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# COMPLETENESS OF THE $k$-TH NULLITY FOLIATIONS

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S. Tachibana and the author [5] have defined the $k$-th nullity distribution of the Riemannian curvature tensor which includes S. S. Chern and N. H. Kuiper's as the 0-th nullity distribution. It is the aim of the present paper to discuss the completeness of the leaves induced from this distribution, when the manifold is complete.

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## 1. Preliminaries and statement of results

Let $M$ be an $n$-dimensional Riemannian manifold, and let $\nabla$, $T_p(M)$ and $\mathfrak{X}(M)$ be the Riemannian connection, the tangent space at a point $p$ of $M$ and the algebra of vector fields on $M$.

Let $K$ be a tensor field of type $(r, s)$. For simplicity, we use the notation

$$(\nabla^k K)(W_{k,\cdots,1}; X_1, \cdots, X_s) \quad \text{or}$$
$$(\nabla^k K)(W_{k,\cdots,i+1}; W_i; W_{i-1,\cdots,1}; X_1, \cdots, X_s)$$

instead of

$$(\nabla^k K)(W_k; \cdots; W_1; X_1, \cdots, X_s) ,$$

where $X_1, \cdots, X_s, W_1, \cdots, W_k \in T_p(M)$, $k$ is a nonnegative integer, and $\nabla^0 K$ means $K$.

The *$k$-th nullity space* $N_p^{(k)}$ of the Riemannian curvature tensor $R$ at $p$ is the subspace of $T_p(M)$ given by

$$N_p^{(k)} = \{X \in T_p(M) \,|\, (\nabla^h R)(W_{h,\cdots,1}; U, V)X = 0 \text{ for any } U, V, W_1, \cdots, W_h \in T_p(M),\ 0 \leq h \leq k\}$$

for a nonnegative integer $k$, and we set $N_p^{(-1)} = T_p(M)$. The 0-th nullity space is the nullity space defined by Chern-Kuiper [2]. We call $\mu^{(k)}(p) = \dim N_p^{(k)}$ the *$k$-th nullity* of $R$ at $p$. The function $\mu^{(k)}$ is upper semi-continuous. We have

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shown that if the function $\mu^{(k)}$ is constant on $M$, then the distribution $N^{(k)}$: $p \to N_p^{(k)}$ is differentiable and involutive, and each maximal integral manifold of $N^{(k)}$ is totally geodesic in $M$.

The following proposition will be needed later.

**Proposition.** *Let $X \in N^{(k)}$, $Y \in N^{(k-1)}$ and $U, V, W_1, \cdots, W_k \in \mathfrak{X}(M)$. Suppose that $\nabla_X Y = 0$ and $[X, U] = [X, V] = [X, W_i] = 0$ $(i = 1, \cdots, k)$. Then*

$$\nabla_X((\nabla^k R)(W_{k,\ldots,1}; U, V)Y) = 0$$

*for a nonnegative integer $k$.*

*Proof.* We shall prove the proposition by mathematical induction on $k$. For the case where $k=0$, using the Bianchi identity $\mathfrak{S}_{X,U,V} (\nabla_X R)(U, V)Y = 0$, where $\mathfrak{S}_{X,U,V}$ denotes cyclic summation over $X, U$ and $V$, we have

$$\underset{X,U,V}{\mathfrak{S}} \{\nabla_X(R(U, V)Y) - R([X, U], V)Y - R(U, V)\nabla_X Y\} = 0 .$$

Thus we get $\nabla_X(R(U, V)Y) = 0$.

Next we assume that the proposition is true for $k - 1$. For any $X \in N^{(k)}$ and $Y \in N^{(k-1)}$ such that $\nabla_X Y = 0$, we have

$$\nabla_X((\nabla^k R)(W_{k,\ldots,1}; U, V)Y) = \nabla_X((\nabla^{k-1} R)(W_{k-1,\ldots,1}; U, V)\nabla_{W_k} Y) .$$

Since $\nabla_{W_k} Y \in N^{(k-2)}$ by Proposition 1 of [5] and $\nabla_X \nabla_{W_k} Y = \nabla_{W_k} \nabla_X Y = 0$, the right hand side of the above equation vanishes by the induction assumption.

q.e.d.

