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A SEMI-LOCAL COMBINATORIAL FORMULA FOR THE SIGNATURE OF A $4k$-MANIFOLD

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Let M be a compact oriented and triangulated manifold of dimension $4k$. If we orient each simplex, we obtain geometric bases for the chains and cochains, which are thus identified

$$C^i = C_i \ .$$

The classical boundary and coboundary operators

$$\partial : C_i \to C_{i-1} \ , \qquad d : C_i \to C_{i+1}$$

are transposes of one another. We shall describe below a symmetric transformation

$$* : C_i \to C_{4k-i} \ .$$

If M has no boundary, we then form the symmetric transformation

$$\begin{pmatrix} * & \partial \\ d & 0 \end{pmatrix} : C \to C \ ,$$

where $C = C_{2k} \oplus C_{2k+1}$. If M has a boundary, we form the transformation

$$\begin{pmatrix} * & \partial \\ d & 0 \end{pmatrix} : C^{\cdot} \to C^{\cdot} \ ,$$

where $C^{\cdot}$ is the d-sub (or ∂-quotient) space of C consisting of the chains which vanish (or have no coefficient) on the boundary. In either case we have

Theorem. signature M = signature $\begin{pmatrix} * & \partial \\ d & 0 \end{pmatrix}$.

The operation $*$ is defined as follows: say two oriented simplices σ and τ are complementary with sign $*(\sigma, \tau)$ if

(i) σ and τ are of complementary dimensions,
(ii) σ and τ span a $4k$-simplex η,
(iii) orientation $\sigma \cdot$ orientation $\tau = *(\sigma, \tau) \cdot$ orientation η.

Then let

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$$*: C_i \to C_{4k-i}\,;\ \sigma \mapsto i!(4k-i)!\ \sum_{\tau} *(\sigma,\tau)\tau\ ,$$

where τ ranges over the complementary simplices of σ. This is a combinatorial analogue of the $*$ operator on a Riemannian manifold. We are indebted to Walter Neumann for pointing out that the factor $i!(4k-i)!$ is required here for (i) below to be satisfied. However, the factor may be omitted in the statement of the theorem, because the matrix identity

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & B \\ B^* & 0 \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} = \lambda\begin{pmatrix} \lambda A & B \\ B^* & 0 \end{pmatrix}$$

shows that

$$\text{signature}\begin{pmatrix} A & B \\ B^* & 0 \end{pmatrix} = \text{signature}\begin{pmatrix} \lambda A & B \\ B^* & 0 \end{pmatrix} \in Z$$

for any strictly positive number λ.

Note that $\begin{pmatrix} * & \partial \\ d & 0 \end{pmatrix}$ is local in character. The value of this transformation on a chain depends only on the simplicial geometry near the chain. However, our formula is not solely a function of the geometry around each vertex (plus a boundary term if M is not closed) which is the real goal here. (Such a formula is known in Riemannian geometry, using curvature and the Thom-Hirzebruch signature formula.)

We shall describe the proof in the closed case. The proof in the bounded case is identical, after replacing C by $C^{\cdot}$.

There are three steps:

(i) Establish a commutative diagram

$$\begin{tikzcd} C_{2k-1} \arrow[r, "d"] \arrow[d, "*"'] & C_{2k} \arrow[r, "d"] \arrow[d, "*"] & C_{2k+1} \arrow[d, "*"] \\ C_{2k+1} \arrow[r, "\partial"] & C_{2k} \arrow[r, "\partial"] & C_{2k-1} \end{tikzcd}$$

yielding a transformation

$$*: H^{2k} \to H_{2k}\ .$$

(ii) Identify this transformation with the cup product pairing

$$(x,y) \mapsto \int_M x \cup y$$

on H^{2k} defining the signature of M. (Recall that the signature of a manifold with boundary is defined by this procedure using $H^{2k}(M,\partial M)$. Thus $C^{\cdot}$ is relevant in the bounded case.)

