading term of $\phi_{k+1}(t)$. Repeat the process. If the process terminates after finitely many steps the induction is completed.

The map $\psi_k(t) = (\phi_1(t), \cdots, \phi_k(t))$ is one-to-one if and only if the characteristic exponents of the map have greatest common divisor 1. Consider the following cases:

First Case. ψ_k not one-to-one. The gcd of char exp of $\psi_k \ne 1$ but the gcd of char exp of $\phi = 1$ so there must be some more char exp in $\phi_{k+1}, \cdots, \phi_n$. Hence the above process can not continue idenfinitely. More precisely, if the process does not terminate, then $\phi_{k+1}, \cdots, \phi_n$ are formal power series in ϕ_1, ϕ_k;

let $\phi_{k+i} = g_i(\psi_k)$ and $g = (g_1, \cdots, g_{n-k})$. If ψ is not one-to-one, then $t \to (\psi_k(t), g(t))$ is not one-to-one either.

Second Case. ψ_k is one-to-one. Then ψ_k is itself the normalization of a curve, and letting R be the subring of $C\{t\}$, the ring of convergent power series in t, of convergent series in $\phi_1(t), \cdots, \phi_k(t)$ it is well known that R contains all power series of high order. (The ideal J of universal denominators has locus just the origin, so by the Nullstellensatz rad $J = m$, the maximal ideal of $C\{t\}$. Hence there exists $N > 0$ such that for $l > N$, $t^l C\{t\} \subset R$, so $t^l \in R$.) Now if the above process goes on for $N(n-k)$ steps, ord $\phi_{k+1} \geq N$ and in one more step (subtracting off the corresponding convergent power series in R) we can make $\phi_{k+1} \equiv 0$, which contradicts the fact that V is imbedded in minimal possible dimension.

Proposition 1. *Let V be irreducible, then there exists $k > 0$ such that $T(V, C_0^k) = T(V, \mathcal{O}_0)$.*

Proof. Since V is imbedded in minimal dimension, coordinates on C^n can be chosen so that the conclusion of Lemma 2 holds. Then it is sufficient to pick $k = [q_n/q_1] + 1$ where $[r]$ for any real number r is the greatest integer less than or equal to r. Given $f \in I(V, C_0^k)$, need to show $d_0 f = 0$. Now $f(z) - P_k(z) = o(|z|^k)$ on V. Write

$$P_k = L_k + H_k\ , \quad L_k = \sum_{i=1}^{n} a_i z_i\ , \quad a_i = \frac{\partial f}{\partial z_i}(o)\ ,$$

$$H_k = \sum_{2 \leq |\alpha| \leq k} a_\alpha z^\alpha\ , \qquad \sum_{i=1}^{n} a_i t^{q_i} + \sum_{2 \leq |\alpha| \leq k} a_\alpha \phi(t)^\alpha = o(t^{q_1 k})\ .$$

Let a_j be the first nonzero coefficient in sum: if it exists we have a contradiction since $q_j \leq q_n < q_1 k$, and $a_j t^{q_j}$ cannot be cancelled by one of the higher order terms since $H_k(\phi(t))$ cannot have leading term order equal to q_j.

Remark. Any linear map $z \to \sum_{i=1}^{n} c_i z_i$ gives a branched covering of V of some sheeting order. It is easy to see that $q_1, \cdots, q_n$ are exactly all the possible sheeting orders.

Propositian 2. *Let V be a reducible curve, $V = V_1 \cup \cdots \cup V_m$ such that $T(V, \mathcal{O}_0)$ is precisely the complex linear span of $T(V_1, \mathcal{O}_0), \cdots, T(V_m, \mathcal{O}_0)$, then there exists integer $k > 0$ such that $T(V, C_0^k) = T(V, \mathcal{O}_0)$.*

Proof. It is always the case that $T(V, \mathcal{O}) \supset$ Complex Span $\{\bigcup_{i=1}^{m} T(V_i, \mathcal{O})\}$, but in general $T(V, \mathcal{O})$ might be larger. Similarly for all k and i, $T(V_i, C^k) \subset T(V, C^k)$ so Real Span $\{\bigcup_{i=1}^{m} T(V_i, C^k)\} \subset T(V, C^k)$ since $T(V, C^k)$ is a real vector space. Now pick $k_i > 0$ so that $T(V_i, C^{k_i}) = T(V_i, \mathcal{O})$ and $k = \max_i \{k_i\}$. Then for each i, $T(V_i, C^k)$ is a complex vector space so Real Span $\{\cup\ T(V_i C^k)\}$ = Complex Span $\{\cup\ T(V_i, C^k)\}$ and $T(V, C^k) = T(V, \mathcal{O})$.

Remark. It is clear from the proof that k can be picked to be less than the maximal sheeting multiplicity of V.

Proposition 3. *Let V be any curve, then there exists $k > 0$ such that*

$T(V, C_0^k) = T(V, \mathcal{O}_0)$.

Proof. Let $\tilde{\mathcal{O}}_p$ be the germs at the point p of weakly holomorphic functions. An element $u \in \mathcal{O}$ is said to be a universal denominator if $u\tilde{\mathcal{O}}_p \subset \mathcal{O}_p$. Let I be the ideal of $\mathcal{O}_p$ of all functions vanishing on Sing (V) and J be the ideal of universal denominators at p. Then locus $(J) \subset$ Sing V, [10, p. 56], so by the Hilbert Nullstellensatz there is a positive integer N, called the conductor number, such that $I^N \subset J$. We shall show that $k \leq N + 1$.

Let $V = V_1 \cup \cdots \cup V_n$ be the decomposition into irreducible components. If V_i has normalization $\phi_i(t)$, the coordinate with minimal exponent is $C_3(V_i) = v_i$; let $w = \sum a_i v_i$ be a real linear combination of the v_i with each $a_i \neq 0$. Now take a new basis of C^n with w as the first element, $w = z_1$; then $o(|w|) = o(|z|)$ on each component of V, hence on all of V. Also $w \in$ Real Span $(\cup C_3(V_i)) \subset$ Real Span $(\cup C_5(V_i)) \subset$ Real Span $(\cup T(V_i, C^k)) \subset T(V, C^k)$. If $f \in I(V, C^k)$, then $\partial f/\partial w = 0$ since $w \in T(V, C^k)$. Now by Lemma 1, we have $P_k(z) = L_k(z) + H_k(z) = o(|w|^k)$ and $L_k(z)$ has no w term. Hence $P_k(z)/w^k$ is a weakly holomorphic function. Furthermore since V imbedded in minimal possible dimension, w does not divide $P_k(z)$ in $\mathcal{O}$. (Suppose $P_k(z) = wg(z)$. Then $\psi(z) = L_k(z) + H_k(z) - wg(z) \in I(V, \mathcal{O})$ and $d_0\psi = L_k - (g(o), 0, \cdots, 0)$, since L_k has no w term, $d_0\psi \neq 0$ (unless $L_k = 0$) and $T(V, \mathcal{O}) \neq C^n$, a contradiction.) Finally w^N is a universal denominator so $w^N(P_k(z)/w^k)$ is holomorphic. Hence $k \leq N$ or $L_k = 0$.

3. Examples

The estimates given for k in § 2 are, in general, the best possible (Example 1), but are not always the precise minimal values for k (Example 2). There exist space curves requiring an arbitrary large k (Example 3).

Example 1. Let V be the irreducible space curve given by the image of $\phi(t) = (t^3, t^4, t^5)$. Then $T(V, \mathcal{O}_0) = C^3$ because there is no first order f vanishing on V since any such $f = I + H, I$ initial part, H higher order part, $0 = f(\phi(t)) = I(t^3, t^4, t^5) + H(t^3, t^4, t^5)$, order $I = 3, 4$, or 5, and order $H \geq 6$. Now the estimates given for k are [maximum multiplicity/minimum multiplicity] + $1 = [5/3] + 1 = 2$ and the conductor number $+ 1 = 2$: Since the semigroup of Z generated by 3, 4, and 5 contains all integers ≥ 3, the holomorphic functions considered as a subset of the weakly holomorphic functions $\phi^*({}_V\mathcal{O}) \subset {}_C\mathcal{O}$, which are generated by t^3, t^4, and t^5 contain all t^k, $k \geq 3$, and ${}_C\tilde{\mathcal{O}}/\phi^*({}_V\mathcal{O})$ is generated by t and t^2. Hence $z_1 = t^3$, $z_2 = t^4$, $z_3 = t^5$ are all universal denominators, $z_i {}_C\mathcal{O}/\phi^*({}_V\mathcal{O}) = 0$, so conductor number $= 1$.

By either of the above estimates, $T(V, C_0^2) = C^3$. Now we show $T(V, C_0^1) = C^2$ (first two coordinates)—to do this we use Bloom's result $T(V, C_0^1) =$ complex linear span of $C_5(V, 0)$. This is easily computed [5] to be C^2.

Example 2. Let $q < p < r$ be three prime integers such that $q > 5$, $3q < r$, $r < 2p$, and q divides none of $2p$, $2r$, $r - p$, and $r + p$, and r is not in the semigroup of Z generated by q and p; for instance $q=7$, $p=13$, $r=23$. Let V be the image in C^3 of $\phi(t)=(t^q,t^p,t^r)$; then $T(V,\mathcal{O}_0)=C^3$, [max multi/min multi] $+1\geq 4$, conductor number ≥ 4, $T(V,C_0^1)=$ complex linear span of $C_5(V,0)=C^2$, but $T(V,C_0^2)=C^3$. only this last assertion will be verified here.

Let $f\in I(V,C_0^2)$ and show $d_0f=(\partial f/\partial z_1,\partial f/\partial\bar z_1,\partial f/\partial z_2,\partial f/\partial\bar z_2,\partial f/\partial z_3,\partial f/\partial\bar z_3)$ $=(0,0,0,0,0,0)$. Now approximating by Taylor series:

$$f(z)-\sum_{|\alpha|\leq 2}\frac{(z-w)^\alpha}{\alpha!}D^\alpha f(w)=o(|z-w|^2)\,,\qquad z,w\in C^3\,.$$

Composing with the normalization, $w=\phi(t)$, $z=\phi(s)$, writing $f_\alpha(t)=D^\alpha f(\phi(t))$, and realizing the second derivative part of the Taylor series is bounded in comparison to $|z-w|^2$:

$$\begin{aligned}&(s^q-t^q)f_{z_1}(t)+\overline{(s^q-t^q)}f_{\bar z_1}(t)+(s^p-t^p)f_{z_2}(t)\\&\quad+\overline{(s^p-t^p)}f_{\bar z_2}(t)+(s^r-t^r)f_{z_3}(t)+\overline{(s^r-t^r)}f_{\bar z_3}(t)\\&\quad=0([|s^q-t^q|+|s^p-t^p|+|s^r-t^r|]^2)\,.\end{aligned}$$

Now let ω be a primitive qth root of unity, $\omega=e^{2\pi i/q}$, and restrict the above equation to the lines $s=\omega^k t$, $k=1,\cdots,q-1$ to yield

$$\begin{aligned}&(\omega^{kp}-1)t^pf_{z_2}(t)+\overline{(\omega^{kp}-1)}\bar t^pf_{\bar z_2}(t)\\&\quad+(\omega^{kr}-1)t^rf_{z_3}(t)+\overline{(\omega^{kr}-1)}\bar t^rf_{\bar z_3}(t)\\&\quad=0(t^{2p}(|\omega^{kp}-1|^2+|\omega^{kp}-1|\cdot|\omega^{kr}-1|+|\omega^{kr}-1|^2))=0(t^{2p})\,.\end{aligned}$$

