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REMOVABILITY OF SINGULAR POINTS ON SURFACES OF BOUNDED MEAN CURVATURE

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In this note we shall be concerned with a continuous surface, or 2-submanifold, which is smoothly immersed except perhaps at one point. A typical example of such a surface would be a surface of revolution in euclidean space whose generating curve meets the axis at an acute angle. If the curvature of the generating curve is bounded near that point, then one may readily compute that the mean curvature of the surface increases beyond bound as the singular point is approached. That this must occur even when the generator has unbounded curvature is rather less obvious. We shall show that if there is no limiting tangent plane at the singular point, then the length of the mean curvature vector is unbounded. Observe, for contrast, that the gaussian curvature may remain bounded, or even constant, near an isolated singularity.

This question was raised in connection with the study of ramified branched immersions of surfaces of prescribed mean curvature with injective boundary mapping (see [5]). If a ramified boundary branch point is not equivalent to an interior point, then the unramified quotient surface has a corresponding deleted interior point. In the general context of branched immersions with a unique continuation property, this deleted point may well be a singular point of the unramified quotient surface. If the original surface has prescribed smooth mean curvature, however, it is now shown that the quotient surface extends across the deleted point as a smooth branched immersion with prescribed mean curvature. This fact will be exploited in a forthcoming study of ramification of solutions to the Plateau problem for surfaces of higher topological type, [4].

We are indebted to Friederich Tomi for a stimulating discussion.

Notation. For a domain G of $\boldsymbol{R}^n$, we write $C^k(G)$, $C^{k,\alpha}(G)$ and $W^k_p(G)$ for the spaces of functions whose (weak) kth partial derivatives are continuous, Hölder continuous with exponent α, and of class $L_p(G)$, respectively. For a vector function $x(u, v)$, $Dx = (x_u, x_v)$ denotes the matrix of its first partial derivatives. A variable point (u, v) or (ξ, η) of $\boldsymbol{R}^2$ is also denoted $w = u + iv$ or $\zeta = \xi + i\eta$. Except as specified in Lemma 2, B denotes the open unit disk of $\boldsymbol{R}^2$, and $B' = B\backslash\{0\}$. In a riemannian manifold M, g_{ij} are the components of the metric tensor, and Γ^k_{ij} the components of the Levi-Civita connection ∇.

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Theorem. *Suppose Σ is an immersed surface of class $C^{2,\beta}$ for some $\beta > 0$, having the topological type of the annulus, in a riemannian manifold M of class C^3. Suppose that at one boundary component, Σ has a unique limit point P in M, and that its mean curvature vector has bounded length. Then there is a mapping $x\colon B \to M$ of class $C^{1,\alpha}(B)$ for all $\alpha < 1$, satisfying the conformality conditions*

$$\sum_{i,j} g_{ij}(x_u^i x_u^j - x_v^i x_v^j) = \sum_{i,j} g_{ij} x_u^i x_v^j = 0 \,, \tag{1}$$

and such that the restriction of x to B' is a $C^{2,\beta}$ parameterization of Σ.

Remarks. 1. We shall prove, in fact, that x may be taken to be any conformal parameterization of Σ, and is a weak solution of system (4) below in all of B.

2. If the mean curvature vector of Σ is given as a function of class C^α of $x(w)$ and of the tangent plane, then x is of class $C^{2,\alpha}(B)$ and satisfies system (4) below in the classical sense (cf. [9, p. 383]).

3. The conclusion of the theorem is that some parameterization extends across the singularity as a smooth mapping, although not necessarily as an immersion. However, it does enjoy those aspects of regularity associated with branched immersions. For example, the tangent plane to Σ at a point Q tends to a limiting plane as Q approaches P. In fact, x satisfies an asymptotic relation analogous to (6) below (see [6, proof of Theorem I]).

4. In the case $H = 0$ of a minimal surface in euclidean n-space, the theorem follows from a result of Osserman [11, Theorem 1].

We will need to know that the area of Σ is finite. This is a consequence of a general theorem of Harvey and Lawson [7, Theorem 3.1]; we give a proof here in the interest of completeness.