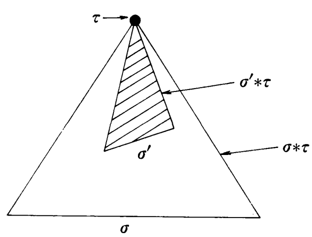
(iii) Appeal to the algebraic lemma below to obtain

$$\text{signature } (H^{2k}, *) = \text{signature } \left(C_{2k} \oplus C_{2k+1}, \begin{pmatrix} * & \partial \\ d & 0 \end{pmatrix}\right) \in \boldsymbol{Z}\,.$$

We describe two ways to establish (i) and (ii). First one can directly establish (i) by a combinatorial argument and then identify $*$ with the cup product by appealing to Whitney's discussion [4, p. 363] and [5]. This method is elementary but not completely conceptual. Perhaps a more satisfying method (and one which is useful in a variety of contexts) is to use Whitney's embedding of the cochains on a complex into the forms on the complex. For an elementary cochain σ with vertices $x_0, x_1, \cdots, x_r$ Whitney defines ([4, p. 229], see also [3]) the differential form

$$\omega_\sigma = r\,! \sum_{i=0}^{r} (-1)^i x_i dx_0 \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_r\ ,$$

thinking of the x_i as barycentric coordinates with nonzero values in the stars of vertices of σ. The form ω_σ is nonzero only in the open star of σ, and its integral over any (tiny) simplex σ' in the star has appealing geometric interpretation; namely, consider the figure:



(* means join here.)

Then

$$\int_{\sigma'} \omega_\sigma = \frac{\text{volume } \sigma' * \tau}{\text{volume } \sigma * \tau}\ ,$$

where $\sigma' \subset$ interior $\sigma * \tau$.

The linear extension of this formula embeds cochains into the forms on the polyhedron, it commutes with d, and it yields an isomorphism between simplicial cohomology and de Rham cohomology (see [4], [5] for real coefficients and [2] for rational coefficients).

The cup product on cohomology is represented by the exterior product of forms. For example, if σ and τ are of complementary dimensions, then $\omega_\sigma \wedge \omega_\tau$ is nonzero precisely when σ and τ span a $4k$-simplex η. We can calculate $\int_\eta \omega_\sigma \wedge \omega_\tau$ to obtain a nonzero number with the sign of $*(\sigma, \tau)$ (see the appendix at the end of this paper). This shows (ii), namely, the $*$-pairing from diagram (ii) is a positive multiple of the cup-product pairing. The commutativity of diagram (i) follows by integrating the equation among differential forms

$$d(\omega \wedge \eta) = d\omega \wedge \eta \pm \omega \wedge d\eta$$

over the manifold. In the closed case we find

$$*d = \pm \partial * .$$

(The other relation $*\partial = \pm d*$ is quite false.) In the bounded case we also find this relation among the chains which vanish on the boundary $C^{\cdot}$. Thus (i) and (ii) are proved.

Algebraic lemma. *Let*

$$\begin{array}{ccccc} U & \xrightarrow{f} & V & \xrightarrow{g} & W \\ \Big\downarrow{\scriptstyle\theta} & & \Big\downarrow{\scriptstyle\varphi} & & \Big\downarrow{\scriptstyle\theta^*} \\ W^* & \xrightarrow{g^*} & V^* & \xrightarrow{f^*} & U^* \end{array}$$

be a commutative diagram of finite-dimensional vector spaces and morphisms with

$$gf = 0 \in \operatorname{Hom}(U, W) , \qquad \varphi^* = \varphi \in \operatorname{Hom}(V, V^*) .$$

Then the bilinear symmetric form $(H, \bar{\varphi})$ *induced by* φ *on the homology space* $H = \ker g / \operatorname{im} f$ *is such that*

$$\text{signature}\,(H, \bar{\varphi}) = \text{signature}\left(V \oplus W^*, \begin{pmatrix} \varphi & g^* \\ g & 0 \end{pmatrix}\right) \in \boldsymbol{Z} .$$

Proof. The signature $\sigma(V, \varphi) \in \boldsymbol{Z}$ of (V, φ) is (by definition) the signature of the associated nondegenerate form $(V/\ker \varphi, \hat{\varphi})$.

