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COBORDISM THROUGH ONE-CODIMENSIONAL FOLIATIONS

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In a number of problems in differential geometry involving smooth structures on manifolds with boundary, a useful approach is to construct a related cobordism theory. For example, in the theory of one-codimensional foliations, the following two types of problem arise. First, given a compact connected oriented boundary X, one can ask if X is a leaf of one-codimensional transversely oriented smooth foliation on any compact manifold which it bounds. Or, given a transversely oriented smooth one-codimensional foliation F on X, one can ask if there exists a transversely oriented smooth one-codimensional foliation F' on any compact oriented manifold bounded by X which restricts to F on X. The cobordism theory based on the first type of problem was introduced by Reinhart [1] and involves the calculation of foliated cobordism monoids M_F^n in dimension n. The second cobordism theory is originally due to Thom and involves the calculation of foliated bordism monoids $F\Omega_1^n$. Underlying both these theories are respectively vector cobordism with associated vector cobordism groups Ω_V^n and vector bordism with associated bordism groups $V\Omega_1^n$. In [2], Reinhart introduced and completely determined the vector cobordism groups Ω_V^n, showing that two closed compact oriented manifolds are vector cobordant if and only if they are cobordant in the usual oriented sense and if, in dimensions not congruent to one mod 4, they have the same Euler number or, in dimensions congruent to one mod 4, they together bound a compact oriented manifold with even Euler number. Koschorke [3] has recently determined the oriented vector bordism groups. Both of the above calculations were based on versions of the Poincaré-Hopf theorem for manifolds with boundary.

In this article, we shall compute the foliated cobordism monoids M_F^n using some recent theorems of Thurston [4], [5]. With these powerful results at our disposal, the calculations turn out to be rather elementary. The results are mainly of interest in comparison with some few isolated results on the foliated bordism groups $F\Omega_1^n$. For example, using the Godbillon-Vey class (which is a bordism invariant), Thurston constructed an epimorphism of $F\Omega_1^3$ onto the real line, exhibiting an uncountable infinity of nonbordant one-codimensional foliations on the three-sphere. It turns out that in dimensions greater than two, the monoids M_F^n are infinitely generated, M_F^n being isomorphic to the monoid

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$\Omega_V^n \oplus \Gamma_F^n$ where the monoids Γ_F^n are monoids generated by diffeomorphism classes of closed compact connected oriented manifolds with first Betti number zero. In dimension two, M_F^n is isomorphic to the sum of two copies of the submonoid N of Z of nonnegative integers. M_F^1 is isomorphic to the group Ω_V^1 which is a trivial group.

The monoids M_F^n. In [1], Reinhart formulated the following definition.

Definition [1]. Two closed compact oriented manifolds X_0^+ and X_1^+ are called cobordant through a one-codimensional foliation (F-cobordant), if there exists a compact connected oriented manifold Z^+ with oriented boundary $\partial_0 Z^+ = X_0^+ \cup X_1^-$, and Z admits an oriented smooth one-codimensional integrable tangent plane field ξ^+ such that $\xi^+ | X_0 = t_{X_0}^+$ and $\xi^+ | X_1 = t_{X_1}^-$, where t_X denotes the tangent bundle of X.

F-cobordism is an equivalence relation, and the relation obtained by dropping the word "integrable" in the above definition is also an equivalence relation: Reinhart's oriented vector cobordism (V-cobordism). Suppose that $*$ denotes either F or V. Then the set M_* of $*$-cobordism classes is graded by dimension into commutative monoids M_*^n of $*$-cobordism classes of n-manifolds with addition induced by the disjoint sum of manifolds, the $*$-cobordism class of a manifold X being denoted by $|X|_*$. The monoids M_V^n are in fact groups. We shall use the following theorems of Thurston to compute the monoids M_F^n.

Theorem 1 (*Thurston* [4]). *Let X be a compact connected smooth manifold. Then X has a smooth one-codimensional foliation tangent to ∂X if and only if X has a one-codimensional tangent plane field tangent to ∂X and either*

(i) *for each connected component ∂X_i of ∂X, $\beta_1(\partial X_i) \neq 0$, or*

(ii) *X is diffeomorphic to $Y \times D^1$ or to $(Y \times D^1)/Z_2$ where Z_2 acts on the disc D^1 via the antipodal map and by an arbitary free action on Y.*

In the above theorem, case (ii) is to be interpreted in the light of Thurston's improved version of the Reeb stability theorem, [5].

Theorem 2 (*Generalised Reeb stability theorem*). *Let F be a one-codimensional transversely oriented C^1 foliation of a compact manifold X with a compact leaf L such that $\beta_1(L) = 0$. Then all leaves of F are diffeomorphic to L, and the leaves are the fibres of a fibration of X over D^1 or S^1. It is assumed that if ∂X is nonempty, ∂X is the union of leaves of F.*

Because of the seperation of manifolds with zero or nonzero first Betti number in the above theorems, we define $\bar{\Gamma}_F^n$ as the submonoid of M_F^n generated by F-cobordism classes of closed compact connected oriented n-manifolds with nonzero first Betti number and Γ_F^n as the monoid generated by F-cobordism classes of similar manifolds with zero first Betti number. Using Thurston's theorem, it follows that $\bar{\Gamma}_F^n \cap \Gamma_F^n = 0$.

