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ALMOST REGULAR CONTACT MANIFOLDS

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If M^{2n+1} is a C^∞-manifold such that a 1-form ω of class C^∞ is defined over M with the property that $\omega \wedge (d\omega)^n = \omega \wedge d\omega \wedge \cdots \wedge d\omega \neq 0$, then M is said to be a contact manifold and ω a contact form. Classical examples are provided by the $(2n+1)$-sphere S^{2n+1} and the total space of the cotangent sphere bundle of any $(n+1)$-dimensional manifold X. Indeed the latter arises as a level of constant energy in the momentum phase space of a Hamiltonian system, see for example [12], and historically has provided the incentive for the study of contact structures in general. In 1958 W. M. Boothby and H. C. Wang introduced the notion of a *regular* contact form; this, loosely speaking, is such that every point x of M has a neighborhood U which is pierced exactly once by any integral curve of the vector field Z dual to ω, [1]. Regularity is equivalent to the existence of a free S^1-action on M, whose orbit space is a symplectic manifold with integral fundamental class. In this paper we weaken the definition of regularity to allow an integral curve to pierce U a finite number of times, and prove that this implies the existence of a C^∞ S^1-action on M without fixed points, and with only finitely many isotropy groups. The manifold M can be fibred in the sense of Seifert, and assuming the M/S^1 is a C^∞-manifold, it is easy to see that the quotient space of principal orbits admits an integral symplectic form. But perhaps the more interesting result is the converse (Theorem 3 below) which allows us to construct an almost regular contact form on the total space of a suitable Seifert fibration. We apply this in particular, when the base is a projective algebraic variety, and obtain examples of contact forms on $(n-1)$-connected $(2n+1)$-manifolds, $n = 2$ or odd.

The existence of a contact form ω imposes restrictions on the tangent bundle $\tau(M)$. On $\boldsymbol{R}^{2n+1}$ the form $\omega = dz + x_1dx_2 + \cdots + x_{2n-1}dx_{2n}$ is contact, and by Darboux' Theorem [8, p. 132], every point x of the contact manifold M has a coordinate neighborhood on which ω can be expressed in this way. It follows that M has an atlas for which the coordinate transformations are compatible with this standard contact form, and hence that the structural group of $\tau(M)$ reduces to $U(n) \oplus 1$, at least when M is orientable [3, 2.3.2]. This provides the definition of an almost contact manifold, and an almost structure is integrable if it is induced by a contact form ω on M. On an *open* manifold

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the general theory of Gromov (see [4] for example) shows that there is a 1-1 correspondence between homotopy classes of sections of the bundle associated to $\tau(M)$ with fibre $SO(2n+1)/U(n)$, and C^∞-homotopy classes of contact structures. In short for open manifolds the integrability problem disappears. Let us consider the example of 1-connected 5-manifolds, which originally motivated this paper. The only obstruction to an almost structure is the third integral Stiefel-Whitney class, which is the image of $w_2(M)$ under the Bockstein homomorphism. Hence, if the latter class vanishes, $\tau(M)$ has structural group $U(2) \oplus 1$ and ($M-$point) is an open contact manifold. The closed manifold is the connected sum of prime manifolds M_k, $2 \leq k \leq \infty$, where $M_\infty = S^2 \times S^3$ and otherwise M_k has second homology group of order k^2. We prove below that, unless 3 divides the order of k, M_k is a contact manifold. Unfortunately it is not clear that the forms ω_k are compatible with connected sums, so Theorem 6 below is only a weak analogue to the results for open manifolds. However both it and Martinet's cleaner result for orientable 3-manifolds [9] suggest that contact forms, like codimension-1 foliations, are differential geometric structures which extend from almost closed to closed manifolds.

The organization of the paper is as follows. In the first section we define an almost regular contact form, and prove that the integral curves of the associated vector field are the orbits of a smooth S^1-action. Our argument is modelled on that of D. B. A. Epstein in [2], except that the almost regular condition easily implies the existence of a global bound on the point-wise periods. In the second section we construct examples of almost regular contact forms, the orbit spaces of whose associated S^1-actions are C^∞-manifolds. The proofs are similar to those in [1], even though near an exceptional orbit a Seifert fibration need not be locally trivial. The obvious 1-form fails to satisfy the contact condition at points of an exceptional orbit, but we are able to modify it, without destroying invariance with respect to the group action. The last section applies the theory of higher dimensional Seifert fibrations over complex manifolds developed in [10] to (i) homotopy spheres in bP_{4n+2}, (ii) $2m$-connected $(4m+3)$-manifolds and (iii) 1-connected 5-manifolds. In dimension 3 the orbit space of the associated S^1-action is always a surface, and the theory of almost regular contact forms is equivalent to the classical Seifert theory; compare [2] and [9].

