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MINIMAL EMBEDDINGS OF THE TORUS IN 3-MANIFOLDS

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1. Introduction

The study of minimal surfaces has historically concentrated on the Plateau problem—more recently, much attention has been given to the possibility of realizing homology classes as minimal submanifolds. Our result here is in the latter direction.

Let M be a smooth manifold, and g a Riemannian metric on M. We say a smooth embedding $\iota: N \to M$ of a submanifold $\iota(N)$ is *minimal* if the volume of N induced by $\iota^* g$ is less than or equal to that induced by any other smooth embedding $\bar{\iota}: N \to M$, with $\bar{\iota}$ homotopic to ι. We prove here the following.

Theorem. *Let M be a smooth 3-manifold with Riemannian metric g. If (M, g) has everywhere negative curvature, there is no minimal embedding of the torus into the interior of M.*

2. Notation and conventions

Let M be a smooth manifold of dimension m, and let $x \in M$. We denote the tangent space to M at x by $T_x M$, and the cotangent space by $T_x^* M$. If N is a smooth manifold of dimension n, and $f: M \to N$ a smooth map, then we denote the associated maps by $f_*: T_x M \to T_{f(x)} N$, $f^*: T_{f(x)}^* N \to T_x^* M$.

Let $\mathscr{F}(M)$ be the frame bundle of M, and $\pi: \mathscr{F}(M) \to M$ the natural projection. Each $u \in \mathscr{F}(M)$ may be considered as a linear isomorphism $u: \boldsymbol{R}^m \to T_{\pi(u)} M$.

Let $GL(m)$ be the general linear group on $\boldsymbol{R}^m$. Then there is a right action $R_a: \mathscr{F}(M) \to \mathscr{F}(M)$ defined by $R_a u(A) = u(aA)$ $\forall\, u \in \mathscr{F}(M)$, $A \in \boldsymbol{R}^m$, $a \in GL(m)$. Let $\{E_1, \cdots, E_m\}$ be a standard basis of $\boldsymbol{R}^m$, and $(\ , \ )$ the standard inner product. We define horizontal canonical one-forms θ^i on $\mathscr{F}(M)$ by the rule $\theta^i(X) = i$th component of $u^{-1} T_* X$ for $X \in T_u \mathscr{F}(M)$.

If g is a Riemannian metric on M, we set $0(M, g) = \{u \in \mathscr{F}(M) \,|\, (A, B) = g(uA, uB)\ \forall\, A, B \in \boldsymbol{R}^m\}$.

On $0(M, g)$ we may define the matrix of forms ω_j^i, $i, j = 1, \cdots, n$, of the Levi-Civita connection on $0(M, g)$, satisfying the following properties:

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(C1) $$\omega^i_j = -\omega^j_i \ ,$$

(C2) $$\{\theta^1, \cdots, \theta^m, \omega^i_j, \ i > j, \ i, j = 1, \cdots, m\}$$ form a global coframe of $0(M, g)$,

(C3) $$R^*_a \omega^i_j = \sum_{k,r=1}^{m} a^{-1}{}^i_k \, \omega^k_r a^r_j \ ,$$

(C4) $$d\theta^j = \sum_{i=1}^{m} \theta^i \wedge \omega^j_i \ .$$

We define the Riemannian curvature form Ω^i_j by

$$\Omega^i_j = d\omega^i_j + \sum_{k=1}^{m} \omega^i_k \wedge \omega^k_j \ .$$

Then $\Omega^i_j = \sum_{k,r=1}^{m} \frac{1}{2}\Omega^i_{jkr}\theta^k \wedge \theta^r$ for some functions Ω^i_{jkr} such that $\Omega^i_{jkr} = -\Omega^i_{jrk}$. The sectional curvature of the plane spanned by $u(E_i) \wedge u(E_j)$ is $\Omega^i_{jij}(u)$.

3. Proof of the theorem

Let M be a smooth manifold of dimension 3, and g a Riemannian metric on M. Let T be the two-dimensional torus, and suppose there exists a minimal immersion $\iota\colon T \to M$. Since T is parallelizable, we may establish a section $\sigma\colon \iota(T) \to 0(M, g)$ such that $\iota^*\sigma^*\theta^3 = 0$. Then the area $A_\iota(T)$ of the image is given by

$$A_\iota(T) = \int_T \iota^*\sigma^*(\theta^1 \wedge \theta^2) \ .$$

Let $\mathfrak{X}$ be a vector field on M, and $\phi_t\colon M \to M$ the associated one-parameter group of diffeomorphisms. We may vary the map $\iota\colon T \to M$ by the variation $\iota_t = \phi_t \circ \iota$. For fixed t, on each image $\iota_t(T)$ we establish a section $\sigma_t\colon \iota_t(T) \to 0(M, g)$ such that $\iota_t^*\sigma_t^*\theta^3 = 0$. On a regular neighborhood N of T we may find a one-parameter group of diffeomorphisms $\tilde{\phi}_t\colon 0(N, g) \to 0(N, g)$ such that $\sigma_t \circ \iota_t = \tilde{\phi}_t \circ \sigma \circ \iota$. We let $\tilde{\mathfrak{X}}$ be the vector field associated to $\tilde{\phi}_t$. Then $\pi_{*\tilde{\mathfrak{X}}} = \mathfrak{X}$. Let $\mathscr{L}_{\tilde{\mathfrak{X}}}$ denote the Lie derivative with respect to $\tilde{\mathfrak{X}}$. Then since ι is minimal,

$$0 = \int_\pi \iota^*\sigma^*(\mathscr{L}_{\tilde{\mathfrak{X}}}(\theta^1 \wedge \theta^2)) \ .$$

Now suppose that, on $\iota(T)$, $\pi_*\tilde{X} = \kappa\sigma(E_3)$ for some function κ. Then

$$0 = \int_T \iota^*\sigma^*(\kappa\tilde{E}_3 \lrcorner \, (d(\theta^1 \wedge \theta^2)))$$

$$= \int_T \iota^*\sigma^*({}_\kappa\tilde{E}_3 \lrcorner (\theta^3 \wedge \omega^1_3 \wedge \theta^2 - \theta^1 \wedge \theta^3 \wedge \omega^2_3))$$

$$= \int_T \iota^*\sigma^*\kappa(\omega^1_3 \wedge \theta^2 + \theta^1 \wedge \omega^2_3) \ .$$

If we set $\iota^*\sigma^*\omega^i_j = \Gamma^i_{jk}\iota^*\sigma^*\theta^k$ then, since κ is arbitrary,

$$\Gamma^3_{11} = -\Gamma^3_{22} \ .$$

Since $d\iota^*\sigma^*\theta^3 = 0$, $\Gamma^3_{12} = \Gamma^3_{21}$.

We set $dA = \iota^*\sigma^*(\theta^1 \wedge \theta^2)$. Then

$$\int_T \iota^*\sigma^*\Omega^1_{212} dA = \int_T \iota^*\sigma^*\Omega^1_2 = \int_T \iota^*\sigma^*(d\omega^1_2 + \omega^1_3 \wedge \omega^3_2)$$

$$= \int_T (-\Gamma^3_{11}\Gamma^3_{22} + \Gamma^3_{12}\Gamma^3_{21}) dA \ .$$

Thus $\int_T \iota^*\sigma^*\Omega^1_{212} dA \geq 0$ so that the sectional curvature of (M, g) must be non-negative somewhere on the image $\iota(T)$.

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