Now multiply this equation by t^{-r}, and let $g_1(t)=(t^p/t^r)f_{z_2}(t)$, $g_2(t)=(\bar t^p/t^r)f_{\bar z_2}(t)$, $g_3(t)=f_{z_3}(t)$, and $g_4(t)=(\bar t/t)^rf_{\bar z_3}(t)$. It suffices to show each $a_i=\lim_{t\to 0}g_i(t)$ is zero. Now the g_i satisfy the equations:

$$0=\lim_{t\to 0}(\omega^{kp}-1)g_1(t)+\overline{(\omega^{kp}-1)}g_2(t)+(\omega^{kr}-1)g_3(t)+\overline{(\omega^{kr}-1)}g_4(t)\,,$$

so it suffices to show the following matrix is nonsingular:

$$\begin{bmatrix}\omega^p-1 & \overline{\omega^p-1} & \omega^r-1 & \overline{\omega^r-1}\\ \omega^{2p}-1 & \overline{\omega^{2p}-1} & \omega^{2r}-1 & \overline{\omega^{2r}-1}\\ \omega^{3p}-1 & \overline{\omega^{3p}-1} & \omega^{3r}-1 & \overline{\omega^{3r}-1}\\ \omega^{4p}-1 & \overline{\omega^{4p}-1} & \omega^{4r}-1 & \overline{\omega^{4r}-1}\end{bmatrix}.$$

To compute the determinant, first factor out ω^p-1 from the first column, $\bar\omega^p-1$ from the second column, etc., and then perform row operations to bring it to the Vander Monde form:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \omega^p & \bar{\omega}^p & \omega^r & \bar{\omega}^r \\ \omega^{2p} & \bar{\omega}^{2p} & \omega^{2r} & \bar{\omega}^{2r} \\ \omega^{3p} & \bar{\omega}^{3p} & \omega^{3r} & \bar{\omega}^{3r} \end{bmatrix} .$$

Hence

$$\begin{aligned} \text{determinant} &= (\omega^p - 1)(\bar{\omega}^p - 1)(\omega^r - 1)(\bar{\omega}^r - 1)(\omega^p - \bar{\omega}^p)(\omega^r - \bar{\omega}^r) \\ &\quad \cdot (\omega^r - \omega^p)(\bar{\omega}^r - \omega^p)(\omega^r - \bar{\omega}^p)(\bar{\omega}^r - \bar{\omega}^p) \end{aligned}$$

which is nonzero since $\bar{\omega} = \omega^{-1}$ and $\omega^l = 1$ if and only if q divides l.

Example 3. Given any integer $k > 0$, there exists a curve in $\boldsymbol{C}^3$ such that $T(V, \mathcal{O}) = \boldsymbol{C}^3$ and $T(V, C^k) = \boldsymbol{C}^2$. Pick integers $q < p < r$ as follows: $q > 4k + 2, p = q + 1,\ r = (2k + 1)p - q(k + 1)$, and let V be the image of the map $\theta(t) = (t^q, t^p, t^r)$. By the Whitney extension theorem, one can show the existence of a C^k function ψ vanishing on V with $\partial\psi/\partial z_3(0) \neq 0$; we can also find another function in $I(V, C^k)$ whose partial with respect to $\bar{z}_3$ is nonzero. We need to choose continuous functions ψ_α on V, $\psi_0 = 0$ on V so that

$$\psi_\alpha(x) - \sum_{|\beta| \leq k - |\alpha|} \frac{1}{\beta!}(x - y)^\beta \psi_{\alpha+\beta}(y) = o(|x - y|^{k-|\alpha|})$$

for $x, y \in V$, $|\alpha| \leq k$, and $\psi_{(0,0,0,0,1,0)}(0) \neq 0$; ψ_α is supposed to represent the restriction to V of $D^\alpha\psi$ where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$, $D^\alpha = \partial_{z_1}^{\alpha_1}\partial_{\bar{z}_1}^{\alpha_2}\partial_{z_2}^{\alpha_3}\partial_{\bar{z}_2}^{\alpha_4}\partial_{z_3}^{\alpha_5}\partial_{\bar{z}_3}^{\alpha_6}$. Since ψ_α are to be defined only on V, and θ is a homeomorphism it suffice to define $\psi_\alpha(\theta)$ which simplifies notation. Start off by choosing

$$\psi_\alpha(\theta(t)) = \begin{cases} 1, & |\alpha| = 1 \text{ and } \alpha_5 = 1\ , \\ 0, & |\alpha| = 1 \text{ and } \alpha_5 = 0\ , \\ 0, & |\alpha| > 1 \text{ and either } \alpha_5 > 1 \text{ or } \alpha_6 > 0\ , \\ f_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}(t), & \alpha_5 = 1 \text{ and } \alpha_6 = 0\ , \end{cases}$$

where the f_α's are yet to be determined (except for $f_{(0,0,0,0)}\theta(t) = t^r$). In this notation, the limiting condition becomes: letting $m = k - 1$, $x = \theta(s)$, $y = \theta(t)$,

$$f_\alpha(s) - \sum_{|\beta| \leq m - |\alpha|} (\theta(s) - \theta(t))^\beta f_{\alpha+\beta}(t)/\beta! = o(|\theta(s) - \theta(t)|^{m-|\alpha|})$$

for all $s, t \in \boldsymbol{C}$, $|\alpha| \leq m$. Now the data will be chosen and different reasons given for the limit above to go to zero near the origin and away from the origin.

There are two notations of C^k on Reg V one given by Whitney's theorem and the other given by the differential structure of X as a complex manifold—

we want to know that these are the same. (Generalizing Lemma 4.2 of [5].) Suppose $f \in C^k(\mathrm{Reg}\,(V))$ and we are given data f_α which satisfy the chain rules:

$$\frac{\partial}{\partial t} f_\alpha(\theta) = \frac{\partial \theta_1}{\partial t} f_{\alpha+(1,0,0,0)} + \frac{\partial \theta_3}{\partial t} f_{\alpha+(0,0,1,0)} ,$$

$$\frac{\partial}{\partial \bar{t}} f_\alpha(\theta) = \frac{\partial \theta_2}{\partial \bar{t}} f_{\alpha+(0,1,0,0)} + \frac{\partial \theta_4}{\partial \bar{t}} f_{\alpha+(0,0,0,1)} ,$$

where $\theta(t) = (\theta_1, \theta_2, \theta_3, \theta_4) = (t^q, \bar{t}^q, t^p u(t), \overline{t^p u(t)})$. Then $f\,|\,\mathrm{Reg}\,(V) \in C^k$ and satisfies

$$(**) \qquad \frac{\partial^{i+j}}{\partial^i t \partial^j \bar{t}} f(\theta) = \sum_{0<|\beta|\le i+j} C_\beta f_\beta \sum_{\substack{\beta_1+\cdots+\beta_m=\beta \\ |\beta_1|+\cdots+m|\beta_m|=|\alpha|}} \prod_{l=1}^{m} (\theta^{(l)})^{\beta_l} ,$$

where $l = |\alpha| - |\beta| + 1$, and C_β is an integer constant. Thus

$$\lim_{\substack{t,s\to p \\ t\ne s}} \left\{ \left[D^\alpha(f(\theta)) - \sum_{|\beta|\le k-|\alpha|} \frac{(t-s)}{\beta\,!} D^\beta(f(\theta)) \right] \Big/ |t-s|^{k-|\alpha|} \right\} = 0 ,$$

and this limit can be seen to be the same as that in Whitney theorem by substituting $(**)$. See [11, Chapter 1, § 6]. This can best be understood by considering the analogous computation with functions of one variable and $k = 2$. Suppose $g = f(\theta)$ so $g' = f'\theta'$ and $g'' = f''\theta'\theta' + f'\theta''$. We are given that

$$\lim\,[|g(t) - g(s) - (t-s)g'(s) - \tfrac{1}{2}(t-s)^2 g''(s)|\,|t-s|^{-2}] = 0 ,$$

and want to show

$$\lim\,[|f(x) - f(y) - (x-y)f'(y) - \tfrac{1}{2}(x-y)^2 f''(y)|\,|x-y|^{-2}] = 0 .$$

Since $\lim |(\theta(t) - \theta(s))/(t-s)|$ exists and is nonzero, we can replace the later limit by

$$\lim\,[|g(t) - g(s) - (\theta(t)-\theta(s))f'(\theta(s)) - \tfrac{1}{2}(\theta(t)-\theta(s))^2 f''(\theta(s))|\,|t-s|^{-2}]$$

to see that this converges to zero, subtract it from out given limit, substitute for g' and g'', and regroup terms to get

$$\lim\,[f''(s)|\theta(t) - \theta(s) - \theta(t-s)\theta'(s) - \tfrac{1}{2}(t-s)^2\theta''(s)|\,|t-s|^{-2}]$$
$$+ \lim \frac{f''(s)}{2}\left[\frac{(\theta(t)-\theta(s))^2}{|t-s|^2} - \theta'(s)\theta'(s)\right] = 0 .$$

Lest the choice of data appear altogether magical, we first show that for the case $q = 3$, $p \ge 7$, the data are unique and lead naturally to the general choices. Now for $f = w^2/z = t^{2p-3}$, $m = [p/3] \le 2$, so f is supposed to be at least twice differentiable and

$$\frac{|f(s) - f(t) - (s^q - t^q)\partial_z f(t) - \overline{(s^q - t^q)}\partial_{\bar z} f(t) - (s^p - t^p)\partial_w f(t) - \overline{(s^p - t^p)}\partial_{\bar w} f|}{|s^q - t^q|^2 + |s^p - t^p|^2}$$

is bounded as $t, s \to 0$. Letting $s = \omega t$ or $\omega^2 t$, where ω is a primitive cube root of unity, we have $t^q = s^q$ and

$$(\omega^r - 1)t^{r-2p} - (\omega^p - 1)t^{-p}\partial_w f(t) - \overline{(\omega^p - 1)}(\bar t/t)^p \partial_{\bar w} f(t)$$

and

$$(\omega^{2r} - 1)t^{r-2p} - (\omega^{2p} - 1)t^{-p}\partial_w f(t) - \overline{(\omega^{2p} - 1)}(\bar t/t^2)^p \partial_{\bar w} f(t)$$

are bounded as $t \to 0$. Multiplying by t^{2p-r} yields the matrix equation

$$\begin{bmatrix} \omega^r - 1 \\ \omega^{2r} - 1 \end{bmatrix} = \lim_{t\to 0} \begin{bmatrix} \omega^p - 1 & \overline{\omega^p - 1} \\ \omega^{2p} - 1 & \overline{\omega^{2p} - 1} \end{bmatrix} \begin{bmatrix} \partial_w f(t) & t^{p-r} \\ \partial_{\bar w} f(t) & \bar t^p / t^r \end{bmatrix} .$$

Since p and q are relatively prime, the above 2×2 matrix is nonsingular and we can solve for $\lim_{t\to 0} \partial_{\bar w} f(t)\bar t^p/t^r = (\omega^r - \omega^p)(1 - \omega^r)/((\bar\omega^p - \omega^p)(1 - \bar\omega^p)) = 1$ and $\lim_{t\to 0} \partial_w f(t) t^{p-\sigma} = (1 - \omega^r)(\omega^r - \bar\omega^p)/((\omega^p - \bar\omega^p)(1 - \omega^p)) = 0$. Thus choosing $\partial_w f \equiv 0$ and $\partial_{\bar w} f = t^r/\bar t^p$, the chain rules

$$r t^{r-1} = q t^{q-1}\partial_z f + p t^{p-1}\partial_w f , \qquad 0 = q\bar t^{q-1}\partial_{\bar z} f + p\bar t^{p-1}\partial_{\bar w} f$$

imply that $\partial_z f = (r/q)t^{r-q}$ and $\partial_{\bar z} f = (-p/q)t^r/\bar t^q$.