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1. Almost regular contact manifolds

If ω is a contact form on M^{2n+1}, then $d\omega$ is a 2-form of rank $2n$, and hence $(\tau_x M)_0 = \{X : X \in \tau_x M \text{ and } d\omega(X, \tau_x M) = 0\}$ has dimension 1. It follows that to ω we may associate a smooth vector field Z such that

(i) $\omega(Z) \equiv 1$, (ii) $i_Z d\omega = 0$.

By the Picard existence theorem for ordinary differential equations, near each point $x \in M$, Z defines a flow which, if M is compact, extends to a smooth global action of the real numbers $\mu: M \times \boldsymbol{R} \to M$ [8, IV, 2]. The orbit $\mu(x \times \boldsymbol{R})$ is the integral curve of Z, which passes through x at time 0. We shall often abbreviate $\mu(x, t)$ to $t \cdot x$.

Definition. The contact form ω is said to be *almost regular*, if there exists a positive integer R, and each point $x \in M$ has a cubical coordinate neighborhood $U = (z, x^1, \cdots, x^{2n})$ such that

(i) each integral curve of Z passes through U at most R times, and

(ii) each component of the intersection of an integral curve with U has the form $x^1 = a^1, \cdots, x^{2n} = a^{2n}$, with a^i constant.

If $R = 1$ the form is said to be *regular*, conforming with the definition in [1]. Intuitively each point of the manifold has a neighborhood pierced by each orbit in at most R segments parallel to the z-axis. That this is a very restrictive condition follows from the first main result.

Theorem 1. *If ω is an almost regular contact form on the compact connected manifold M^{2n+1}, the associated global flow μ is equivalent to a flow inducing an effective C^∞ action on M by the circle group S^1. This action has no fixed points, and the finitely many distinct isotropy subgroups are contained in some finite subgroup A of S^1.*

Proof. That the orbits of $\boldsymbol{R}$ are simple closed curves follows from the fact that M is covered by finitely many good neighborhoods, each containing at most R linear segments of the orbit. Since each orbit is closed we may define the period function $\lambda: M \to \boldsymbol{R}$ by $\lambda(x) = \inf_{0<t} (\mu(x, t) = x)$. Clearly λ is constant on each orbit. The period function λ is lower semicontinuous, hence bounded away from zero; by reparametrising $\boldsymbol{R}$ if necessary we may suppose $\lambda(x) \geq 1$. It is convenient to break the rest of the argument up into several steps.

Lemma 1. *λ has a global upper bound on M, i.e., $1 \leq \lambda(x) \leq \Lambda$ for all $x \in M$.*

Proof. This is again a simple consequence of compactness and the almost regularity condition. M is covered by finitely many closed cubical sets $F_i \subset U_i$, and the natural metrics on U_i may be pieced together by means of a partition of unity to give a global metric on M, restricting to the natural metric on F_i. It follows that a particle started from x at time $t = 0$ spends a total time interval in F_i bounded above by RT_i, where T_i equals the t-width of the cube F_i. The existence of Λ is now clear.

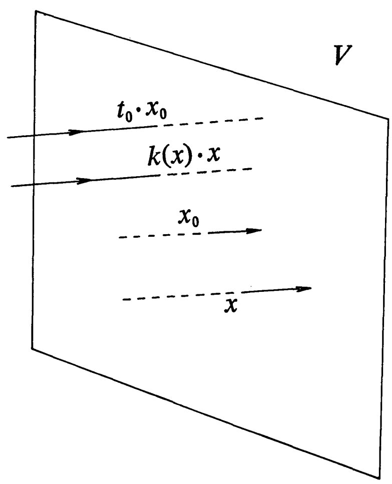
Let U be a regular neighborhood of the arbitrary point $x_0 \in M$, so that x_0 has coordinates $(0, \cdots, 0)$. There is a $2n$-cubical set V with coordinates $(x^1, \cdots, x^{2n})$ transversal to the orbits of μ near x_0. We define $\kappa: V \to \boldsymbol{R}$ by the conditions

(i) $\kappa(x) > 0$,

(ii) $x_t \notin V$ for $0 < t < \kappa(x)$,

(iii) $x_{\kappa(x)} \in V$.
Since each orbit is closed, κ is defined at each point of V; we may think of it as defining the time of first return of an orbit to a neighborhood of its position at time 0. Observe that $\kappa(x) \leq \lambda(x)$ for all $x \in V$, and that $y = \kappa(x)\cdot x$ has z-coordinate 0 in U. Write $t_0 = \kappa(x_0)$.

