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THE DE RHAM COHOMOLOGY OF SUBCARTESIAN SPACES

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The notion of differentiable subcartesian space is a generalization of that of differentiable manifold. Arbitrary subsets of $\boldsymbol{R}^n$ are special examples as well as differentiable manifolds with boundary, or corners, and analytic or semi-analytic spaces. In [7] we constructed the category of C^∞-subcartesian spaces and introduced the calculus of tensor fields and differential forms. In this sequel to [7] we study the cohomology algebra formed from those differential forms.

In § 1 we define the de Rham cohomology of a C^∞-subcartesian space. In § 2 we establish the Eilenberg-Steenrod axioms on an appropriate admissible category of pairs of subcartesian spaces. In § 3 we show by example that the de Rham and Čech cohomologies are distinct. We then establish a spectral sequence which has its E_2-terms in sheaf cohomology and which converges in the de Rham cohomology. We introduce a graded-sheaf invariant $\mathscr{H}(S)$ of a differentiable subcartesian space S, the *de Rham sheaf of S*, whose vanishing in higher degrees is sufficient for the de Rham cohomology to be naturally isomorphic to the sheaf cohomology with coefficients in $\mathscr{H}^0(S)$. If S is locally contractible, then $\mathscr{H}^0(S) = \boldsymbol{R}$ and $\mathscr{H}^k(S) = 0$ for $k > 0$, thus giving a natural isomorphism of the de Rham and sheaf- theoretic cohomology theories. We finish with an appendix on the C^k-cohomology, showing that it is not a topological invariant.

It is perhaps worth while to compare the cohomology theory developed here with those of [10], [11], and [12]. In [10] Schwartz constructed a cohomology theory which coincides with Čech cohomology on finite dimensonal compact spaces. Example 3.1 shows that this is not always the case for our theory. In [11] Smith constructed an exterior differential algebra and cohomology theory for each pair $(X, \mathscr{F})$, where X is a topological space and $\mathscr{F}$ is a set of continuous $\boldsymbol{R}$-valued functions on X. One might expect our theory to follow as a special case of Smith's when X is a C^∞-subcartesian space and $\mathscr{F} = C^\infty(X)$, but Example 3.15 shows that this is not the case. In [12] Spallek considered several notions of differential forms on differentiable spaces and stated a de Rham isomorphism theorem. In [7] we showed that the differential forms as defined for subcartesian spaces and the differential forms of [12] are different. Whether

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the corresponding cohomology theories are isomorphic is an open question.

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1. Definition of the de Rham cohomology

For each C^∞-subcartesian spaces S we shall denotes the graded $C^\infty(S)$-algebra of alternating covariant tensors (also called *forms*) by $F(S) = \{F^k(S) \mid k \in \boldsymbol{Z}\}$, the subalgebra of forms having differentials by $D(S) = \{D^k(S) \mid k \in \boldsymbol{Z}\}$, the graded ideal of differentials of 0 by $\mathfrak{m}(S) = \{\mathfrak{m}^k(S) \mid k \in \boldsymbol{Z}\}$, and the graded algebra of differential forms by $A(S) = \{A^k(S) \mid k \in \boldsymbol{Z}\}$. All of the homogeneous submodules of grades $k < 0$ are 0 by definition. If $f: S \to R$ is a C^∞-mapping of subcartesian spaces, then $f^*: F(R) \to F(S)$ maps $D(R) \to D(S)$, $\mathfrak{m}(R) \to \mathfrak{m}(S)$, and hence induces the pull-back $A(R) \to A(S)$ (also denoted f^*). Thus the following systems with their corresponding systems of pull-backs of inclusions are presheaves of modules over the presheaf C_S^∞ of C^∞-functions:

$$
(1.1) \qquad \begin{aligned} F_S &:= \{F(U) \mid U \text{ open in } S\}\,, & D_S &:= \{D(U) \mid U \text{ open in } S\}\,, \\ \mathfrak{m}_S &:= \{\mathfrak{m}(U) \mid U \text{ open in } S\}\,, & A_S &:= \{A(U) \mid U \text{ open in } S\}\,. \end{aligned}
$$

All four satisfy the sheaf axiom F_1 of Godement [5] for arbitrary S, and F_S satisfies sheaf axiom F_2. If S is paracompact, then all four satisfy both F_1 and F_2. In any case we denote the corresponding generated sheaves (*espaces etales*) by $\mathscr{F}_S, \mathscr{D}_S, \mathscr{M}_S$ and $\mathscr{A}_S$. Note that $\mathscr{F}_S = \mathscr{D}_S$ and $\mathscr{A}_S = \mathscr{F}_S/\mathscr{M}_S$.

Since exterior differentiation commutes with pull-backs [7], $(A_S, \boldsymbol{d})$ is a differential graded presheaf, and $(\mathscr{A}_S, \boldsymbol{d})$ is a differential graded sheaf. A C^∞-mapping $f: S \to S'$ induces f-cohomorphisms (cf. Bredon [3]) $F_{S'} \to F_S$, $D_{S'} \to D_S$, $\mathfrak{m}_{S'} \to \mathfrak{m}_S$, $A_{S'} \to A_S$ and hence an f-cohomorphism $f^*: \mathscr{A}_{S'} \to \mathscr{A}_S$. Each of these is compatible with differentiation.