More generally, we extend the above data by recalling that f_α is supposed to represent $D^\alpha F = (\partial^{\alpha_1}/\partial z)(\partial^{\alpha_2}/\partial \bar z)(\partial^{\alpha_3}/\partial w)(\partial^{\alpha_4}/\partial \bar w)F$ and defining higher derivatives inductively: any f_α with $\alpha_3 \neq 0$ or $\alpha_4 \geq 2$ is identically zero; terms f_α with $\alpha_4 = 1$ satisfy the formula $f_\alpha(t) = f_{(\alpha_1,\alpha_2,0,0)}(t)/\bar t^p$; $f_{(\alpha_1+1,\alpha_2,0,0)}$ and $f_{(\alpha_1,\alpha_2+1,0,0)}$ are determined by the chain rules. Hence we have

$$f_\alpha = \begin{cases} 0 , & \text{if } \alpha_3 > 0 \text{ or } \alpha_4 > 1 , \\ C_\alpha t^{r-q\alpha_1}/\bar t^{q\alpha_2 + p\alpha_4} , & \text{otherwise} , \end{cases}$$

where $C_\alpha = \prod_{i=1}^{\alpha_1}(r/q - i + 1)\prod_{j=1}^{\alpha_2}(-p/q - j + 1)$. By the inequalities $r - mq > \sigma - (m-1)q - p > 0$, note that f_α is bounded on V if and only if $|\alpha| \leq m$.

Let

$$g_\alpha(s, t) = f_\alpha(s) - \sum_{|\beta| \leq m - |\alpha|} \frac{(\theta(s) - \theta(t))^\beta}{\beta!} f_{\alpha+\beta}(t) .$$

We must show that $g_\alpha(s, t) = 0(|s^q - t^q|^{m-|\alpha|} + |s^p - t^p|^{m-|\alpha|})$ uniformly in s and t. Choose a real constant $c > 0$ so small that the set $\{\lambda : |\lambda^q - 1| \leq c\}$ consists of q connected components about the qth roots of unity. We will treat the cases $|s^q - t^q| \leq c|t|^q$ and $|s^q - t^q| \geq c|t|^q$ separately.

Case A. $|s^q - t^q| \geq c|t|^q$. We have

$$|s|^q \leq |s^q - t^q| + |t^q| \leq (1 + 1/c)|s^q - t^q| ,$$
$$|s^p - t^p| \leq |s|^p + |t|^p \leq M|s^q - t^q|^{p/q} , \qquad M > 0 .$$

Thus

$$f_\alpha(t) = 0(|t|^{r - q(\alpha_1 + \alpha_2) - p\alpha_4}) = 0(|s^q - t^q|^{r/q - (\alpha_1 + \alpha_2) - p\alpha_4/q}) ,$$
$$|\theta(s) - \theta(t)|^\beta = 0(|s^q - t^q|^{\beta_1 + \beta_2 + p\beta_4/q}) , \qquad (\beta_3 = 0) .$$

Hence $g_\alpha(s, t) = O|s^q - t^q|^{r/q - (\alpha_1 + \alpha_2) - p(\alpha_4/q)} = o(|s^q - t^q|^{m - |\alpha|})$ by inequalities (*).

Case B. $|s^q - t^q| \leq c|t|^q$. We set $s = \lambda\omega^t$ where $\omega^q = 1$ and $|\lambda - 1| \leq \frac{1}{2}$. Note that if $\alpha_4 + \beta_4 \leq 1$, we have

$$f_{\alpha+\beta}(t) = \frac{f_\alpha(t)}{t^{q\beta_1}\bar{t}^{q\beta_2 + p\beta_4}} \prod_{i=1}^{\beta_1} (\sigma/q - \alpha_1 - i + 1) \prod_{j=1}^{\beta_2} (-p/q - \alpha_2 - j + 1) .$$

Thus if we set

$$\binom{\gamma}{\delta} = \frac{1}{\delta!} \prod_{i=1}^{\delta} (\gamma - i + 1) ,$$

we have

$$g_\alpha(s, t) = f_\alpha(s) - f_\alpha(t) \sum_{\substack{|\beta| \leq m - |\alpha| \\ \alpha_4 + \beta_4 \leq 1}} \binom{r/q - \alpha_1}{\beta_1}\binom{-p/q - \alpha_2}{\beta_2} \frac{(\theta(s) - \theta(t))^\beta}{t^{q\beta_1}\bar{t}^{q\beta_2 + p\beta_4}} ,$$

$$g_\alpha(\lambda\omega t, t) = C_\alpha(t^{r - q\alpha_1}/\bar{t}^{q\alpha_2 + p\alpha_4})\Big[\omega^{r + p\alpha_4}\lambda^{r - q\alpha_1}/\bar{\lambda}^{q\alpha_2 + p\alpha_4}$$
$$- \sum_{\substack{|\beta| \leq m - |\alpha| \\ \alpha_4 + \beta_4 \leq 1}} \binom{r/q - \alpha_1}{\beta_1}\binom{-p/q - \alpha_2}{\beta_2}(\lambda^q - 1)^{\beta_1}(\bar{\lambda}^q - 1)^{\beta_2}(\bar{\omega}^p\bar{\lambda}^p - 1)^{\beta_4}\Big] .$$

Consider the function (with γ_1 and γ_2 real) $h(z, w) = z^{\gamma_1}w^{\gamma_2}$ where $h(1, w) = w^{\gamma_2}$ and $h(z, 1) = z^{\gamma_1}$. The Taylor expansion gives

$$h(z, w) = \sum_{|\delta| \leq l} \binom{\gamma_2}{\delta_1}\binom{\gamma_2}{\delta_2}(z - 1)^{\delta_1}(w - 1)^{\delta_2} + 0(|z - 1|^{l+1} + |w - 1|^{l+1}) ,$$

provided that $|z - 1| \leq \frac{1}{2}$ and $|w - 1| \leq \frac{1}{2}$. Then letting $\phi(\lambda) = h(\lambda^q, \bar{\lambda}^q) = \lambda^{q\gamma_1}\bar{\lambda}^{q\gamma_2}$, we have

$$\lambda^{q\gamma_1}\bar{\lambda}^{q\gamma_2} = \sum_{|\delta| \leq l} \binom{\gamma_1}{\delta_1}\binom{\gamma_2}{\delta_2}(\lambda^q - 1)^{\delta_1}(\bar{\lambda}^q - 1)^{\delta_2} + 0(|\lambda^q - 1|^{l+1}) .$$

Applying this to the above:

first when $\alpha_4 = 1$, then $r + p\alpha_4 \equiv 0 \pmod q$,

$$\begin{aligned} g_\alpha(\lambda wt, t) &= C_\alpha(t^{r-q\alpha_1}/\bar{t}^{q\alpha_2+p})[\lambda^{r-q\alpha_1}/\bar{\lambda}^{q\alpha_2+p} - \lambda^{r-q\alpha_1}/\bar{\lambda}^{q\alpha_2+p} \\ &\qquad + 0(|\lambda^q - 1|^{m-|\alpha|+1})] \\ &= 0(t^{r-(m-1)q-p}|(\lambda\omega t)^q - t^q|^{m-|\alpha|}) . \end{aligned}$$

So in this case (using *), $g_\alpha(s, t) = o(|s^q - t^q|^{m-|\alpha|})$.

Now when $\alpha_4 = 0$, $\sigma \equiv -p \pmod q$, so

$$\begin{aligned} g_\alpha(\lambda\omega t, t) &= C_\alpha(t^{r-q\alpha_1}/\bar{t}^{q\alpha_2})[\bar{\omega}^p\lambda^{r-q\alpha_1}/\bar{\lambda}^{q\alpha_2} \\ &\quad - \sum_{|\beta|\le m-|\alpha|}\binom{r/q-\alpha_1}{\beta_1}\binom{-p/q-\alpha_2}{\beta_2}(\lambda^q-1)^{\beta_1}(\bar{\lambda}^q-1)^{\beta_2} \\ &\quad - (\bar{\omega}^p\bar{\lambda}^p - 1)\sum_{|\beta|\le m-|\alpha|}\binom{r/q-\alpha_1}{\beta_1}\binom{-p/q-\alpha_2}{\beta_2}(\lambda^q-1)^{\beta_1}(\bar{\lambda}^q-1)^{\beta_2}] \\ &= 0(|t|^{r-q|\alpha|}[\bar{\omega}^p\lambda^{r-q\alpha_1}/\bar{\lambda}^{q\alpha_2} - \lambda^{r-q\alpha_1}/\bar{\lambda}^{q\alpha_2+p} + |\lambda^q - 1|^{m-|\alpha|+1} \\ &\quad - (\bar{\omega}^p\bar{\lambda}^p - 1)\lambda^{r-q\alpha_1}/\bar{\lambda}^{q\alpha_2+p} + |\bar{\omega}^p\bar{\lambda}^p - 1|\,|\lambda^q - 1|^{m-|\alpha|}]) \\ &= 0(|t|^{r-qm}|(\lambda t)^q - t^q|^{m-|\alpha|} + |t|^{r-q(m-1)-p}|(\lambda t)^q - t^q|^{m-|\alpha|-1}|(\omega\lambda t)^p - t^p|) , \\ g_\alpha(s, t) &= 0(|t|^{r-qm}|s^q - t^q|^{m-|\alpha|} + |t|^{r-q(m-1)-p}|s^q - t^q|^{m-|\alpha|-1}|s^p - t^p|) \\ &= o(|s^q - t^q|^{m-|\alpha|} + |s^p - t^p|^{m-|\alpha|}) . \end{aligned}$$

Finally $T(V, C^{k+1}) = T(V, \mathcal{O})$ follows from the estimate given in Proposition 1 since [max multi/min multi] $+ 1 = [(kq + 2k + 1)/q] + 1 = k + 1$.