Lemma 2. *There is a neighborhood $V_3 \subset V$ of x such that $\kappa | V_3$ is a C^∞-function.*



There exists $V_1 \subset V$ such that for $\varepsilon > 0$, $t\cdot x \in U$ for all $x \in V_1$ and $|t - t_0| < \varepsilon$. Moreover ε may be chosen so small that for $x \in V_1$, $\mu((t_0 - \varepsilon, t_0 + \varepsilon), x)$ is connected. For t in this interval $x^i(t\cdot x) = x^i(y)$, and $z(t\cdot x) = g(t, z(x), x^1(x), \cdots, x^{2n}(x))$ is a function of class C^∞ such that

$$g(t_0, 0, \cdots, 0) = 0 .$$

If $|t - t_0|$ and $|t'|$ are both sufficiently small,

$$\begin{aligned} g(t, z(t'\cdot x), x^1(t'\cdot x), \cdots, x^{2n}(t'\cdot x)) &= z(t\cdot t'\cdot x) = z(t + t'\cdot x) \\ &= g(t + t', z(x), x^1(x), \cdots, x^{2n}(x)) , \end{aligned}$$

by the definition of a flow. Since each orbit is normal to the plane $z = 0$, we may choose a new coordinate system near $x_0 = (0, \cdots, 0)$ by setting

$$\bar{x}^i(x) = x^i(y) - x^i(y_0) , \qquad \bar{z}(x) = g(t_0, z(x), \cdots, x^{2n}(x)) .$$

If Z is the vector field associated to ω, and we compute the components of Z_{x_0} with respect to the new coordinates, then the only possible nonzero component is

$$Z_{x_0}(\bar{z}) = \lim_{t\to 0} [g(t_0, z(t\cdot x_0), \cdots, x^{2n}(t\cdot x_0)) - g(t_0, z(x_0), \cdots, x^{2n}(x_0))/t]$$

$$= \lim_{t\to 0} [g(t + t_0, z(x_0), \cdots, x^{2n}(x_0)) - g(t_0, z(x_0), \cdots, x^{2n}(x_0))/t]$$

$$= \frac{\partial g}{\partial t}(t_0, z(x_0), \cdots, x^{2n}(x_0)) \ .$$

Since $\langle \omega, Z \rangle \equiv 1$, $(\partial g/\partial t)|_{(t_0,0)} \neq 0$. Therefore by the implicit function theorem there is a neighborhood of x_0, $V_2 \subset V_1$ on which there is defined a C^∞-function $t = \varphi(x^1(x), \cdots, x^{2n}(x))$ such that

(i) $g(\varphi, 0, x^1(x), \cdots, x^{2n}(x)) = 0$, (ii) $t_0 = \varphi(0, \cdots, 0)$.

The final step is to show that near $(0, \cdots, 0)$, $\kappa(x) = \varphi(x^1(x), \cdots, x^{2n}(x))$. Let x be some point with z-coordinate 0 and $(x^1(x), \cdots, x^{2n}(x)) \in V_2$. If $|t - t_0|$ is sufficiently small, $t\cdot x = \varphi(x^1, \cdots, x^{2n})\cdot x$ has coordinates

$$x^i(t\cdot x) = x^i(y) \ , \qquad z(t\cdot x) = g(\varphi, 0, x^1(x), \cdots, x^{2n}(x)) = 0 \ .$$

Therefore $\varphi(x^1(x), \cdots, x^{2n}(x)) = h\cdot\kappa(x)$, where h is some real number ≥ 1 and dependent on x. We claim that $h(x) \equiv 1$ inside some neighborhood $V_3 \subset V_2$. Otherwise there is a sequence of points $\{x_m\}$ in V_2 converging to x_0 such that $h(x_m) > 1 + \eta$ for some positive constant η. (The difference between the times of first and second return is bounded aways from zero.) Then

$$0 \neq \kappa(x_0) = \lim_{m\to\infty} \varphi(x^1(x_m) \cdots x^{2n}(x_m)) = \lim_{m\to\infty} h(m)\cdot\kappa(x_m) \geq (1 + \eta)\kappa(x_0)$$

by the continuity of φ. Since this is a contradiction, there exists $V_3 \ni (0, \cdots, 0)$ such that $\kappa\colon V_3 \to \boldsymbol{R}$ is of class C^∞. Following [2], define an associated C^∞-map $T\colon V_3 \to V$ by $T(x) = \kappa(x)\cdot x$.