Lemma 1. *Let P be a point of a riemannian manifold M, and suppose Σ is a properly immersed k-submanifold of $M\backslash\{P\}$. If Σ has bounded mean curvature, then it has finite k-volume in some neighborhood of P.*

Proof. Let H be the mean curvature vector of Σ, and H_0 an upper bound for its length. Denote by $\rho(Q)$ the riemannian distance from P to Q in M. Choose $\varepsilon_0 > 0$ such that $U_0 = \rho^{-1}([0, \varepsilon_0])$ is compact and ρ^2 is smooth on U_0. Write $Y = \operatorname{grad}\rho$. Let b^2 be a positive upper bound, and a^2 a lower bound for sectional curvatures of M on U_0 (a may be imaginary). Now choose $\varepsilon_1 \le \varepsilon_0$ with $b\varepsilon_1 < \pi$ and $2\varepsilon_1(H_0 + 1) \le 1$. For $\varepsilon < \varepsilon_1$, denote $S_\varepsilon = \rho^{-1}(\varepsilon)$. Given any vector X tangent to S_ε, let $c(X)$ denote the normal curvature of S_ε in the X direction, with respect to the normal $-Y$. Then $c(X) \ge a \cot a\varepsilon$ (cf. [1, p. 251]). In particular, if ε_1 is chosen small enough, then $c(X) > \varepsilon^{-1} - 1$ for all $\varepsilon \in (0, \varepsilon_1)$.

We shall examine the variation of k-volume of Σ upon flowing along the vector field $T = \rho Y$. Let B denote one-half of the Lie derivative of the metric tensor with respect to T. At any point of Σ, decompose T into orthogonal components: $T = T^t + T^n$, where T^t is tangential to Σ and T^n is normal. Then for any tangential vector field X along Σ,

$$\begin{aligned} B(X, X) &= \langle \nabla_X T, X\rangle = \langle \nabla_X T^t, X\rangle + \langle \nabla_X T^n, X\rangle \\ &= \langle \nabla_X T^t, X\rangle - \langle \nabla_X X, T^n\rangle . \end{aligned}$$

Let $E_1, \cdots, E_k$ be an orthonormal frame field for Σ on some open subset. Then the above formula gives

$$\sum_{i=1}^{k} B(E_i, E_i) = \operatorname{div}_\Sigma T^t - k\langle H, T\rangle , \tag{2}$$

where the divergence is the intrinsic divergence of Σ. The integral of this function over Σ is the first variation of k-volume.

Now suppose Z is a unit vector orthogonal to Y. Then

$$B(Z, Z) = \langle \nabla_Z(\rho Y), Z\rangle = \rho\langle \nabla_Z Y, Z\rangle = \rho c(Z) \geq 1 - \rho .$$

Meanwhile, since $Y = \operatorname{grad} \rho$ is autoparallel and has unit length, we may compute

$$\begin{aligned} B(Y, Z) &= \tfrac{1}{2}(\langle \nabla_Z T, Y\rangle + \langle \nabla_Y T, Z\rangle) \\ &= \tfrac{1}{2}(\rho\langle \nabla_Z Y, Y\rangle + Z(\rho)\langle Y, Y\rangle + \rho\langle \nabla_Y Y, Z\rangle + Y(\rho)\langle Y, Z\rangle) = 0 . \end{aligned}$$

Finally,

$$B(Y, Y) = \langle \nabla_Y T, Y\rangle = \rho\langle \nabla_Y Y, Y\rangle + Y(\rho)\langle Y, Y\rangle = Y(\rho) = 1 .$$

Thus the symmetric bilinear form B has all eigenvalues $\geq 1 - \rho$.

In particular, (2) now implies

$$\operatorname{div}_\Sigma T^t \geq k(1 - \rho) - k\rho\,|H| \geq k(1 - \varepsilon_1 - \varepsilon_1 H_0) \geq \tfrac{1}{2}k . \tag{3}$$

For some fixed $\eta < \varepsilon_1$, denote $\Sigma_\varepsilon = \Sigma \cap \rho^{-1}((\varepsilon, \eta))$ and $\Gamma_\varepsilon = \Sigma \cap S_\varepsilon$. We shall assume that η and $\varepsilon < \eta$ are regular values of the restriction of ρ to Σ; this allows arbitrarily small values of ε, according to Sard's theorem. Then Σ_ε is a compact manifold with smooth boundary $\Gamma_\varepsilon \cup \Gamma_\eta$. Let ν denote the outward unit normal to Σ_ε at its boundary; then $\langle T, \nu\rangle \leq 0$ on Γ_ε. Integrating relation (3) with respect to k-volume over Σ_ε, we have