Note. The exact statement of the Whitney extension theorem being employed here is: Let A be a closed set in $\boldsymbol{R}^n$ and f a continuous real valued function on A. Then a necessary and sufficient condition that f have a C^k extension to some open subset of $\boldsymbol{R}^n$ containing A is that there exist continuous real valued functions $f = f_0, f_\alpha, |\alpha| \le k$ on A such that for all $\alpha, p \in A$

$$\lim_{\substack{x,y\to p\\ x\ne y}} \frac{f_\alpha(x) - \sum_{|\beta|<k-|\alpha|}\left(\frac{x-y}{\beta!}\right)^\beta f_{\alpha+\beta}(y)}{\|x-y\|^{k-|\alpha|}} = 0 .$$

Example 4. A one-dimensional Stein space X such that there is a holomorphic homeomorphism of X into $\boldsymbol{C}^3$, there is no holomorphic embedding of X into any $\boldsymbol{C}^n$, but for every k there is a C^k embedding $X \to \boldsymbol{C}^{2k+1}$. We will give curves X_k in $\boldsymbol{C}^k$ such that for $l \le k-1$, $T(X_k, C^l) = \boldsymbol{C}^{l+1}$. Let $q > 4k + 2$, $r_l = (2l + 1)(q + 1) - (l + 1)q = lq + 2l + 1$, for $0 \le l \le k-2$, and X_k the curve in $\boldsymbol{C}^k$ given by $t \to (t^q, t^{q+1}, t^{r_1}, t^{r_2}, \cdots, t^{r_{k-2}})$. The methods of Example 3 applied to curve $\pi_i(X_k)$, $i = 3, \cdots, k$ show that $T(X_k, C^l) = \boldsymbol{C}^{l+1}$, where $\pi_i: \boldsymbol{C}_k \to \boldsymbol{C}^3$, $\pi_i(X_1, \cdots, X_n) = (x_1, x_2, x_i)$. Now patch these X_k together away from the singular points to form a noncompact irreducible one-dimensional complex space X. By [Gunning & Rossi, Theorem IX, B. 10] X is a Stein space, so [Gunning & Rossi, Theorem VII, C. 10] there is a holomorphic homeomorphism of X into $\boldsymbol{C}^3$. The obstruction to the existence of a holomorphic embedding of X in some C^n is that the local holomorphic embedding dimension be bounded. However one can embed X in C^{2k+1} with C^k functions by using a partition of unity to patch together local embeddings. Alternately one can construct a weakly holomorphic homeomorphism $X \to C^{2k+1}$ by usual method for Stein space.

Remark. Of course if there is a fixed n such that for every k there is a C^k embedding of a Stein space X into C^n, then there is a holomorphic embedding into some C^n.

4. Equisingular case

For a special class of varieties, we can show that the conductor number is an upper bound for least k such that $T(V, C_p^k) = T(V, \mathcal{O}_p)$. For a variety V of pure dimension r in C^n, let $C = \text{Sing}\,(\text{Sing}\,V) \cup \{p \in V \mid \dim C_4(V, p) > r\} \cup \{p \in V \mid \dim C_5(V, p) > r + 1\}$ where $C_4(V, p)$ and $C_5(V, p)$ are the fourth and fifth Whitney tangents cones to V at p, [24], [25]. Then C is an analytic subset of V of codimension at least two [22, Prop. 3.6] and every $p \in V - C$ has an open neighborhood so that after a local biholomorphic change of coordinates the following hold (and V is said to be equisingular at p):

(i) For each irreducible component V_i of V, $V_i \cap \text{Sing}\,V = \text{Sing}\,V_i = C^{r-1}$, [22, Props. 2.10, 2.12, and 4.5].

(ii) Each component has a one-to-one nonsingular normalization [22, Prop. 4.2] $\phi: D \to V_i$ given by $\phi(t_1, \cdots, t_r) = (t_1, \cdots, t_{r-1}, t_r^q, \phi_{r+1}(t), \cdots, \phi_n(t))$, where q is the sheeting order of $\pi \mid V_i$ and $\pi(x_1, \cdots, x_n) = (x_1, \cdots, x_r)$. The branching set of this projection is just $\phi(\{t_r = 0\}) = C^{r-1}$.

Now let $\text{Cond}_p(V)$ denote the conductor number of the variety at the point p. If V_i is a component of V, it is clear that any universal denominator for V is a universal denominator for V_i and since $\text{Sing}\,V_i = \text{Sing}\,V$, we have that $\text{Cond}_p(V) \geq \text{Cond}_p(V_i)$. For any fixed $s = (t_1, \cdots, t_{r-1})$ consider the curve W_s in V_i given by $t_r \to \phi(s, t_r)$. Since this curve W_s lies in $E_s = s \times C^{n-r+1}$, weakly holomorphic functions on W_s extend to weakly holomorphic functions on V_i by ignoring the first $r - 1$ variables. Hence any universal denominator for V_i is a universal denominator for W_s and $\text{Cond}_p(V_i) \geq \text{Cond}_p(W_s)$.

Note that for s in a neighborhood of p, $\text{Cond}_s(V) \leq \text{Cond}_p(V)$. (The ideal sheaf of J is coherent [7, Theorem 22] because it is the kernel of $\mathcal{O} \to \text{Hom}_{\mathcal{O}}(\hat{\mathcal{O}}, \tilde{\mathcal{O}}/\mathcal{O})$, hence $I(Sg(V))/J$ is coherent; the index of nilpotence of a coherent sheaf is an upper semi-continuous function.) We will show that for $k > \text{Cond}_p(V)$, $T(V, C_p^k) = T(V, \mathcal{O}_p)$. Then defining $k(p)$ to be the minimal such k, we have a function $V \to \boldsymbol{Z}$. Then k is bounded on compact sets.

Now we need to prove

Lemma 2. *There is an analytic set $A' \subset C^{r-1}$ such that $\bigcup_{s \notin A'} T(V, \mathcal{O}_s)$ is a complex vector bundle over $C^{r-1} - A'$ such that for $s \notin A'$, $E_s \cap T(V, \mathcal{O}_s) = T(W_s, \mathcal{O}_s)$ and $T(V, \mathcal{O}_s) = T(W_s, \mathcal{O}_s) \oplus C^{r-1}$.*

Then letting $N = \mathrm{Cond}_p(V) + 1$, for all $s \notin A'$, s near p, $\mathrm{Cond}_s(V) < N$ so $T(W_s, C_s^N) = T(W_s, \mathcal{O}_s)$ by Proposition 3. Now $C^{r-1}, W_s \subset V$, so $C^{r-1} = T(C^{r-1}, C_s^N) \subset T(V, C_s^N)$ via [18, Satz 1.2.1] and $T(W_s, C_s^N) \subset T(V, C_s^N)$. Hence $T(V, C_s^N) \supset T(W_s, C_s^N) \oplus T(C^{r-1}, C_s^N) = T(W_s, \mathcal{O}_s) \oplus C^{r-1} = T(V, \mathcal{O}_s)$, so $T(V, C_s^N) = T(V, \mathcal{O}_s)$.

But Lemma 2 just follows from a sequence of results of an earlier work [3, § 2]. Let U be a polydisk in $\mathbf{C}^m$ centered at 0 and (z_i) coordinates in U. Given $f_1, \cdots, f_m \in \Gamma(U, \mathcal{O}_U^l)$ we denote by R the sheaf of relations among (f_j). For any integer q, $0 \leq q \leq n$, we may write $U = U_{n-q} \times U_q \subset C^{n-q} \times C^q$. For $a \in U_{n-q}$ we set $U_q^a = \{b \in U : b - a \in U_q\}$. We denote by $R \mid U_q^a$ the restriction of R to U_q^a and by $R(U_q^a)$ sheaf of relations among $(f_j \mid U_q^a)$.

Lemma α. *For each integer q there is a negligible set $A_q \subset U_{n-q}$ such that each point $p \in (U_{n-q} - A_q) \times (0)$ has a neighborhood N_p on which there are $\alpha_1, \cdots, \alpha_k \in \Gamma(N_p, R)$ with the property that $(\alpha_i \mid N_p \cap U_q^a)$ generate $R(U_q^a) \mid N_p \cap U_q^a$. Hence $R(U_q^a)$ and $R \mid U_q^a$ agree off of A^q.*

Lemma β. *Let U be as in Lemma α and let X be a pure r-dimensional analytic subset of U. Assume for some fixed q that given any $a \in U_{n-q}$, a is contained in every irreducible component of $X \cap U_q^a$. Denote by I_X the sheaf of germs of holomorphic functions vanishing on X. Then there is a negligible set $A \subset U_{n-q}$ such that given $p \in U_{n-q} - A$ there are a neighborhood N_p of p and $h_1, \cdots, h_m \in \Gamma(N_p, I_X)$ with the following properties:*

(a) *(h_i) generates $I_X \mid N_p$,*

(b) *for any $a \in U_{n-q} - A$, $(h_i \mid U_q^a \cap N_p)$ generates $I_{X \cap U_p^a} \mid U_q^a \cap N_p$.*

In these lemmas, "negligible" means the countable union of local analytic varieties. However it can be seen from the proofs that the set being removed is analytic in the event that the slices of the variety are one-dimensional. These proofs can be found at the end of the section.

Now $T(V, \mathcal{O}_s) = \{a : a \cdot d_p f = 0 \text{ for all } f \in I(V, \mathcal{O}_s)\}$ but it is unnecessary to use all $f \in I(V, \mathcal{O}_s)$, any finite set of generators $h_1, \cdots, h_m$ for $I(V, \mathcal{O}_s)$ over $\mathcal{O}_s$ will suffice $f = (f_1, \cdots, f_m)$, $f_i \in {}_V\mathcal{O}$ on V among $(dh_1, \cdots, dh_m)$ and $\mathcal{S}$ = the sheaf of relations $(f, g) = (f_1, \cdots, f_m, g_1, \cdots, g_m)$, $f_i \in \mathcal{O}$, $g_i \in \mathcal{O}^m$ on C^n among $(dh_1, \cdots, dh_m, h_1, \cdots, h_m)$. Define $\pi : \mathcal{S} \to R$ by $\pi(f, g) = f$. Clearly π is onto and $R \mid \{s\} = T(V, \mathcal{O}_s)$. By Lemma α, there exists analytic $A_1 \subset C^{r-1}$ so that for $a \notin A_1$, $\mathcal{S}(U_q^a) = \mathcal{S} \mid U_q^a$ and hence $R(U_q^a) = R \mid U_q^a$. Then for $a \notin A_1$, $R(U_q^a) \mid \{s\} = (R \mid U_q^a) \mid \{s\}$ and by definition $(R \mid U_q^a) \mid \{s\} = E_s \cap T(V, \mathcal{O}_s)$. For $a \notin A_1 \cup A$, by definition $R(U_q^a) \mid \{s\} = T(W_s, \mathcal{O}_s)$. Thus letting $m = \max_{p \in C^{r-1}} \{\mathrm{rank}_p\,(dh_1, \cdots, dh_m)\}$ and $A_2 = \{p \in C^{r-1} : \mathrm{rank}_p\,(dh_1, \cdots, dh_m) < m\}$, $A' = A_2 \cup A_1 \cup A$ is the required set.

Remark. It is actually unnecessary to remove the set A_2, and the reason for this is extremely revealing for what is going on in the above discussion. Really Lemma 2 states that there exists a curve C in V, $C = W_s \cup$ coordinate axis in C^{r-1}, such that $T(C, \mathcal{O}) = T(V, \mathcal{O})$; hence $T(V, C^k) \supset T(C, C^k) = T(C, \mathcal{O}) = T(V, \mathcal{O})$. At point $s \in A_2$ however, to get such a curve it is not sufficient to just take intersections of V with linear subspaces. This is illustrated by the following example.