Lemma 3. *There exists a neighborhood W of x in V, so that with respect to suitable coordinates $T|W$ is conjugate to a periodic linear map.*

Proof. If Λ is the global bound on the pointwise periods from Lemma 1, set $N = [\Lambda + 1]$. By induction there exist neighborhoods W_i of x_0 in V such that

$$TW_{i+1} \subseteq W_i \qquad (1 \leq i \leq N! = q) \ .$$

Since $1 \leq \lambda(x) \leq \Lambda$, for each $x \in W_q$, $T^r x = x$ for some r, $1 \leq r \leq N$, and $T^q = 1$ on W_q. The neighborhood $W = \bigcap_{i=1}^q T^i W_q$ is invariant under T, and by averaging the usual metric on V restricted to W, we obtain a T-invariant metric.

The lemma now follows from the commutativity of the diagram

$$\begin{array}{ccc} \boldsymbol{R}^{2n} & \xrightarrow{T_*} & \boldsymbol{R}^{2n} \\ {\scriptstyle \exp^{-1}}\uparrow & & \uparrow{\scriptstyle \exp^{-1}} \\ W & \xrightarrow{T} & W \ . \end{array}$$

Returning to κ, we see that $q\kappa(x) = p\lambda(x)$, where p is an integer depending on x. Covering M by a finite number of regular coordinate neighborhoods of the form $W \times (-\varepsilon, \varepsilon)$ we may construct a C^∞-map $F: M \to \boldsymbol{R}$ with the property that on each orbit F is an integral multiple of λ.

The flow $\nu: M \times \boldsymbol{R} \to \boldsymbol{R}$ given by

$$\nu(x, t) = \mu(x, tF(x))$$

has the same orbits as μ, but $\nu | M \times \boldsymbol{Z}$ is trivial, hence ν induces a smooth S^1-action on M. By the general theory of transformation groups, see for example [6], there are only finitely many orbit types. The group S^1 cannot be an isotropy subgroup, since the period of any orbit is positive. The manifold M is connected, hence the isotropy subgroup corresponding to the principal orbit type is the unique maximal vertex in the "slice diagram" [6, § 4]. Since the action is effective, it follows that the principal isotropy subgroup is the identity. The subgroup A in the conclusion of the theorem may be taken to be the sum of the remaining isotropy subgroups.

2. Seifert $(2n + 1)$-manifolds

If ω is an almost regular contact form on M^{2n+1}, Theorem 1 implies that the associated vector field Z generates an S^1-action, with respect to which M^{2n+1} has the structure of a Seifert fibration, see [10]. Globally, if $\pi: M \to M/S^1 = B$ denotes the projection onto the space of orbits, by first factoring out the action of A, π decomposes as

$$\pi = \pi_2 \circ \pi_1 ,$$

where $\pi_1: M \to N$ is a branched covering map, and $\pi_2: N \to B$ is a principal S^1-fibration. In particular, there is an open dense subset of M, the union of the principal orbits, which has the structure of a principal S^1-fibre bundle. In the neighborhood of an arbitrary point of x, the description of the action is almost as nice. Thus, if S^1_x is the isotropy subgroup of the point $x \in M$, and V is a slice through x, carrying a representation τ of S^1_x, then the quotient $S^1 \times_{S^1_x} V$ is mapped diffeomorphically onto some open neighborhood of the orbit of x [6, 1.3]. Locally, if $\Gamma: C \to S^1$ is a section over some neighborhood of the identity in S^1/S^1_x, then $(c, v) \mapsto \Gamma(c)\cdot v$ is a diffeomorphism of $C \times V$ onto a neighborhood of x.

From now on we assume that B, the space of orbits, has the structure of a compact complex manifold. This will be the case, for example, if S^1 acts analytically on the complex manifold M, with a single nontrivial isotropy subgroup A, such that the slice representation is the sum of a trivial and a 1-dimensional (complex) representation. In particular the union of the exceptional orbits has real codimension two, and all the examples discussed in the final section will be of this type.

We recally that the $2n$-dimensional manifold B is called *symplectic*, if it admits a closed 2-form Ω of maximal rank.

Theorem 2. *If ω is an almost regular contact form on M^{2n+1}, such that B, the space of orbits of the associated S^1-action, is a C^∞-manifold, then the open submanifold B_0 defined by the principal orbits admits an integral symplectic form Ω. Furthermore over B_0*

$$d\omega = \pi^*\Omega .$$

Proof. Since the S^1-action, defined by the associated vector field Z, is free on the union of the principal orbits, this result follows from the main theorem in [1]. The point is that the form ω is S^1-invariant, defines a connection on $\pi^{-1}(B_0)$, whose curvature form is the pullback of the 2-form Ω, defined on the base. Furthermore the cohomology class defined by Ω is integral, and classifies the S^1-bundle structure on $\pi^{-1}(B_0)$. As in the case of regular contact forms this remark is the clue to the converse theorem.