$$\int_{\Gamma_\eta} \langle T, \nu\rangle dV_{k-1} + \int_{\Gamma_\varepsilon} \langle T, \nu\rangle dV_{k-1} \geq \frac{1}{2} k \operatorname{vol}_k(\Sigma_\varepsilon) ,$$

where dV_{k-1} is the integrand of $(k-1)$-volume. This implies

$$\operatorname{vol}_k(\Sigma_\varepsilon) \leq \frac{2}{k} \int_{\Gamma_\eta} \langle T, \nu\rangle dV_{k-1}$$

and hence is bounded as $\varepsilon \to 0$. Therefore $\mathrm{vol}_k\,(\Sigma_0)$ is finite. q.e.d.

We shall have need of a special case of the following lemma, which we state in a natural degree of generality. Certain cases for $f = 0$ are well known (see, for example, [10], Theorems 1 and 4, and Remark 2]). We give a direct proof in the interest of simplicity.

Lemma 2. *Consider $x \in W_p^1(B)$ and $f \in L_1(B)$, where $1 \leq p \leq \infty$ and B is a bounded domain in $\boldsymbol{R}^m$. Denote $B' = B \backslash \boldsymbol{R}^{m-q}$, where $q \geq 1$. Suppose f is a weak Laplacian of x in B'. If $p^{-1} + q^{-1} \leq 1$, then $\Delta x = f$ holds weakly in B.*

Proof. Since B has finite measure, we may decrease p if necessary to obtain $p^{-1} + q^{-1} = 1$. For $\varepsilon > 0$, let Ω_ε denote the set of points of $\boldsymbol{R}^m$ at distance less than ε from $\boldsymbol{R}^{m-q}$, and $B_\varepsilon = B \cap \Omega_\varepsilon$. Choose an arbitrary test function $\varphi \in C_0^\infty(B)$. There is a constant N such that $|\varphi| \leq N$, $|D\varphi| \leq N$ and for all $\varepsilon < 1$, $\mathrm{vol}\, B_\varepsilon \leq (N\varepsilon)^q$. For each $\varepsilon \in (0, 1)$, choose a smooth function η with support in Ω_ε, such that $\eta = 1$ on $\Omega_{\varepsilon/2}$, $|\eta| \leq 1$ everywhere, and $|D\eta| \leq 3/\varepsilon$. Writing $\varphi_1 = \varphi\eta$, we have $\varphi = \varphi_1 + \varphi_2$, where $\varphi_2 \in C_0^\infty(B')$. Therefore

$$I = \int_B Dx \cdot D\varphi + f\varphi = \int_B (Dx \cdot D\varphi_1 + f\varphi_1) + \int_B (Dx \cdot D\varphi_2 + f\varphi_2) = I_1 + I_2 \,.$$

We need to show $I = 0$. Since $\Delta x = f$ weakly in B', we have $I_2 = 0$. Now I is independent of ε, so it suffices to show that I_1 is arbitrarily close to zero when ε is chosen small enough. We have

$$\left|\int_B f\varphi_1\right| = \left|\int_B f\varphi\eta\right| \leq N\int_{B_\varepsilon} |f| \to 0 \,,$$

as $\varepsilon \to 0$, since $f \in L_1(B)$. Meanwhile, for $\varepsilon < 1$, we have

$$\begin{aligned}\left|\int_B Dx \cdot D\varphi_1\right| &\leq \int_{B_\varepsilon} (|Dx \cdot D\varphi| + N|Dx \cdot D\eta|) \leq N\left(1 + \frac{3}{\varepsilon}\right)\int_{B_\varepsilon} |Dx| \\ &\leq \frac{4N}{\varepsilon}\left\{\int_{B_\varepsilon} |Dx|^p\right\}^{1/p} \{\mathrm{vol}\, B_\varepsilon\}^{1/q} \leq 4N^2\left\{\int_{B_\varepsilon} |Dx|^p\right\}^{1/p} ,\end{aligned}$$

which tends to zero as $\varepsilon \to 0$, since $x \in W_p^1(B)$. This shows that $I_1 \to 0$, which forces $I = 0$.