Let V be the image in C^4 of $\phi(s, t) = (s, t^3, t^4, st^5)$; none of ϕ_i is a power series in $\{\phi_j\}_{j \neq i}$, so $T(V, \mathcal{O}) = C^4$. Now if we restrict as above to the slice $x_1 = s = 0$, we get $(0, t^3, t^4, 0)$ whose tangent space is $C_{x_2x_3}$, not $C_{x_2x_3x_4}$ as needed. Taking a nonsingular slice back in the normalization, $s = ct^k$, c constant, $k > 0$, yields $(ct^k, t^3, t^4, t^{k+5})$; since $k + 5$ is in the semigroup generated by 3 and 4, the tangent space is $C_{x_2x_3}$ if $k > 5$, C_{x_1} if $k = 1$, $C_{x_1x_2}$ if $k = 2$, and a two-dimensional subspace of $C_{x_1x_2x_3}$ if $k = 4$ or 3—in any case nothing in the x_4 direction. If instead one trys a linear section $ax_1 + bx_2 + cx_3 + dx_4 = 0$ in the ambient space, one gets $(-(bt^3 + ct^4)/(a + dt^5), t^3, t^4, -(bt^8 + ct^9)/(a + dt^5))$; the tangent space is a two-dimensional subspace of $C_{x_1x_2x_3}$. Hence we resort to nonsingular sections in the normalization $s^p = t^q$, $q < p$, p and q relatively prime integers, which itself has normalization $\lambda \to (\lambda^q, \lambda^p)$. Composing gives $\lambda \to (\lambda^q, \lambda^{3p}, \lambda^{4p}, \lambda^{5p+q})$ so it is possible for these to be all independent since $5p + q < 6p$ (the semigroup generated by $3p$ and $4p$ does not contain integers between $5p$ and $6p$). In fact $q = 7$ and $p = 11$ works.

Attempted proof of Lemma α (which does not quite work). Use induction on q—the relation of $f_1, \cdots, f_m$ will be reduced to several relations of the type $R(g_1, \cdots, g_m)$ in ${}_{n-1}\mathcal{O}^m$ in a manner which commutes with restriction to U_q^a for each $a \in U_{n-q}$.

We may assume that at least one f_i, say f_m, which is not independent of $z_{n-q+1}, \cdots, z_n$ (or else we get trivially an isomorphism of $R \mid U_{n-q}$ and $R(U_{n-q})$ and we are immediately reduced to the case $q = 0$). Choose coordinates in U_q so that f_m is not identically zero in the z_n direction and writing $z = (x, y) \in U_{n-q} \times U_q$, let $A_1 = \{x \in U_{n-q}\colon (\partial^k f_m/\partial z_n^k)(x, 0) = 0 \text{ for all } k > 0\}$ and $A_2 = \{x \in U_{n-q}\colon \text{for each } f_i \text{ not independent of } z_n, (\partial^k f_i/\partial z_n^k)(x, 0) = 0 \text{ for all } k > 0\}$; each is a proper analytic subset of U_{n-q} and $A' = (A_1 \cup A_2) \times U_q$. If $p \notin A'$, each f_i is regular in the z_n direction so by the Weierstrass preparation theorem, there is a neighborhood N_p of p, unit u and holomorphic polynomial $g_i \in {}_{n-1}\mathcal{O}[z_n]$ so that $f_i = u_i g_i$ in N_p. Now the lemma is a local result and permits multiplication by units so we can replace the f_i's by the g_i's.

A relation $(\alpha_1, \cdots, \alpha_m) \in R$ is said to be a polynomial relation if each $\alpha_i \in {}_{n-1}\mathcal{O}[z_n]$; then R is generated over ${}_n\mathcal{O}$ by the polynomial relations. Let $\alpha \in R$ and for each $i = 1, \cdots, m - 1$ write $\alpha_i = u_i g_m + r_i$ by the division theorem where $u_i \in {}_n\mathcal{O}$ and $r_i \in {}_{n-1}\mathcal{O}[z_n]$ has degree $< \deg g_m$. Let r_m be defined by the equations:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \vdots \\ \alpha_m \end{bmatrix} = u_1 \begin{bmatrix} g_m \\ 0 \\ \vdots \\ 0 \\ -g_1 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ g_m \\ \vdots \\ 0 \\ -g_2 \end{bmatrix} + \cdots + u_{m-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ g_m \\ -g_{m-1} \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_{m-1} \\ r_m \end{bmatrix}.$$

It remians only to show r_m is a holomorphic polynomial. Clearly $(r_1, \cdots, r_m) \in R$ so $r_m g_m = -\sum_{i<m} r_i g_i \in {}_{n-1}\mathcal{O}[z_n]$. By the algebraic divisor algorithm $r_m g_m = Q g_m + R$ where Q, $R \in {}_{n-1}\mathcal{O}[z_n]$ and $\deg R < \deg g_m$. But then R/g_m is holomorphic, g_m vanishes to order $\deg g_m$ in the z_n direction because it is Weierstrass, and R vanishes to lower order. Thus $R \equiv 0$ and $r_m = Q$. Note that each entry of the above relations has degree $< \max \deg g_i = k$.

Next we reduce the polynomial relations among the g_i's to finitely many holomorphic relations among the coefficients of the g_i's. Let $\alpha = (\alpha_1, \cdots \alpha_m)$, $\alpha_i = \sum_{\nu=1}^{k} c_\nu^i z_n^\nu$, $g_i = \sum_{\nu=1}^{k} a_\nu^i z_n^\nu$; then $\alpha \in R$ if and only if $\sum_{i=1}^{m} \sum_{j=0}^{\nu} a_{\nu-j}^i c_j^i = 0$ in ${}_{n-1}\mathcal{O}$ for $\nu = 0, \cdots, 2k$. This means the element

$$[c^i] = (c_0^1, \cdots, c_k^1, c_0^2, \cdots, c_k^2, c_0^{m-1}, \cdots, c_k^{m-1}, \cdots, c_k^m) \in {}_{n-1}\mathcal{O}^{m(k+1)}$$

is a relation between the finitely many sections $\nu = 0, \cdots 2k$:

$$s_\nu = (a_\nu^1, \cdots, a_{\nu-k}^1, a_\nu^2, \cdots, a_{\nu-k}^2, \cdots, a_\nu^{m-1}, \cdots, a_{\nu-k}^{m-1}, a_\nu^m, \cdots, a_{\nu-k}^m) ,$$

where $a_\nu^i = 0$ if $\nu < 0$.

Remark 1. Each s has as entries either all the tail coefficient a_0^i or $a_k^m = 1$. (For $\nu \le k$, it contains a_0^i and for $\nu \ge k$ it contains a_k^m.)

Now one completes the induction by applying the previous construction to each s_ν removing another analytic set so that in the complement each coefficient of s_ν can be written locally as the product of a unit and a holomorphic polynomial. However there is one irrepairable error in this construction: While there is an obvious isomorphism of relations between $(u_1 g_1, \cdots, u_m g_m)$ and relations between $(g_1, \cdots, g_m)$ in the second stage of the induction there are several equations to be satisfied, each inducing an isomorphism of the relations with and without the units, but these isomorphisms are not all the same and the composition of two of them are unlikely to preserve the polynomial relations so it seems impossible to reduce to several relations on ${}_{n-2}\mathcal{O}$. The only way around this difficulty seems to make the following assumption: Each f_i is a holomorphic polynomial, at least one is monic in z_n (hence $A' = \emptyset$) and at each stage of the induction each s has holomorphic polynomials as entries with at least one of them a Weierstrass polynomial. While this assumption might seem rather arbitrary, it is in fact satisfied in the application to the ideal sheaf of a variety.

Proof of Lemma α. We proceed by induction on q. When $q = 0$, U_q^a is just the point $\{a\}$ and $R(U_q^a)$ is the vector subspace of C^l spanned by the vectors $(f_j^i(a))$. The matrix $F = [f_j^i]$ defines a map $F: \mathcal{O}_U^m \to \mathcal{O}_U^l$ such that $R = \ker F$. The set A^0 of points where the rank of (f_j) is less than maximal is analytic and on $U - A^0$, R is the sheaf of sections of a vector bundle so our conclusion holds there. This takes care of $q = 0$.

Next, suppose $q > 0$. We denote by M the field of meromorphic functions on U, and set r equal to the rank of F over M. We may find a matrix B over

M and a holomorphic function h such that

$$hB\cdot F = \left[\begin{array}{c|c} hI & \tilde{f}^i_j \\ \hline 0 & 0 \end{array}\right] .$$

Here I is the $r\times r$ identity matrix, and the $\tilde{f}^i_j$ are holomorphic. Now $\alpha=(\alpha_j)$ $\in R$ if and only if

$$\alpha_i h + \sum_{j>r} \tilde{f}^i_j \alpha_j = 0 , \qquad i=1,\cdots,r .$$

Notice that if we multiply h be any holomorphic function, all of the above remains valid. Writing $z=(x,y)\in U_{n-q}\times U_q$, expand h as $h(x,y)=\sum_\alpha h_\alpha(x)y^\alpha$, and set $A' = \{z\in U\colon h_\alpha(x) = 0\}$ for all α with $|\alpha|\neq\{0\}$. Changing h if necessary we may assume that A' is a proper analytic subset of U. A' is also homogeneous in the last q variables so $A'\cap U_{n-q}$ is a proper analytic subset A'' of U_{n-q} and $A' = A''\times U_q$. If $p\in(U_{n-q} - A'')\times(0)$, then after changing our last q coordinates h will be regular in z_n at p. Now we can find a neighborhood N_p of p such that $h = uh'$, with u a unit and h' a Weierstrass polynomial. If we write $\tilde{f}^i_j = f^i_{j,1}h' + f^i_{j,2}$ and $\alpha_j = \alpha_{j,1}h' + \alpha_{j,2}$, then our previous equations can be written in the form

$$\tilde{\alpha}_i h' + \sum_{j>r} f^i_{j,2}\alpha_{j,2} = 0 , \qquad i=1,\cdots,r .$$

All of the entries in this equation are polynomials of bounded degree in z_n. Thus we may view these last equations as a larger set of equations involving functions of $n-1$ variables. These may be thought of as defining a system of relations equivalent to the restriction to N_p of those we began with. Because of the lack of dependence on z_n we may view these last equations as defining relations on $N_p\cap(C^{n-1}\times(p_n))$. All of the above commutes with restriction to U^a_q, so we may assume inductively that our lemma holds on N_p. If we cover $U_{n-q} - A''$ with a locally finite set (N_{p_i}), then it is easy to see from [13] that A'', together with the union of the negligible sets in each N_{p_i}, forms a negligible subset of U_{n-q}, and in its complement the conditions of the lemma are satisfied. This completes the proof.

Remark 2. If $q=1$ or 0, the set A_q is analytic—because we avoid having to use the divisor theorem in the complement of where we already used it.

Our proof of Lemma 1 is modeled closely on Spallek's work in [13], [14], [15] in which he proved the following converse to our result: If F is a finitely generated subsheaf of $\mathcal{O}$, there is an analytic set A^q of dimension at most $n-q-1$ such that if $g\in\mathcal{O}(U)$ and for every $a\in U_{n-q}$, $g\,|\,U^a_q\in F\,|\,U^a_q$, then $g\in F(U - A^q)$. Because our applications do not permit the type of coordinate changes employed in [13], our result in Lemma α is weaker than the corresponding result in [13].