Let L be a nonsingular hypersurface in the projective algebraic variety B, and N the principal S^1-bundle over B defined by the 2-form Ω coming from the natural Kähler structure. Let M be the total space of a Seifert fibration over B, with a single nontrivial isotropy subgroup A, such that

(i) π_2 is the projection map of N, and

(ii) the branching locus of the finite covering π_1 is defined by $\pi_2^{-1}L$.

Theorem 3. *If the Seifert fibration $\pi : M \to B$ is defined by the data above, M admits an S^1-invariant, and hence almost regular, contact form ω.*

Proof. Recall the construction of Kobayashi in [7]. If B_0^{2n} is the base space of a principal S^1-fibration, determined by an integral symplectic form Ω, there is a scalar valued 1-form ω_0 on the total space with the property that $d\omega_0 = \pi^*\Omega$. Since the fibration is principal, and Ω has maximal rank, so does $d\omega_0$. Furthermore, if the vector field $Z = d/dt$ is taken as a basis for the Lie algebra of the group S^1, it determines a "vertical" vector Z^+ at each point x of the total space. By construction $\omega_0(Z^+) \neq 0$, and since $(d\omega_0)^n$ cannot vanish on the element determined by the "horizontal" vectors, the contact condition $\omega_0 \wedge (d\omega_0)^n \neq 0$ is satisfied. Finally, if the vector field X satisfies $d\omega_0(X, \tau_x M_0) = 0$, then

$$\Omega(\pi X, \tau_{\pi x} B_0) = 0 ,$$

implying that X is vertical, and Z the vector field associated to the 1-form ω_0.

In the present case M is a Seifert fibration, in which the nonprincipal orbits map to the hypersurface L in B. On $M_0 = \pi^{-1}(B - L)$ there is a contact form ω_0, but because of the branching along $\pi_2^{-1}L$, the pullback of the contact form ω on the intermediate fibre space N does not define a contact form on M. Indeed the product $\pi_1\omega \wedge (\pi_1\omega)^n$ fails to be a volume form along the nonprincipal orbits. We must therefore modify $\pi_1\omega$ in a neighborhood of the union $\pi^{-1}L$ of the exceptional orbits.

As a subvariety of B, L inherits an integral symplectic form Ω_L, of maximal rank equal to $n-1$. Furthermore the 2-codimensional subvariety $\pi^{-1}L$ of M admits a contact form ω_L (say) by the Kobayashi construction applied over L. If z denotes a normal coordinate in the complex tubular neighborhood V of $\pi^{-1}L$ in M, set

$$\omega_V = \omega_L + \frac{i}{2}(zd\bar{z} - \bar{z}dz) .$$

At interior points of the $(2n+1)$-dimensional manifold V, ω_V satisfies the contact condition. Thus, if $z = u + iv$,

$$\frac{i}{2}(zd\bar{z} - \bar{z}dz) = udv - vdu ,$$

which goes to a multiple of the usual volume form on $\boldsymbol{R}^2$ under exterior differentiation. Furthermore

$$\begin{aligned}(\omega_V) \wedge (d\omega_V)^n &= (\omega_L + udv - vdu)(2ndu \wedge dv \wedge (d\omega_L)^{n-1}) \\ &= \pm 2ndu \wedge dv \wedge \omega_L \wedge (d\omega_L)^{n-1} ,\end{aligned}$$

which is a volume form, given that ω_L satisfies the contact condition on $\pi^{-1}L$. It is possible to choose the forms ω_0 and ω_V compatibly in the sense that where each of them defines a volume form (in M_0 and near $\pi^{-1}L$ respectively), these belong to the same component of the complement of the zero section in the $(2n+1)$st exterior power of τ_M. Assuming the choice of a metric on the fibres of V, we let $\bar{V}_r$ denote the closed tubular neighborhood of $\pi^{-1}L$ such that each fibre is a 2-disc of radius r. If (u, v) are Cartesian coordinates in a generic fibre, we write (r, θ) for the corresponding polar coordinates. With these conventions, define a global 1-form ω_M as follows:

$$\omega_M = \begin{cases} \omega_0 \text{ on } M - \bar{V}_2 , \\ \eta\omega_V \text{ at interior points of } V_1 , \\ (2-r)\eta\omega_V(\underline{x}, r, \theta) + (r-1)\omega_0(\underline{x}, r, \theta) \\ \qquad\qquad \text{on the set } \bar{V}_2 - \operatorname{Int}(V_1),\ \eta \text{ small} . \end{cases}$$

The continuous form so obtained may have to be smoothed at the boundary of V_2 or V_1, but then satisfies the contact condition, because ω_V and ω_0 give rise to compatible volume elements. Invariance with respect to the group action follows, since ω_0 and ω_L are defined using the bundle structure.