The form (Z, ξ) to which φ restricts on $Z = \ker g$ has the same associated nondegenerate form as $(H, \bar{\varphi})$, so that

$$\sigma(H, \bar{\varphi}) = \sigma(Z, \xi) \in \boldsymbol{Z} .$$

Choosing direct complements X, Y to Z, fX in V, W respectively, we can write

$$\varphi = \begin{pmatrix} \xi & \psi^* \\ \psi & \xi \end{pmatrix} : V = Z \oplus X \to Z^* \oplus X^* ,$$

$$f = \begin{pmatrix} 0 & \iota \\ 0 & 0 \end{pmatrix} : V = Z \oplus X \to W = fX \oplus Y ,$$

where $\iota \in \operatorname{Hom}(X, fX)$ is an isomorphism. Then there is defined an isomorphism of forms

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \psi & \frac{1}{2}\xi & \iota^* & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$: \left(V \oplus W^*, \begin{pmatrix} \varphi & g^* \\ g & 0 \end{pmatrix}\right) = \left[Z \oplus X \oplus fX^* \oplus Y^*, \begin{bmatrix} \xi & \psi^* & 0 & 0 \\ \psi & \xi & \iota^* & 0 \\ 0 & \iota & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right]$$

$$\to (Y, \xi) \oplus \left(X \oplus X^*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \oplus (Y^*, 0)\ ,$$

and so

$$\begin{aligned} \sigma\left(V \oplus W^*, \begin{pmatrix} \varphi & g^* \\ g & 0 \end{pmatrix}\right) &= \sigma(Z, \xi) + \sigma\left(X \oplus X, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) + \sigma(Y^*, 0) \\ &= \sigma(Z, \xi) = \sigma(H, \bar{\varphi}) \in \boldsymbol{Z}\ , \end{aligned}$$

proving the lemm a.

The above lemma comes from an algebraic theory of surgery (see [1]) in the following special case. Let $(C, d, *)$ be a $4k$-dimensional cochain complex (C, d) with a symmetric pairing

$$* : (C, d) \to (C, \partial)$$

to the dual chain complex, involving the usual dimension shift:

$$\begin{array}{ccccccccccc} C^0 & \xrightarrow{d} & C^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & C^{2k} & \xrightarrow{d} & \cdots & \xrightarrow{d} & C^{4k} \\ {\scriptstyle *}\downarrow & & \downarrow{\scriptstyle *} & & & & \downarrow{\scriptstyle *} & & & & \downarrow{\scriptstyle *} \\ C_{4k} & \xrightarrow{\partial} & C_{4k-1} & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & C_{2k} & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & C_0\ . \end{array}$$

The cohomology $H^{2k+1}(C)$ is killed by replacing $(C, d, *)$ with the triple $(C', d', *')$ defined by

$$\begin{array}{ccccccccccccccc} C^0 & \xrightarrow{d} & C^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & C^{2k-1} & \xrightarrow{\begin{pmatrix} d \\ 0 \\ \varepsilon* \end{pmatrix}} & C^{2k} \oplus Z \oplus Z^* & \xrightarrow{(d\ e\ 0)} & C^{2k+1} & \xrightarrow{d} & \cdots & \xrightarrow{d} & C^{4k} \\ {\scriptstyle *}\downarrow & & \downarrow{\scriptstyle *} & & & & \downarrow{\scriptstyle *} & & \downarrow{\scriptstyle \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}} & & \downarrow{\scriptstyle *} & & & & \downarrow{\scriptstyle *} \\ C_{4k} & \xrightarrow{\partial} & C_{4k-1} & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & C_{2k+1} & \xrightarrow[\begin{pmatrix} \partial \\ \varepsilon \\ 0 \end{pmatrix}]{} & C_{2k} \oplus Z^* \oplus Z & \xrightarrow[(\partial\ 0\ *e)]{} & C_{2k-1} & \xrightarrow{d} & \cdots & \xrightarrow{d} & C_0 \end{array}$$