Proof of Lemma β. In order to apply Lemma α we need to express I_X as a sheaf of relations in a manner which commutes with restriction. To do this we recall Cartan's proof of the coherence of I_X[9]. Since $U_{n-q} \subset X$, we may change our last $n-q$ coordinates so that projection on the first r coordinates induces a u-sheeted branched covering with branch locus B. (At this stage we may have to shrink U. We will only use the local form of this lemma.) Near each point of $D - \Pi(B)$ the map Π has u local inverses of the form $w_j(x) = (x_1, \cdots, x_r, w_{j,r+1}(x), \cdots, w_{j,n}(x))$. Using these we form $\prod_{j=1}^{u}(z_i - w_{j,i}(z))$, and this extends to a polynomial $P_i(z) \in \mathcal{O}(D)[z_i]$. By a linear change of the last $n-r$ coordinates we may insure that the discriminant δ of P_{r+1} is not identically 0. Let $C = \{z \in U : \delta(z) = 0\}$. For $i = r+2, \cdots, n$ we define polynomials $Q_i(z)$ as follows: if $z \in U - C$, then near z

$$Q_i(z) = \sum_{k=0}^{u-1} a_k(x) z_{r+1}^k ,$$

$$a_k(x) = \Delta \cdot \det [1, w_{j,r+1}(x), \cdots, w_{j,r+1}(x)^{k-1}, \delta(x) w_{j,i}(x), \\ w_{j,r+1}(x)^{k+1}, \cdots, w_{j,r+1}(x)^{u-1}] .$$

Here $\Delta^2 = \delta$ and $x = (z_1, \cdots, z_r)$. Q_i extends to an element of $\mathcal{O}(D)[z_{r+1}]$.

Let $\rho : U \to U_{n-q}$ be the canonical projection associated with our choice of coordinates. $A' = \{a \in U_{n-q} : \dim_a C \cap U_q^a = \dim X \cap U_q^a \equiv r - n + q$ is a proper analytic subset of U_{n-q} since $C \cap U_q^a = (p \mid C)^{-1}(a)$, $\{b \in X : \dim_b f^{-1}(f(b)) \geq l\}$ is an analytic set for any holomorphic map $f : X \to Y$ and any integer l, and if $\dim C \cap U_q^a = \dim X \cap U_q^a$ then $C \cap U_q^a$ contains a component of $X \cap U_q^a$ so $a \in C \cap U_q^a$ by hypothesis and $\dim_a C \cap U_q^a = r - n + q$. Notice that for $a \in U_{n-q}$ restriction gives $\Pi : X \cap U_q^a \to D \cap U_q^a$. Since $B \subset C$, if $a \in U_{n-q} - A'$ then all of the above constructions commute with restriction to $X \cap U_q^a$. From [9] we know that $g \in I_{X,a}$ if and only if for all sufficiently large N, $\delta^N g$ is in the ideal generated by the germs of $P_{r+1}, \cdots, P_n, \delta z_{r+2} - Q_{r+2}, \cdots, \delta z_n - Q_n$ at a, and similarly for $I_{X \cap U_q^a, a}$. Thus $I_{X,a}$ is the identified set of germs which appear as the first element in $\alpha \in \mathcal{O}_{X,a}^{n-r+1}$, defining a relation between the germs of δ^N, P_{r+1}, $\delta z_i - Q_i$ at a. Now we can apply Lemma α to complete the proof.

Now returning the proof of Lemma 2 (to show analyticity of the removed set), we must study the bad set (ala Lemma α) of the relations among $P_{r+1}, \cdots, P_n, z_{r+2}\delta - Q_{r+2}, \cdots, z_n\delta - Q_n$, which arises in reducing the relations from C^n to C^{n-q}. Since we are assuming each slice of the variety is one-dimensional, $n - q = r - 1$, $q = n - r + 1$. For the first $n - r$ steps of the induction in Lemma α it is possible to use the method of the first proof and hence get no bad set (each P_i, $z_i\delta - Q_i$ is a holomorphic polynomial in ${}_r\mathcal{O}[z_{r+1}, \cdots, z_n]$, and after each reduction to less variables each s_ν has all entries holomorphic polynomials and by Remark 1, at least one entry a Weierstrass polynomial—either the leading coefficient of P_j, equal to 1, or all the P_i, $i < j$).

Then for the last two steps of the induction one can use the method of the second proof of Lemma α and by Remark 2, remove an analytic set.

5. Homogeneous case

Now consider the case of a homogeneous algebraic variety, which is a set $V =$ common locus in $\boldsymbol{C}^n$ of finitely many homogeneous polynomials. Here it is easy to find an analytic curve (reducible) C in V such that $T(V, \mathcal{O}_0) = T(C, \mathcal{O}_0)$. For analytic set V, let $L(V)$ denote the complex linear span. First construct a curve C in V such that $L(C) = L(V)$ as follows: pick finitely many points $v_1, \cdots, v_k \in V$ and let $C_k = L(v_1) \cup \cdots \cup L(v_k)$; clearly $C_k \subset V$ so $L(C_k) \subset L(V)$. If $L(C_k) \neq L(V)$, then $V \not\subset L(C_k)$; pick $v_{k+1} \in V - L(C_k)$ and let $C_{k+1} = L(v_{k+1}) \cup C_k$. Then $\dim L(C_{k+1}) > \dim L(C_k)$ so eventually for some m, $L(C_m) = L(V)$.

Now applying Lemma 3 below to both C and V, we have $T(V, \mathcal{O}) = L(V) = L(C) = T(C, \mathcal{O})$.

Lemma 3. *If V is homogeneous, then $L(V) = T(V, \mathcal{O})$.*

Proof. Any $f \in I(V, \mathcal{O})$ is the sum of homogeneous polynomials which all vanish on V, so V is the common locus of the initial terms which are linear; hence $V \subset T(V, \mathcal{O})$. Since $T(V, \mathcal{O})$ is linear, $L(V) \subset T(V, \mathcal{O})$. On the other hand, dim $T(V, \mathcal{O})$ is the minimal embedding dimension of V, so dim $T(V, \mathcal{O}) \leq \dim \mathscr{L}(V)$.

Remark. It is not at all surprising that the result is so easy for homogeneous varieties since the critical degree of differentiability is just $k = 1$: By the methods [3, Lemma 3] of Lemma 3 one easily sees that $L(C_5(V)) = T(C_5(V), \mathcal{O}) \supset T(C_3(V), \mathcal{O}) = T(V, \mathcal{O})$, but $C_5(V) \subset T(V, C^1)$ so $T(V, \mathcal{O}) \subset L(C_5(V) \subset L(T, C^1)) = T(V, C^1)$ because $T(V, C^1)$ is a complex vector space. Hence $T(V, \mathcal{O}) = T(V, C^1)$.

Alternately, one can see that the critical degree of differentiability is just one as follows: Suppose $T(V, C_0^1) \neq T(V, \mathcal{O}) =$ ambient space, then some differentiable function vanishing on V has a nonzero partial derivative at the origin, so considering the Taylor expansion of f restricted to V we have $z_i/|z| \to 0$ on V as $|z| \to 0$, for some $z_i \not\equiv 0$ on V. But V is homogeneous and $|\lambda z_i|/|\lambda z| = |z_i|/|z|$, so the values of $|z_i|/|z|$ do not change as $|z| \to 0$.

6. General case

Theorem 1. *For any point $p \in V$, a complex analytic variety, there exists an integer $k > 0$ such that $T(V, C_p^k) = T(V, \mathcal{O}_p)$. If $k(p)$ is defined to be the smallest such integer, then the function $k: V \to \boldsymbol{Z}$ is bounded on compact subsets of V and bounded for algebraic varieties.*

The first statement follows from Theorem 2, as pointed out in the remark at the end of the last section. The second statement follows from the proof of Theorem 2.

Theorem 2. *For every $p \in V$, there is a complex analytic curve C in V passing through p such that $T(C, \mathcal{O}_p) = T(V, \mathcal{O}_p)$.*

Proof. This was inspired by [4, § 4] where it is shown that every differential operator on a variety is the finite sum of differential operators on curves in the variety. Unfortunately the proof given there does not seem to guarantee that first order operators are the sum of first order operators on curves.

Proposition 4. *Let V be an analytic variety with* $\dim V > 1$, $p \in V$. *Then there is an analytic variety $W \subset V$ with* $\dim W < \dim V$ *such that $T(W, \mathcal{O}_p) = T(V, \mathcal{O}_p)$. It is clear that* Theorem 2 *follows from* Proposition 4 *by induction. Before starting on the proof we review some well known facts about completion of modules* [26].

Let A be a local noetherian ring with maximal ideal m, and E a finitely generated A module. Then E is given the structure of a topological group with the fundamental system of neighborhoods m^kE, called the *natural topology*. If F is a closed submodule of E, the natural topology of E induces on F the natural topology of F, and the quotient E/F also has the natural topology. The completion (via Cauchy sequences) of E in this topology is $\hat{E} = \varprojlim E/m^kE$ and also has the natural topology given by the fundamental system of neighborhoods $\hat{m}^k\hat{E}$. If $\bigcap_k m^kE = \{0\}$, the canonical map $E \to \hat{E}$ is injective, E is considered as a dense subset of $\hat{E}$, and $\hat{E}$ is complete, that is, $\hat{\hat{E}} = \hat{E}$. If $0 \to F \to E \to G \to 0$ is an exact sequence of finitely generated A modules, then $0 \to \hat{F} \to \hat{E} \to \hat{G} \to 0$ is an exact sequence of finitely generated $\hat{A}$ modules, consequently $E/F = \hat{E}/\hat{F}$, $\hat{F} \cap E = F$, and F is closed in E. Next $\hat{E} = \hat{A}E$, so if a, b are any two ideals of A, $\hat{a}\hat{b} = \hat{A}a\hat{A}b = \hat{A}ab = \widehat{ab}$. If $\tilde{a}$ is any ideal of $\hat{A}$, then $(A \cap \tilde{a})^\wedge = \hat{A}\cdot(A \cap \tilde{a}) \subset \hat{A}A \cap \hat{A}\tilde{a} = \hat{A}\tilde{a} = \tilde{a}$, in summary $(A \cap \tilde{a})^\wedge \subset \tilde{a}$. If a, b are ideals of A and $\hat{a} = \hat{b}$, then $a = \hat{a} \cap A = \hat{b} \cap A = b$. If $\{F_i\}$ is a finite family of submodules of E, then $(\cap F_i)^\wedge = \cap \hat{F}_i$. For an infinite family, we have $(\cap F_i)^\wedge \subset \hat{F}_i$ since the latter is a closed set. For any submodule F of E, $\bigcap_{k=1}^{\infty} (F + m^kE) = \hat{F}$.

If A is the ring of convergent power series over the complexes, $C\{X_1, \cdots, X_n\}$, then $\hat{A} = C[[X_1, \cdots, X_n]]$ the ring of formal power series over C, and every ideal of either ring is closed. By an analytic ring we mean $C\{X_1, \cdots, X_n\}/I$ where I is an ideal. If an analytic ring A is an integral domain, so is its completion $\hat{A}$, [10, Theorem 1], hence the completion of a prime ideal is again prime. Conversely if $\hat{A}$ is an integral domain, then A is an integral domain since it is a subring of $\hat{A}$; if $\tilde{p}$ is prime in $\hat{A}$, then $\tilde{p} \cap A$ is prime in A.