Clearly Theorem 3 is a special case of a more general result, in which we allow more than one nontrivial isotropy group, that is, the total space M decomposes as the union of more than two strata. We have confined ourselves to the simplest nonprincipal case, since this is all we shall require in the next section.

3. Applications

In this section we shall apply the construction of almost regular contact forms to highly connected manifolds. The base space B will be $CP(n)$, and the total space M^{2n+1} a branched covering of the standard sphere S^{2n+1}. In the language of the previous section let the hypersurface L have degree δ, and if x belongs to $\pi^{-1}L$, let α be the order of the isotropy subgroup S^1_x. The slice representation is described by the matrix

$$\left(\begin{array}{c|c} e^{2\pi i\nu/\alpha} & 0 \\ \hline 0 & I_{2n-2} \end{array}\right),$$

and we may define β uniquely by

$$0 < \beta < \alpha,\ (\alpha, \beta) = 1 , \quad \text{and} \quad \nu\beta \equiv 1 \bmod \alpha ,$$

In [10, Theorem 3.15] it is shown that M is determined up to orientation-preserving equivariant diffeomorphism by the integral orbit invariants $(\delta\,; \alpha, \beta)$, except that, for n even, $(\delta\,; \alpha, \beta) \sim (\delta\,; \alpha, \alpha - \beta)$. The manifold M is simply connected if and only if $1 = \pm(\alpha + \delta\beta)$. If this condition is satisfied M is $(n - 1)$-connected, and

$$H_n(M, Z) \cong \underbrace{Z/\alpha + \cdots + Z/\alpha}_{\kappa(n)} ,$$

where $\kappa(\delta, n) = [(\delta - 1)^{n+1} - (-1)^{n+1}]/\delta + (-1)^{n+1}$, [10, Theorem 4.9]. By Theorem 3 above a Seifert fibration built up from this data admits an almost regular contact form. The definition of $\kappa(\delta, n)$ yields the following table of values:

$\delta \backslash n$	$n = \text{odd}$	$n = \text{even}$	$n = 2$
1	0	0	0
2	1	0	0
3	$\frac{1}{3}(2^{n+1} - 1) + 1$	$\frac{1}{3}(2^{n+1} + 1) - 1$	2

As an important special case M^{2n+1} is a homotopy sphere if and only if $\delta = 1$ ($\delta = 1$ or 2) for n odd (n even). Most of the examples below can also be succinctly described as Brieskorn varieties [5, § 14]; we use the notation $V(a_0, \cdots, a_{n+1})$ for $\{\underline{z} \in C^{n+2}\colon f(\underline{z}) = z_0^{a_0} + \cdots + z_{n+1}^{a_{n+1}} = 0\}$ and $V_1(a_0, \cdots, a_{n+1}) = V(a_0, \cdots, a_{n+1}) \cap S^{2n+3}$. If d is the least common multiple of the a_i, there is a C^*-action on V, given by $t(z_0, \cdots, z_{n+1}) = (t^{d/a_0}\cdot z_0, \cdots, t^{d/a_{n+1}}z_{n+1})$ with respect to which V is invariant. This C^*-action on V induces an S^1-action on V_1, which thus acquires a Seifert fibre structure.

Lemma 4. *The orbit space of $V(2k+1, 2, \cdots, 2) - (0, \cdots, 0)$ under the action of C^* is $CP(n)$.*

Proof. Project $V - 0$ to $C^{n+1} - 0$ by mapping $(z_0, \cdots, z_{n+1})$ to $(z_1, \cdots, z_{n+1})$. The map is compatible with the C^*-action on the image given by $z_i \mapsto t^{2k+1}z_i$, and V_1 is a branched covering of S^{2n+1}. The equivalence classes in $C^{n+1} - 0$ are lines through the origin, hence the final orbit space is $CP(n)$. The induced S^1-action on V_1 clearly has the same image.