In the next section we shall prove

**Theorem 1.** *Suppose that $M$ contains an open subset $G$ on which $\mu^{(h)}$ is constant for $0 \le h \le k$, and that $\gamma: [0, s_*] \to M$ is a geodesic satisfying $\gamma(s) \in G$ and $\dot\gamma(s) \in N^{(k)}$ for all $s \in [0, s_*)$. Then $N^{(h)}$ is parallel along $\gamma([0, s_*])$.*

We define $G^{(k)}$ to be the nonempty open subset of $M$, on which $\mu^{(k)}$ assumes its minimum for $M$, and $G^{(h)}$ to be the nonempty open subset of $G^{(h+1)}$, on which $\mu^{(h)}$ assumes its minimum for $G^{(h+1)}$, $0 \le h \le k - 1$. Setting $G = G^{(0)}$, from Theorem 1 we obtain immediately

**Theorem 2.** *If $M$ is complete, then the leaves of the $k$-th nullity foliation of $R$ induced on $G$ are complete.*

## 2. Proof of Theorem 1

The fundamental idea in our proof is similar to that of Abe [1]. We shall prove the theorem by mathematical induction on $k$. For the case where $k = 0$, the theorem has been proved [1], [3], [4]. Then we assume that it is true for $k - 1$.

Let $L$ be a leaf of $N^{(k)}$ in $G$, and $p$ a point in $L$. Since $N^{(k)} \subset N^{(k-1)}$, $L$ is a submanifold of a leaf $L'$ of $N^{(k-1)}$ in $G$ through the point $p$. Consider a unit speed geodesic $\bar{\gamma}\colon [0, s_*) \to L$. Since $L$ is totally geodesic in $M$, $\bar{\gamma}$ extends to a complete geodesic in $M$, and $N^{(h)}$, $0 \leq h \leq k - 1$, is parallel along $\bar{\gamma}([0, s_*])$ by the induction assumption. It suffices to show that $N^{(k)}$ is parallel along $\bar{\gamma}([0, s_*])$. Suppose $p_* = \bar{\gamma}(s)$ for any $s \in [0, s_*]$.

Let $B(p_*, \varepsilon)$ be an $\varepsilon$-ball with $p_*$ as its center such that for any $x$ in $B(p_*, \varepsilon)$, $\mathrm{Exp}_x$ of $T_x(M)$ in $M$ gives a diffeomorphism of the $2\varepsilon$-ball in $T_x(M)$ with its image. Let us take a point $q = \bar{\gamma}(t) \in L \cap B(p_*, \varepsilon)$ and reparametrize $\bar{\gamma}$ to get a new unit speed geodesic $\gamma$ such that $\gamma(0) = q$ and $\gamma(t_*) = p_*$ for some $t_*$.

For convenience, we assume that the indices run over the following ranges:

$$i, j = 1, \cdots, m: \quad \text{nullity indices,}$$
$$a, b = m + 1, \cdots, n: \quad \text{nonnullity indices,}$$
$$I, J = 1, \cdots, n: \quad \text{unrestricted indices,}$$

where $m$ denotes the value of the function $\mu^{(k)}$ on $G$.

Now let $\zeta = (x^I)$ be a Frobenius coordinate system on a neighborhood $U (\subset G \cap B(p_*, \varepsilon))$ of $q$ such that $\zeta(q) = (0, \cdots, 0) \in \boldsymbol{R}^n$, $\partial/\partial x^I$ are orthogonal at $q$, and $(x^i)$ are coordinates of slices by the leaves of $k$-th nullity. Let $\sum$ be the slice determined by $x^i = 0$, and let $E_i$ be $m$ orthonormal vector fields in $N^{(k)}$ on $\sum$ such that $E_1(q) = \dot{\gamma}(q)$.

Denote by $\phi$ the restriction of $\zeta$ to $\sum$. Then $\phi$ gives a diffeomorphism of $\sum$ onto a neighborhood $W$ of the origin $(0, \cdots, 0) \in \boldsymbol{R}^{n-m}$. Define a $C^\infty$ mapping $F\colon \boldsymbol{R}^m \times W \to M$ by

$$F(t^1, \cdots, t^m, x) = \mathrm{Exp}_{\phi^{-1}(x)} \left( \sum_{i=1}^{m} t^i E_i(\phi^{-1}(x)) \right) ,$$

where $x = (x^a)$ is a point in $\boldsymbol{R}^{n-m}$ such that $\zeta \circ \phi^{-1}(x) = (0, \cdots, 0, (x^a))$. $F$ is of class $C^\infty$.

We set

$$H_a = \{(t^1, 0, \cdots, 0, 0, \cdots, 0, x^a, 0, \cdots, 0) \in \boldsymbol{R}^m \times W\} ,$$

where $x^a$ occurs in the $a$-th component in $\boldsymbol{R}^m \times W \subset \boldsymbol{R}^n$. Let $V_a$ be the restriction of $F$ to $H_a$. Then for each $a$, $V_a(t, x)$ defines a geodesic variation along the geodesic $\gamma(t) = V_a(t, 0)$. We denote by $X_a$ the associated Jacobi field for each $a$. Then we have $\nabla_{\dot{\gamma}}^2 X_a = 0$. Thus by the same argument as in the proof of Lemma 1.4.2 of Abe [1], we see that $F$ is regular on $H = \{(t^1, 0, \cdots, 0) \in \boldsymbol{R}^m \times W \,|\, 0 \leq t^1 < t_*\}$ except possibly at finitely many points. Let $h_*$ be the greatest value of the first coordinate in $\boldsymbol{R}^m \times W$ at such singular points of $F$ in $H$. Then there exists an open neighborhood $N$ of the set $H' = \{(t, 0, \cdots, 0) \in H \,|\, h_* < t < t_*\}$, where the rank of $F_*$ is $n$ constantly. Thus $F|N$ is an immersion of $N$ into $M$. By the inverse function theorem, at any point $x$ in

$H'$, we have a neighborhood $N_x$ where $F$ becomes a diffeomorphism. Taking $N_x$ small enough, we can assume $N_x \subset G$.