Proposition 1. *M_F^n is isomorphic to $\bar{\Gamma}_F^n \oplus \Gamma_F^n$ for all dimensions n.*

Proof. The addition homomorphism $(+) : \bar{\Gamma}_F^n \oplus \Gamma_F^n \to M_F^n$ sending an element (x, y) into $x + y$ is a monoid isomorphism. First, $(+)$ is clearly onto.

Given a manifold X, one can write X as a disjoint sum $X = \bigcup_{i \in I} X_i$ of connected components. Let I' and I'' label respectively those components of X with nonzero and zero first Betti number. Then $|X|_F = \sum_{i \in I'} |X_i|_F + \sum_{i \in I''} |X_i|_F$ is an element of Im $(+)$. Moreover, it follows from the generalized Reeb stability theorem that if (x, y) and (x', y') are distinct elements of $\bar{\Gamma}_F^n \oplus \Gamma_F^n$, then $x + y \neq x' + y'$. Therefore $(+)$ is a monoid monomorphism.

Proposition 2. *The restriction of the forgetful homomorphism from M_F^n onto Ω_V^n to $\bar{\Gamma}_F^n$ is an isomorphism for $n \neq 2$.*

Proof. By Theorem 1, the restriction of the forgetful homomorphism to $\bar{\Gamma}_F^n$ is one-to-one into Ω_V^n. The restriction is onto Ω_V^n for $n \neq 2$. For, Ω_V^n is generated by V-cobordism classes of closed compact connected oriented n-manifolds. We shall show that if X is such a manifold, there exists a closed compact connected oriented n-manifold X^* which is V-cobordant to X and has nonzero first Betti number. First, suppose that n is not congruent to one mod 4. If n is odd, then a suitable choice for X^* is $X \# (S^1 \times S^{4k+2})$ for $n = 4k + 3$ with $k \geq 0$. In even dimensions greater than two, one can easily verify that the following manifolds satisfy our criteria.

Let Z be a compact closed connected oriented four-manifold with zero signature and Euler number. Then the manifold $X^* = X \# (S^{2k} \times S^{2k}) \# (Z)^k$ serves in dimensions divisible by four, and $X^* = X \# Z'$ serves in dimensions $4k + 2$, $k > 0$ where Z' can be chosen as say, the connected sum of $(S^{2k} \times S^{2k}) \times S^2$ with three copies of $S^2 \times (Z)^k$. In dimensions congruent to one mod 4, we have to find an oriented cobordism through an oriented manifold with even Euler number. In this case, $X^* = X \# (S^1 \times CP^{2k})$ is a suitable choice.

Corollary 1. *If $n \neq 2$, M_F^n is isomorphic to the monoid $\Omega_V^n \oplus \Gamma_F^n$ where Γ_F^n is generated by diffeomorphism classes of closed compact connected oriented n-manifolds with trivial first Betti number.*

Proof. We only need verify the statement about Γ_F^n. But it follows immediately from the generalized Reeb stability theorem that if $|X|_F$ is a nontrivial generating class of Γ_F^n, $|X|_F$ consists solely of manifolds diffeomorphic to X.

Proposition 3. *M_F^2 is isomorphic to the monoid $N \oplus N$, where the first summand corresponds to the submonoid of Ω_V^2 generated by the V-cobordism class of the two-holed torus and the second to $\bar{\Gamma}_F^2$ which is generated by the diffeomorphism class of the sphere.*

Proof. As in Proposition 2, the forgetful homomorphism f from $\bar{\Gamma}_F^2$ into Ω_V^2 is a monoid monomorphism. According to Reinhart [2], the function g sending a V-cobordism class $|X|_V$ into $\frac{1}{2}\chi(X)$ is an isomorphism of Ω_V^2 onto $\boldsymbol{Z}$. Therefore $\bar{\Gamma}_F^2$ is isomorphic to the submonoid Im $(g \circ f)$ of $\boldsymbol{Z}$. But for a compact connected oriented smooth closed surface X, $|X|_V$ is in Im (f) if and only if X has nonzero first Betti number; for if X is V-cobordant to Y which has nonzero first Betti number, $\chi(X) = \chi(Y)$ implying that $\beta_1(X) = \beta_1(Y)$. Therefore $|X|_V$ is in Im (f) if and only if $g(|X|_V)$ is zero or negative, and the monoid homomorphism $-g \circ f$ is an isomorphism of $\bar{\Gamma}^2_F$ onto N. Note that Im (f) is generated by the V-cobordism class of the two-holed torus which has Euler number -2.

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