If A is a local noetherian ring, $\dim(A)$ is the largest integer k such that there exists a strictly increasing chain of prime ideals $p_0 \subset p_1 \subset \cdots \subset p_k = m$ of A. The dimensions of $C\{X_1, \cdots, X_n\}$ and $C[[X_1, \cdots, X_n]]$ are both n. The height of a prime p is the length h of the largest chain of primes $p_1 \subset \cdots \subset p_h \subset p$. The depth of a prime p is the length d of the longest chain of primes $p \subset p_1 \subset \cdots \subset p_d = m$, so that $\text{Length}_A(p) + \text{Depth}_A(p) = \dim A$. Depth and height of a prime and dimension of a ring are both preserved by completion.

Now returning to the proof of Proposition 4, assume V is imbedded in minimal possible dimension, that is, $T(V, \mathcal{O}_p) = C^n$ so $I(V) \subset m^2$, where m is the maximal ideal of $\mathcal{O}$. We want to show that there exists an analytic set $W \subset V$, $\dim W < \dim V$ so that $I(W) \subset m^2$. The most naive idea would be to say $\mathcal{O}_V$ is a unique factorization domain, so let W be the union of two different subvarieties W_1, W_2 of codimension one in V, where W_1 is the locus of f_1 so $I(W_1)$ is generated by f_1. Then any $f \in I(W_1 \cup W_2)$ can be written as $f = f_1 g$, $g \in I(W_2)$ so $\operatorname{ord} f \geq 2$. However this does not work: Let $V =$ locus of $z^3 - xy$ in C^3, $W_1 = \text{locus}\,(x) = y \text{ axis} = \{(0, a, 0)\}$, $W_2 = \text{locus}\,(x - z) = W_1 \cup \{(a, a^2, a)\}$, and $f = x - z$. Hence the proposition will have to be proven by contradiction of assumption that all lower dimensional subvarieties have tangent space not equal to C^n.

Let $\dim V = r$, $V = V' \cup V''$, $\dim V' = r$, $\dim V'' \leq r - 1$. Let $V_1'', \cdots, V_l''$ be the irreducible components of V'', and $V_1', \cdots, V_h'$ the irreducible components of V'. Let $I = I(V, \mathcal{O})$. Then $q_i = I(V_i', \mathcal{O})$ and $p_i = I(V_i'', \mathcal{O})$ are all prime and $I = (\bigcap_{i=1}^h q_i) \cap (\bigcap_{i=1}^l p_i)$. Pick a countable set $W_{l+1}, W_{l+2}, \cdots$ of irreducible subvarieties of codimension one in V' such that $\cup W_i$ is dense in V'. (Take local parameterization $\pi: V' \to C^r$ and a countable dense set $a_i \in CP^{r-1}$, such that each a_i determines a hyperplane H_i normal to it. Then $\cup H_i$ is dense in C^r so $\pi^{-1}(H_i)$ is dense in V since π is a closed map. Let W_i be the irreducible components of $\pi^{-1}(H_i)$.) Then $P_{l+i} = I(W_{l+i}, \mathcal{O})$ is prime and $\bigcap_{i \geq 1} P_i = I$ since ano continuous function vanishing on a dense subset of V is identically zero. For all k let $I_k = P_1 \cap \cdots \cap P_k$. Clearly we have

$$I_1 \supset I_2 \supset \cdots I_k \supset \cdots \supset \bigcap_{k=1}^{\infty} I_k = I\,,$$

$$\hat{I}_1 \supset \hat{I}_2 \supset \cdots \supset \hat{I}_k \supset \cdots \supset \bigcap_{k=1}^{\infty} \hat{I}_k \supset \left(\bigcap_{k=1}^{\infty} I_k\right)^{\wedge} = \hat{I}\,.$$

Now $I \subset m^2$, so $\hat{I} \subset \widehat{m^2} = \hat{m}^2$, and the proposition clearly follows from the below lemmas which imply $\hat{I} \not\subset \hat{m}^2$.

Lemma 4. *If no $I_k \subset m^2$, then $\cap \hat{I}_k \not\subset \hat{m}^2$.*

Lemma 5. $\bigcap_{k=1}^{\infty} \hat{I}_k = (\bigcap_{k=1}^{\infty} I_k)^{\wedge}$.

Proof of Lemma 4. Suppose $f_k \in I_k$, $\operatorname{ord} f_k = 1$ for all k. Let H_k^1 be the complex vector space given the image of the natural map $I_k \to m/m^2$. Then $H_1^1 \supset H_2^1 \supset \cdots$ is a decreasing sequence of finite dimensional vector spaces and hence is stable for large j, say $H_j^1 \supset H_l^1$ for all j. By assumption $H_l^1 \neq 0$, choose $0 \neq h_1 \in H_l^1$. Now define homogeneous polynomials h_k of degree k inductively as follows: Suppose $h_1, \cdots, h_{k-1}$ are defined, $\varphi_{k-1} = h_1 + \cdots + h_{k-1}$, so that for all j, $\exists g_j \in \mathcal{O}$, $\operatorname{ord} g_j \geq k$, and $\varphi_{k-1} + g_j \in I_j$. Let H_j^k be the complex vector space spanned by the image S_j^k of the natural map $I_j \to m/m^{k+1}$, restricted to those elements in I_j whose image in m/m^k is φ_{k-1}. Then $H_1^k \supset$

$H_2^k \supset \cdots$ is a decreasing sequence of finite dimensional vector spaces and is stable for large j, say $H_j^k \supset H_l^k$ for all j. Choose $h_k \in S^k$. Then $h_k \in S_j^k$ for all j—apriori h_k is only in H_j^k, but there exist finitely many $c_i \in \boldsymbol{C}$, h_{ji}, $g_{ji} \in \mathcal{O}$, h_{ji} homogeneous polynomial of degree k, ord $g_{ji} \geq k+1$, $\varphi_{k-1} + h_{ji} + g_{ji} \in I_j$, so that $\varphi_k = \varphi_{k-1} + h_k = \sum c_i(\varphi_{k-1} + h_{ji} + g_{ji}) \bmod m^{k+1}$. Comparing terms of orders $k-1$ and k, we have $\sum c_i = 1$ and $\sum c_i h_{ji} = h_k$. I_j is a vector space so $\sum c_i(\varphi_{k-1} + h_{ji} + g_{ji}) \in I_j$ and its image in m/m^{k+1} is φ_k. Hence for all j, $\varphi_k \in S_j^k$ and there exist $g_j = \sum c_i g_{ji}$ so that $\varphi_k + g_{j\infty} \in I_j$ completing the induction. Let φ be the formal sum $\sum_{k=1}^{\infty} h_k \in \hat{\mathcal{O}}$. Then $\varphi = (\varphi_k + g_j) + (\varphi_k - \varphi) - g_j \in I_j + \hat{m}^k + m^k \subset \hat{I}_j + \hat{m}^k$, $\varphi \in \bigcap_k (\hat{I}_j + \hat{m}^k) = \hat{\hat{I}}_j = \hat{I}_j$, so $\varphi \in \bigcap_j \hat{I}_j$. q.e.d.

Now the result of Lemma 5 does not hold for any set of ideals but depends upon the fact that infinitely many $(j > l)$ $\hat{I}_j$ are height-one primes in $\mathcal{O}_V$. Counterexample to general statement of Lemma 5: Let I_n be the ideal in $\boldsymbol{C}\{x, y\}$ generated by $x + \sum_{k=1}^{n} k!\, y^k$ and y^{n+1}. Then $\bigcap_{n=1}^{\infty} I_n = (0)$, so $(\bigcap_{n=1}^{\infty} I_n)^\wedge = (0)$. But $\hat{I}_n =$ ideal in $\boldsymbol{C}[[x, y]]$ generated by the same two elements and contains the divergent series $x + \sum_{k=1}^{\infty} k!\, y^k$, so $\bigcap_{n=1}^{\infty} \hat{I}_n \neq (0)$.

Proof of Lemma 5. The primes $\hat{P}_i$ are all distinct, and

$$\bigcap_{i=1}^{\infty} \hat{P}_i = \bigcap_{k=1}^{\infty} \left(\bigcap_{i=1}^{k} P_i\right)^\wedge = \bigcap_{k=1}^{\infty} \hat{I}_k \supset \hat{I} = \left(\bigcap_{i=1}^{h} \hat{q}_i\right) \cap \left(\bigcap_{i=1}^{l} \hat{p}_i\right).$$

We need to show $\bigcap_{i=1}^{\infty} \hat{P}_i \subset$ each $\hat{q}_j$. Now each $\hat{p}_{l+i}$ contains some $\hat{q}_j$, and each $\hat{q}_j$ is contained in infinitely many $\hat{P}_{l+i}$; let Q_j be the intersection of all $\hat{P}_{l+i}$ such that $\hat{q}_j \subset \hat{P}_{l+i}$. Clearly $\bigcap_{j=1}^{h} Q_j = \bigcap_{i=1} \hat{P}_{l+i}$, $\hat{q}_j \subset Q_j$, and it suffices to prove equality. But $\hat{q}_j$ has depth r, each $\hat{P}_{l+i}$ has depth $r-1$, $\hat{q}_j \subset Q_j \subset$ infinitely many $\hat{P}_{l+i}$, and no ideal can belong to infinitely many minimal primes, soQ_j is prime and equals $\hat{q}_j$ (via Noether Lasker decomposition).

Now we turn to proving that $k: V \to \boldsymbol{Z}$ is bounded by an upper semicontinuous function, and hence $k(p)$ is bounded on compact subset of V. To see this, we look carefully at the curve C_p in V through p given by Theorem 2. It varies analytically—has a bounded number of components and $\bigcap_p C_p$ can be made into the union of equisingular families of curves of a different variety, and so the conductor number of C_p is locally bounded. More specifically, assume $0 \in V$. Then in some neighborhood of the origin we have

Proposition 5. *There is a fixed analytic set L, $0 \in L$, such that for all $p \in$ Sg V, $\dim V \cap (L + p) = 1$, and $T(V, \mathcal{O}_p) = T(V \cap (L + p), \mathcal{O}_p)$.*

Proposition 6. *There is an upper bound, over all $p \in$ Sg V, for the conductor number of $V \cap (L + p)$ at p.*

Clearly the boundedness of $k(p)$ follows from Propositions 5, 6, and 3.

Proof of Proposition 5. It can easily be seen that there is a fixed set of analytic varieties $\{H_i\}$, in $\boldsymbol{C}^n$ each containing the origin such that the curves $H_i \cap V$ are distinct. (If V is of pure dimension, we can choose the H_i to be linear subspaces; otherwise we must start off by taking the locus of functions vanishing identically on all the lower dimensional irreducible components, but not identically on any top dimensional component.) Let $L_k = \bigcup_{i=1}^k H_i$ and $C_k = L_k \cap V$. Then the proof of Proposision 4 shows that for every $p \in V$, there exists $k > 0$ (depending on p) such that $T(V, \mathcal{O}_p) = T(V \cap (L_k + p), \mathcal{O}_p)$. Now let T be the analytic set $\bigcup_{p \in \mathrm{Sg}\, V} p \times T(V, \mathcal{O}_p) \subset \boldsymbol{C}^{2n}$ and $T_k = \bigcup_{p \in \mathrm{Sg}\, V} p \times T(V \cap (L_k + p), \mathcal{O}_p) \subset \boldsymbol{C}^{2n}$; then $T = \bigcup_{k=1}^\infty T_k$. If we knew each T_k were an analytic set, then Proposition 5 would follow from Lemma 6 since T restricted to a compact neighborhood can have only finitely many irreducible components.