Theorem 4. *Every homotopy sphere $\sum^{2n+1}$ which bounds a parallelisable $2n+2$ manifold admits an almost regular contact structure, $n > 3$, n even.*

Proof. Suppose that n is even, so that $bP_{2n+2} \hookrightarrow Z/2$. By [5, Theorem 11.3] $\sum^{2n+1}$ is represented by $V_1(2k+1, 2, \cdots, 2)$, where for $2k+1 \equiv \pm 1(8)$ we obtain the standard sphere and for $2k+1 \equiv \pm 3(8)$ the Kervaire sphere. In the language of Seifert fibrations over $CP(2n)$ the only nontrivial isotropy group of the S^1-action on V_1 is $Z/(2k+1)$, and the proof of Lemma 4 shows that the exceptional orbits map to a quadric hypersurface. With $\alpha = 2k+1$, $\beta < \alpha$ is determined by the equation

$$1 = (2k+1) + 2\beta, \quad \text{or} \quad \beta = k+1 \qquad \text{modulo } \alpha .$$

The existence of a contact form now follows from § 2.

For the next application we consider $(n-1)$-connected $(2n+1)$-manifolds M for n odd under the simplifying assumptions that M bounds a parallelisable manifold and $H_n(M, Z)$ is a torsion group of odd order (there is nothing special about $n = 3, 7$). Up to the addition of a homotopy sphere M is classified as a C^∞-manifold by $H_n(M, Z)$ and the nonsingular symmetric intersection pairing $H_n(M, Z) \otimes H_n(M, Z) \to Q/Z$ [14, Theorem 7], Moreover M splits as a connected sum of manifolds (i) L_m, $H_n(L_m) \cong Z/m$, $m = p^r$ and $1 \otimes 1 \mapsto c/m$, c a quadratic residue modulo m, and (ii) L'_m, with the same data except that c is a quadratic nonresidue. The decomposition is unique up to relations of the form $L_m \# L_m \cong L'_m \# L'_m$ [13, Theorem 4]. For the prime manifolds we have

Theorem 5. *Let p be an odd prime and $m = p^r$. If $p \equiv 1 \pmod 4$, L_m admits an almost regular contact structure. If $p \equiv 3 \pmod 4$ there is an almost regular contact structure on L'_m if $n \equiv 3 \pmod 4$, and on L_m if $n \equiv 1 \pmod 4$.*

Proof. Consider the Brieskorn variety $V_1(m, 2, \cdots, 2)$ (remember that n is odd!). As a Seifert fibration over $CP(n)$, Z/m is the only nontrivial isotropy group and the exceptional orbits map to a hypersurface of degree 2. Since $\kappa(2, n) = 1$, $H_n(M) \cong Z/m$. If $V^t = \{\underline{z} \in C^{n+2} \mid z_0^{a_0} + \cdots + z_{n+1}^{a_{n+1}} = t\}$, by the original argument of Brieskorn, $V^1 \cong V^t$ $(t \neq 0)$, and for sufficiently small t, $V_1(a_0, \cdots, a_{n+1})$ is diffeomorphic to $V^t \cap S^{2n+3}$ [5, 14.1–14.3]. The cohomology group $H^{n+1}(V^1, Z)$ is free abelian, and the symmetric cup product pairing induces the intersection pairing on $H_n(M)$. From [5, 12.4] it follows that $1 \otimes 1 = -((-1)/m)^{n+1/2}$. If $p \equiv 1 \pmod 4$, -1 is a quadratic residue, if $p \equiv 3$

(mod 4) a quadratic nonresidue, and in the first case the parity of $\frac{1}{2}(n-1)$ is irrelevent.

Theorem 5 admits a limited generalization to composite manifolds. Let A be a finite abelian group of odd order with every Sylow subgroup cyclic, that is,

$$A \cong Z/p_1^{r_1} \oplus \cdots \oplus Z/p_t^{r_t} .$$

Since A is cyclic of order $p_1^{r_1} \cdots p_t^{r_t}$, we may consider the Brieskorn variety $V_1(p_1^{r_1} \cdots p_t^{r_t}, 2, \cdots, 2)$, which fibres over $CP(n)$ as in the prime case. The argument of [5. 12.4] applied to the intersection form shows that

$$M \cong L_{m_1} \# \cdots \# L_{m_t} .$$

(If $n \equiv 3 \pmod 4$ we must replace L_{m_j} by L'_{m_j} in the connected sum above for each prime $p_j \equiv 3 \pmod 4$, which further illustrates the limited power of the construction.) If the Sylow subgroups are not restricted to be cyclic, the position is much worse, since replacing Q_j in the Seifert construction by a hypersurface of higher degree leads to a p_j-Sylow subgroup isomorphic to $(Z/p_j^{r_j})^{\kappa(\cdot, n)}$, and κ increases rapidly with n. The same observation shows that the Seifert construction is an even clumsier instrument for discussing contact structures on M^{2n+1}, when n is even. As we have already observed in the proof of Theorem 4, $\kappa(2, n) = 0$, so in order to obtain a nontrivial homology group $H_n(M, Z)$ we must map the exceptional orbits to hypersurfaces of degree ≥ 3, and $\kappa(3, n) = \frac{1}{3}(2^{n+1} + 1) - 1$. Only in dimension 5, when $\kappa(3, 2) = 2$, does the construction yield a reasonably large class of contact manifolds.