where $e \in \operatorname{Hom}(Z, C^{2k+1})$ is the inclusion of $Z = \ker(d: C^{2k+1} \to C^{2k+2})$ and $\varepsilon = e^* \in \operatorname{Hom}(C_{2k+1}, Z^*)$ is the dual. The nondegenerate forms associated with $(H^{2k}(C'), *')$ and $\left(C^{2k} \oplus C_{2k+1}, \begin{pmatrix} * & \partial \\ d & 0 \end{pmatrix}\right)$ are isomorphic, so that they have the same signature. The algebraic lemma is thus a verification of

$$\text{signature}\,(H^{2k}(C'), *') = \text{signature}\,(H^{2k}(C), *) \in \boldsymbol{Z}\,,$$

that is, "cobordant complexes have the same signature".

Appendix: The calculation of $\int_\eta \omega_\sigma \wedge \omega_\tau$

Let σ, τ be complementary oriented simplices in the triangulation of M. Write $x_0, x_1, \cdots, x_r$ for the vertices of σ, and $y_0, y_1, \cdots, y_s$ for those of τ, with

$$x_0 = y_0\,, \qquad r + s = 4k\,.$$

Let η be the $4k$-dimensional simplex spanned by σ and τ, with the vertices z_0, $z_1, \cdots, z_{4k}$ given by

$$z_i = \begin{cases} x_0 = y_0\,, & i = 0\,, \\ x_i\,, & 0 < i \leq r\,, \\ y_{i-r}\,, & r < i \leq 4k\,. \end{cases}$$

We have the differential forms

$$\omega_\sigma = r!\sum_{i=0}^{r} (-)^i x_i dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_r\,,$$

$$\omega_\tau = s!\sum_{j=0}^{s} (-)^j y_j dy_0 \wedge \cdots \wedge \widehat{dy_j} \wedge \cdots \wedge dy_s\,,$$

and also define

$$\omega = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{4k}\,.$$

The relation

$$\sum_{i=0}^{4k} z_i = 1$$

holds in η (and indeed defines η), so that

$$\sum_{i=0}^{4k} dz_i = 0\,.$$

Using these relations, we obtain

$$
\begin{aligned}
&\left(\sum_{i=0}^{r} (-)^i x_i dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_r\right) \\
&\qquad \wedge \left(\sum_{j=0}^{s} (-)^j y_j dy_0 \wedge \cdots \wedge \widehat{dy_j} \wedge \cdots \wedge dy_s\right) \\
&= x_0 y_0 \omega + (x_0 dx_1 \wedge \cdots \wedge dx_r) \\
&\qquad\qquad \wedge \left(\sum_{j=1}^{s} (-)^j y_j dy_0 \wedge \cdots \wedge \widehat{dy_j} \wedge \cdots \wedge dy_s\right) \\
&\quad + \left(\sum_{i=1}^{r} (-)^i x_i dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_r\right) \\
&\qquad \wedge (y_0 dy_1 \wedge \cdots \wedge dy_s) \\
&= \left(x_0 y_0 + x_0\left(\sum_{j=1}^{s} y_j\right) + \left(\sum_{i=1}^{r} x_i\right) y_0\right)\omega \\
&= z_0\left(\sum_{i=0}^{4k} z_i\right)\omega = z_0 \omega \ .
\end{aligned}
$$

It follows that

$$
\int_\eta \omega_\sigma \wedge \omega_\tau = r!\, s! \int_\eta z_0 \omega = *(\sigma, \tau)\frac{r!\, s!}{(4k+1)!} \ .
$$

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