Lemma 6. *Let $Z_1 \subset Z_2 \subset \cdots$ be an increasing sequence of analytic sets such that $\bigcup_{k=1}^\infty Z_k = Z$ is analytic and has only finitely many irreducible components. Then $Z = Z_k$ for some k.*

Proof. Every irreducible component Z' of Z must be in some Z_k, or else $Z' = \bigcup_{k=1}^\infty (Z' \cap Z_k)$ is the union of countably many analytic subsets of lower dimension. Since Z' is a complete metric space, the Baire category theorem says it cannot be the union of countably many closed nowhere dense subsets.

But T_k is too hard to work with directly, so instead we introduce some new varieties by stringing out the old ones. Define $W_k = \bigcup_{p \in \mathrm{Sg}\, V} p \times V \cap (L_k + p) \subset \boldsymbol{C}^{2n}$. Then W_k is an analytic set in $\boldsymbol{C}^{2n}$: Let V be the locus of f_i, Sg V the locus of g_i, and L_k the locus of l_i. Then W_k is the set of $(a, b) \in \boldsymbol{C}^{2n}$ such that $g_i(a) = 0$, $f_i(b) = 0$, $l_i(b - a) = 0$. Let $W = \mathrm{Sg}\, V \times V \subset \boldsymbol{C}^{2n}$. Consider Sg V to be in each W_k and W as Sg $V \times 0$. Let $S_k = \bigcup_{p \in \mathrm{Sg}\, V} p \times T(W_k, \mathcal{O}_p)$, $S = \bigcup_{p \in \mathrm{Sg}\, V} p \times T(W, \mathcal{O}_p)$. By the proof of Proposition 4, $S = \cup S_k$, and by Lemma 6, $S = S_k$ for some fixed k, so for all $p \in \mathrm{Sg}\, V$, $T(W_k, \mathcal{O}_p) = T(W, \mathcal{O}_p) = T(\mathrm{Sg}\, V, \mathcal{O}_p) \times T(V, \mathcal{O}_p)$. Let $E = \{(0, b) : b \in \boldsymbol{C}^n\} = 0 \times \boldsymbol{C}^n$ in $\boldsymbol{C}^{2n}$ and $E_a = E + a$. Then $E_p \cap W_k = V \cap (L_k + p)$ and $E_p \cap W = V$. Intersecting the above with E_p yields

$$
\begin{aligned}
T(V \cap (L_k + p), \mathcal{O}_p) &= T(E_p \cap W_k, \mathcal{O}_p) \subset E_p \cap T(W_k, \mathcal{O}_p) \\
&= E_p \cap T(W, \mathcal{O}_p) = T(V, \mathcal{O}_p) ,
\end{aligned}
$$

where the inclusion is not always equality. However by Lemma 2 there is an analytic set $A \subset \mathrm{Sg}\, V$, $\dim A < \dim \mathrm{Sg}\, V$ such that for $p \in \mathrm{Sg}\, V - A$, the above inclusion is an equality. If A were null, the proposition would be proven —but since this is rather unlikely we iterate the construction. Let $\tilde{W}_k = \bigcup_{p \in A} p \times V \cap (L_k + p)$, $\tilde{W} = A \times V$, $\tilde{S}_k = \bigcup_{p \in A} p \times T(\tilde{W}_k, \mathcal{O}_p)$, $\tilde{S} = \bigcup_{p \in A} p \times T(W, \mathcal{O}_p)$. Then $\tilde{S} = \cup \tilde{S}_k$ by Proposition 4, $\tilde{S} = \tilde{S}_k$ for some k by Lemma 6, and by Lemma 2 there exists analytic $B \subset A$, $\dim B < \dim A$ so that for all $p \in A - B$, $T(V \cap (L_k + p), \mathcal{O}_p) = T(V, \mathcal{O}_p)$. This finally yields a stratification of Sg V, and an integer k associated to each strata so that for each point in that strata $T(V \cap (L_k + p), \mathcal{O}_p) = T(V, \mathcal{O}_p)$. Just take the largest of this finite set of integers.

Proof of Proposition 6. Let L_0 be the fixed analytic set of Proposition 5 and $W_0 = \bigcup_{p \in \mathrm{Sg}\, V} p \times V \cap (L_0 + p)$. Let $d = \dim \mathrm{Sg}\, V$, choose a projection $\rho_1 : \boldsymbol{C}^n \to \boldsymbol{C}^d$ giving a local parameterization of $\mathrm{Sg}\, V$ near the origin, choose a projection $\rho_2 : \boldsymbol{C}^n \to \boldsymbol{C}^1$ giving a local parameterization of L_0 near the origin, and define $\pi : \boldsymbol{C}^{2n} \to \boldsymbol{C}^{d+1}$ by $\pi(a, b) = (\rho_1(a), \rho_2(b))$. Then $(\pi \,|\, W_0)^{-1}(0, 0) = \rho_1^{-1}(0) \times \rho_2^{-1}(0) = (0, 0)$, so π gives a local parameterization of W_0 such that $\pi(\mathrm{Sg}\, V) \subset \boldsymbol{C}^d$ where $\boldsymbol{C}^d$ is identified with $\boldsymbol{C}^d \times 0$ in $\boldsymbol{C}^{d+1}$. Let B_1 be the branching set of ρ_1. Let B_2 be the branching set of π—each irreducible component of which either contains $\mathrm{Sg}\, V$ or intersects $\mathrm{Sg}\, V$ in a set of dimension less than $\mathrm{Sg}\, V$—and let B_2' be the union of intersection of those irreducible components of B which do not contain $\mathrm{Sg}\, V$. Let Z be the union of the components of $\mathrm{Sg}\, V$ of non-maximal dimension, and Z' the intersection with $\mathrm{Sg}\, V$ of all irreducible components of $\mathrm{Sg}\, W_0$ which do not lie in $\mathrm{Sg}\, V$. Let $A = \mathrm{Sg}\,\mathrm{Sg}\, V \cup B_1 \cup B_2' \cup Z \cup Z'$.

Then for each $p \in \mathrm{Sg}\, V - A$, there is a neighborhood U of p in $\boldsymbol{C}^{2n}$ such that $\pi \,|\, U \cap W_0$ gives a local parameterization with branching set B a manifold contained in $\boldsymbol{C}^n \times 0$, $\pi(B) \subset \boldsymbol{C}^d$. Now $\pi : W_0 - B \to \boldsymbol{C}^{d+1} - \boldsymbol{C}^d$ is a covering projection and induces a map on the first homotopy groups $\pi_* : \pi_1(W_0 - B) \to \pi_1(\boldsymbol{C}^{d+1} - \boldsymbol{C}^d) \simeq \boldsymbol{Z}$. Since $\boldsymbol{Z}$ is a principal ideal domain, image $(\pi_*) \simeq q\boldsymbol{Z}$ for some q. Let D^{d+1} be a unit polydisc in $\boldsymbol{C}^{d+1}$, $D^d = D^{d+1} \cap \boldsymbol{C}^d$, and $\psi(t_1, \cdots, t_d, t_{d+1}) = (t_1, \cdots, t_d, t_{d+1}^q)$. Then $\psi_*\pi_1(D^{d+1} - D^d)) \simeq q\boldsymbol{Z}$. By a standard result in algebraic topology, there exists a map $\phi : D^{d+1} - D^d \to W_0 - B$ such that $\pi\varphi = \psi$. (Given map $\psi : Z \to X$ and covering map $\pi : \tilde{X} \to X$, then there exists map $\phi : Z \to \tilde{X}$ so $\pi\phi = \psi$ if and only if $\psi_*\pi_1(Z) \subset \pi_*\pi_1(\tilde{X})$.) Then ϕ is holomorphic because locally it is $\pi^{-1}\psi$. Since π is a proper map (inverse image of compact sets are compact), ϕ is bounded near D^d, so by the Riemann removable singularities theorem it extends to a holomorphic map on D^{d+1}, $\phi(t) = (t_1, \cdots, t_d, t_{d+1}^q, \phi_{d+2}, \cdots, \phi_{2n})$. Then ϕ is one-to-one because π and ψ are both q to one off D^d. (Another standard result in algebraic topology is that the number of points in the fiber of a covering map $\pi : \tilde{X} \to X$ is the index of subgroup $\pi_*\pi_1(\tilde{X})$ in $\pi_1(X)$.) In summary, each irreducible component of W_0 has a normalization of the above form.

Let $N_1 = \mathrm{Cond}_0\,(W_0)$, for p near 0, $\mathrm{Cond}_p\,(W_0) \leq N_1$. Now we want to show that for $p \in \mathrm{Sg}\, V - A$, $\mathrm{Cond}_p\,(W_0) \geq \mathrm{Cond}\,(W_0 \cap E_p = V \cap (L_0 + p))$,

$$(*) \qquad \begin{aligned} &\text{e.g., } I(\mathrm{Sg}\, W_0)^k \tilde{\mathcal{O}}_p(W_0) \subset \mathcal{O}_p(W_0) \text{ implies} \\ &\qquad I(\mathrm{Sg}\,(W_0 \cap E_p))^k \tilde{\mathcal{O}}_p(W_0 \cap E_p) \subset \mathcal{O}_p(W_0 \cap E_p)\ . \end{aligned}$$

Since for fixed $s \in \boldsymbol{C}^d$, $\phi(s, t_{d+1})$ is the normalization of $W_0 \cap E_p$, the restriction map $\tilde{\mathcal{O}}_p(W_0) \to \tilde{\mathcal{O}}(W_0 \cap E_p)$ is onto: let $h\phi \in \tilde{\mathcal{O}}_p(W_0 \cap E_p)$, $h\phi \in \mathcal{O}_p(D^1) \subset \mathcal{O}_p(D^{d+1})$ extending the function by ignoring the other d variables so $h\phi \in \mathcal{O}_p(W_0)$. Also any element of $I(\mathrm{Sg}\,(W_0 \cap E_p))^k$ is the sum of elements either identically zeo on $W_0 \cap E_p$ or in $I(\mathrm{Sg}\,(W_0))^k$, in either case a universal denominator of $W_0 \cap E_p$; the set of universal denominators is an ideal so line $*$ is valid.

Now we repeat the construction. Let $\tilde{W}_0 = \bigcup_{p\in A} p \times V \cap (L_0 + p)$ and $N_2 = \mathrm{Cond}_0(\tilde{W}_0)$, take a local parameterization of $\tilde{W}_0$, and remove an analytic set A' of strictly lower dimension to make $\tilde{W}_0$ equisingular along $A - A'$; hence $\mathrm{Cond}_p(V \cap (L_0 + p)) \leq N_2$ for all $p \in A - A'$. This finally gives a stratification of Sg V and an integer N_i associated to each strata so that for each point in that strata $\mathrm{Cond}_p(V \cap (L_0 + p)) \leq N_i$. Just take the largest of this finite set of integers.

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