We recall from the introduction that $w_2(M^5) = 0$ is a sufficient condition for the structural group of τM^5 to reduce to $U(2) \oplus 1$. Furthermore, if $w_2 = 0$, M^5 is classified up to diffeomorphism by $H_2(M, Z) \cong rZ + \oplus_{j=1}^{t} (Z/m_j)^2$, [11]; each m_j-factor occurs as a square because of skew-symmetry in the intersection pairing [13, Theorem 3]. Therefore M^5 decomposes uniquely as the connected sum of prime manifolds

$$M \cong r(S^2 \times S^3) \# M_{m_1} \# \cdots \# M_{m_t} .$$

If $m_j = p^{r_j}$ ($p \neq 3$) the following theorem gives a description of M_{m_j} distinct from that in [11], and shows that such a prime 5-manifold can be given a contact structure.

Therom 6. *Let M^5 be a 1-connected 5-manifold such that $w_2(M) = 0$ and $H_2(M, Z) \cong (Z/m)^2$, $3 \nmid m$. Then M^5 admits an almost regular contact structure.*

Proof. We may obtain M^5 as a Seifert fibration over $CP(2)$, and in such a way that Z/m is the unique nontrivial isotropy group, with exceptional orbits mapping to a nonsingular cubic curve in $CP(2)$. Since $\kappa(3, 2) = 2$, the second homology group of the total space is isomorphic to $(Z/m)^2$. Note that the condition that m is not divisible by 3 is essential to the effectiveness of the construction in [10]. Furthermore the Brieskorn variety $V(m, 3, 3, 3)$ has the same orbit invariants, and provides perhaps the most explicit description of M^5_m. As in Lemma 4 we show that the orbits map to complex projective space by first mapping V to C^3-0 by means of the last three coordinates.

We have stated Theorem 6 in sufficient generality to cover the case when $H_2(M, Z) \cong T \oplus T$, T a cyclic group of composite order not divisible by 3. If T is not cyclic, the same sort of complications arise as for higher dimensional manifolds. Possibly more interesting in the light of Gromov theory is the case when $H_2(M, Z) \cong (Z/3^r)^{2s}$. By either performing the Seifert construction with the image of the nontrivial orbits a curve of degree $p > 3$ in $CP(2)$, or by considering the variety $V(3^r, p, p, p)$ we obtain a manifold M such that $2s = ((p-1)^3 + 1)/p - 1$. For example, $s(4) = 3$, $s(5) = 6$, $s(6) = 10$, and the connected sum $M_3 \# M_3 \# M_3$ admits an almost regular contact form.

The method of construction developed in this paper leads to a number of questions, particularly if our results are compared with those on open manifolds obtained in [4].

Problem 1. Given (almost regular) contact structures on M_1 and M_2, does there exist a contact structure of any kind on the connected sum $M_1 \# M_2$? Conversely, does a connected sum with a contact structure split as the connected sum of contact manifolds? A positive answer to the first question would increase the significance of Theorem 5.

The situation with 1-connected 5-manifolds is very interesting, since if $w_2(M) = 0$, we know that M^5-point admits a contact form ω. Since S^3 is parallelisable, its cotangent bundle is trivial and the included S^2-bundle isomorphic to $S^2 \times S^3$. Classically therefore, $S^2 \times S^3 = M_\infty$ in the notation of D. Barden and S. Smale, admits a contact form. If the answer to Problem 1 above is positive, it is natural to ask

Problem 2. Does the prime 5-manifold $M^5_{3^r}$ with second homology group isomorphic to $Z/3^r \oplus Z/3^r$ admit an other than almost regular contact form? Positive answers to problems 1 and 2 would solve

Problem 3. Does a closed 1-connected M^5 with vanishing second Stiefel-Whitney class admit a contact form?

Problem 4. The necessary and sufficient condition for M^5 to admit an almost contact structure is the vanishing of the integral class $W_3(M)$. There exist manifolds, for example the nontrivial S^3 bundle over S^2 with $w_2(M) \neq 0$ and $\beta w_2(M) = W_3(M) = 0$. Is the almost structure integrable in this case?

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