Since $\boldsymbol{R}^m \times W$ has the canonical coordinate frame $N_1, \cdots, N_n$ which is induced from that in $\boldsymbol{R}^m \times \boldsymbol{R}^{n-m} = \boldsymbol{R}^n$, we can introduce a frame field $\partial/\partial x^I = (F_* | N_x)(N_I)$ such that $\partial/\partial x^i$ are tangent to leaves in $N_x^{(k)}$.

Let $Y(t_*)$ be a $k$-th nullity vector at $p_* = \gamma(t_*)$. Parallelly translate $Y(t_*)$ backwards along $\gamma$. Since $p_*$ is in the leaf $L'$ of $N^{(k-1)}$, and $N^{(k-1)}$ is parallel along $x^1$-curves near $\gamma([h_*, t_*])$ by the induction assumption, we can extend $Y$ to a vector field, also denoted by $Y$, on a neighborhood of the set $\gamma([h_*, t_*])$ such that $Y \in N^{(k-1)}$ and $\nabla_{\partial/\partial x^1} Y = 0$. We shall show that

$$(*) \qquad \nabla_{\dot\gamma(t)}((\nabla^h R)(X_{I_h,\cdots,I_1}; X_a, X_b)Y) = 0$$

for $h = 0, 1, \cdots, k$, where $X_I$'s are vector fields along $\gamma$, such that $X_I(\gamma(t)) = (\partial/\partial x^I)(\gamma(t))$ on $H \cap N_x$.

Since the vector field $Y$ is a $(k-1)$th nullity vector field on $\gamma([h_*, t_*])$, we have $(*)$ for $h = 0, 1, \cdots, k-1$. Thus it remains to show $(*)$ for $h = k$. It suffices to show that

$$\nabla_{\partial/\partial x^1}((\nabla^k R)(\partial/\partial x^{I_h}; \cdots; \partial/\partial x^{I_1}; \partial/\partial x^a, \partial/\partial x^b)Y) = 0$$

on $\gamma((h_*, t_*)) \cap F(N_x)$. Since $\partial/\partial x^1$ is in $N^{(k)}$ in $F(N_x)$, $Y \in N^{(k-1)}$, $\nabla_{\partial/\partial x^1} Y = 0$ on a neighborhood of $\gamma$, and $[\partial/\partial x^I, \partial/\partial x^J] = F_*([N_I, N_J]) = 0$, the above is a consequence of the proposition in § 1.

Let $t$ be chosen from $(h_*, t_*)$. Then we claim that

$$(\nabla^k R)(W_{k,\cdots,1}; U, V)Y(t) = 0$$

for all $W_1, \cdots, W_k, U$ and $V$ in $T_{\gamma(t)}M$; i.e., $Y(t)$ is in $N^{(k)}_{\gamma(t)}$. In fact, let

$$U = U' + \sum_{a=m+1}^{n} U^a X_a , \qquad V = V' + \sum_{b=m+1}^{n} V^b X_b ,$$

where $U'$ and $V'$ are the $N^{(k)}_{\gamma(t)}$ components of $U$ and $V$, respectively. Then

$$\begin{aligned}
&(\nabla^h R)(W_{h,\cdots,1}; U, V)Y(t) \\
&\quad = (\nabla^h R)(W_{h,\cdots,1}; U', V')Y(t) \\
&\qquad + (\nabla^h R)(W_{h,\cdots,1}; U', \textstyle\sum V^b X_b)Y(t) \\
&\qquad + (\nabla^h R)(W_{h,\cdots,1}; \textstyle\sum U^a X_a, V')Y(t) \\
&\qquad + (\nabla^h R)(W_{h,\cdots,1}; \textstyle\sum U^a X_a, \sum V^b X_b)Y(t) .
\end{aligned}$$

The first three terms of the right hand side of the above equation vanish by the fact that $U', V' \in N^{(k)}$, and the last term must vanish by $(*)$ and the fact that $Y(t_*) \in N^{(k)}_{\gamma(t_*)}$. Therefore $N^{(k)}$ is parallel along $\gamma | (h_*, t_*]$. It follows that $N^{(k)}$ is parallel along $\bar\gamma([0, s_*])$, as required.

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