Proof of the theorem. It follows from the uniformization theorem that there is a conformal parameterization x of Σ, defined on a plane domain G of the topological type of the annulus, $x \in C^{2,\beta}(G)$. This may be seen as in [2, Chapter II], where local uniformization is given by a classical theorem of Lichtenstein (cf. [3, pp. 350–357]). By means of a further conformal mapping, we may assume that G is the domain bounded by concentric circles C_1 and C_2 of radii r_1 and r_2, respectively, where $0 \leq r_1 < r_2 \leq \infty$, such that as w tends to C_1,

$x(w)$ approaches P. Observe that, since the conclusion of the theorem is a local statement at P, we may take r_2 as close to r_1 as desired, thereby restricting attention to a neighborhood of C_1 in G. In particular, we may assume $r_2 < \infty$. Now the area of Σ is given by one-half the Dirichlet integral of x over G, which is therefore finite. Moreover, $x \in C^0(G \cup C_1)$, so that after reducing r_2 slightly we see that x is uniformly continuous. In particular, $x \in W_2^1(G)$.

We now introduce a system of coordinates for M at P, and write $x = (x^1, \cdots, x^n)$, where n is the dimension of M. The conformality conditions (1) hold, and the components of x satisfy the system of partial differential equations

$$\Delta x^k = -\sum_{i,j} \Gamma^k_{ij}(x^i_u x^j_u + x^i_v x^j_v) + 2H^k |x_u \wedge x_v| \tag{4}$$

in G. Here $\Delta x^k = x^k_{uu} + x^k_{vv}$; $\Gamma^k_{ij}(w)$ are the Christoffel symbols of M at $x(w)$; $H^1, \cdots, H^n$ are the components of the mean curvature vector $H(w)$ of Σ at $x(w)$; and $|x_u \wedge x_v|$ denotes the area of the parallelogram spanned by x_u and x_v. Since M is of class C^3, the Christoffel symbols are of class $C^1(M)$, and in particular define bounded functions on $G \cup C_1$. Since H is also bounded, x satisfies in G a partial differential inequality of the form

$$|\Delta x| \leq K |Dx|^2 \tag{5}$$

for some constant K.

We shall now show that $r_1 = 0$. Supposing to the contrary that $r_1 > 0$, we choose a point w_0 on C_1. For any $\theta > 0$, there is a one-to-one conformal mapping $\rho\colon Z_\theta \to G$, where

$$Z_\theta = \{\varphi\colon |\varphi| < 1,\ |\varphi - 1| < \theta\}\ , \quad \text{with} \quad \rho(1) = w_0\ ,$$

such that the arc $\{\varphi\colon |\varphi| = 1,\ |\varphi - 1| < \theta\}$ is mapped onto an arc of C_1. Moreover, ρ is a smooth diffeomorphism of $\bar{Z}_\theta$ with its image. Denote $y = x \circ \rho$. Then according to a result of Heinz [8, Hilfssatz], $y \in C^{1,\alpha}(\bar{Z}_\varphi)$ for all $\varphi \in (0, \theta)$ and $\alpha \in (0, 1)$. It follows that x is of class $C^{1,\alpha}$ and satisfies the conformality relations (1) on a neighbohood of w_0 in $\bar{G}$. But since all of C_1 is mapped onto P, these relations imply $Dx(w) = 0$ for all w in a neighborhood of w_0 on C_1. Thus in particular $Dy(1) = 0$, and it follows from the same result of Heinz that y satisfies an asymptotic relation of the form

$$y_\xi + iy_\eta = a(\zeta - 1)^r + o(\zeta - 1)^r\ , \tag{6}$$

as $\zeta \to 1$, $\zeta \in \bar{Z}_\theta$, for some nonzero $a \in C^n$ and some integer r. But this implies $Dy(\xi) \neq 0$ for $\zeta \neq 1$ in some neighborhood of 1, which contradicts the fact that $Dy(\zeta) = 0$ for all ζ in a neighborhood of 1 on the unit circle. This shows that $r_1 = 0$.

Therefore, by a scale change if necessary, we may assume $G = B'$. Recall that x is continuous in B and, moreover, is of class $W_2^1(B') = W_2^1(B)$. In particular, the right-hand side of the partial differential equation (4) is a function of classs $L_1(B)$. It now follows from the case $p = q = m = 2$ in Lemma 2 that x is a weak solution of (4) in all of B. This implies that $x \in C^{1,\alpha}(B)$ for all $\alpha \in (0, 1)$ (see [12]).

Added in proof. Our theorem continues to hold if Σ is only a branched immersion of bounded mean curvature. This may be seen from the proof of Lemma 7.1 in [5, II].

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