Lemma 1.2. *Let Σ be a paracompactifying family of supports on S, and let $\mathscr{S}$ be a sheaf of $\boldsymbol{R}$-vector spaces over S. Then $\mathscr{F}_S \otimes \mathscr{S}$, $\mathscr{M}_S \otimes \mathscr{S}$, and $\mathscr{A}_S \otimes \mathscr{S}$ are Σ-soft and Σ-fine.*

Proof. Existence of C^∞-partitions of unity on paracompact subcartesian spaces [7, Proposition 1.2], and Σ being paracompactifying imply that $\mathscr{F}_S^0$ is Σ-fine and Σ-soft. Since $\mathscr{T}_S \otimes \mathscr{S}$, $\mathscr{M}_S \otimes \mathscr{S}$, and $\mathscr{A}_S \otimes \mathscr{S}$ are $\mathscr{F}_S^0$-modules, it follows that each is Σ-soft and Σ-fine. q.e.d.

Let $i: R \subsetneq S$, and let $\mathscr{S}$ be a sheaf of $\boldsymbol{R}$-vector spaces over S. Define $\mathscr{K}(S, R; \mathscr{S})$ to be the sheaf of germs of local sections γ of $\mathscr{A}_S \otimes_{\boldsymbol{R}} \mathscr{S}$ such that $(i^* \otimes |_R)\gamma = 0$ (where $|_R$ is the restriction cohomomorphism $\mathscr{S} \to \mathscr{S}|_R$). Equivalently, $\mathscr{K}(S, R; \mathscr{S})$ is the kernel of the unique homomorphism $j: \mathscr{A}_S \otimes \mathscr{S} \to i(\mathscr{A}_R \otimes \mathscr{S}|_R)$ such that $i^* \otimes |_R$ has the factorization $\mathscr{A}_S \otimes \mathscr{S} \to i(\mathscr{A}_R \otimes \mathscr{S}|_R) \to \mathscr{A}_R \otimes \mathscr{S}|_R$ (cf. [3, p. 9ff]). An elementary argument shows that $\mathscr{K}(S,R;\mathscr{S}) = \mathscr{K}(S,R)\otimes\mathscr{S}$, (where $\mathscr{K}(S,R) := \mathscr{K}(S,R;\boldsymbol{R})$) when $R \subseteq S$ is closed.

Define $\delta := \boldsymbol{d}\otimes_{\boldsymbol{R}}\mathrm{Id}: \mathscr{A}_S\otimes\mathscr{S}\to\mathscr{A}_S\otimes\mathscr{S}$. Then $\delta^2 = 0$ and δ leaves $\mathscr{K}(S,R;\mathscr{S})$ invariant. Thus $(\mathscr{A}_S\otimes\mathscr{S},\delta)$ and $(\mathscr{K}(S,R;\mathscr{S}),\delta)$ are differential graded sheaves. If Σ is a family of supports on S, then the following sequence of complexes of $\boldsymbol{R}$-vector spaces is exact:

$$0\to\Gamma_\Sigma\mathscr{K}(S,R;\mathscr{S})\to\Gamma_\Sigma\mathscr{A}_S\otimes\mathscr{S}\to\Gamma_{\Sigma\cap R}\mathscr{A}_R\otimes(\mathscr{S}|_R)\ . \tag{1.3}$$

Definition 1.4. The complex $\Gamma_\Sigma\mathscr{K}(S,R;\mathscr{S}) =: K_\Sigma(S,R;\mathscr{S})$ is called the de Rham complex of (S,R) with coefficients $\mathscr{S}$ and supports Σ. We define the de Rham cohomology of (S,R) with coefficients $\mathscr{S}$ and supports Σ to be the homology of $K_\Sigma(S,R;\mathscr{S})$, that is,

$$\boldsymbol{H}^k_\Sigma(S,R;\mathscr{S}) := H^kK_\Sigma(S,R;\mathscr{S})\ ,\qquad k\in\boldsymbol{Z}\ .$$

Obviously,

$$\boldsymbol{H}^k_\Sigma(S,R;\mathscr{S}) = 0 \qquad \text{for all } k<0. \tag{1.5}$$

When S is paracompact, $R=\phi$, $\Sigma = \mathrm{cls}$ (all closed subsets of S), and $\mathscr{S} = \boldsymbol{R}$, then $K_\Sigma(S,R;\mathscr{S})$ is naturally isomorphic to the complex of differential forms $A(S)$, and

$$\boldsymbol{H}^k(S) := \boldsymbol{H}^k_{\mathrm{cls}}(S,\emptyset;\boldsymbol{R}) \approx \frac{\mathrm{Ker}\,\boldsymbol{d}: A^k(S)\to A^{k+1}(S)}{\mathrm{Im}\,\boldsymbol{d}: A^{k-1}(S)\to A^k(S)}\ ,$$

i.e., *closed differential k-forms* modulo *exact differential k-forms*. Moreover, if we define a k-form $\zeta\in F^k(S)$ to be *closed* when $0\in F^{k+1}(S)$ is one of its differentials, and *exact* when it is a differential of some $\omega\in F^{k-1}(S)$, then

$$\boldsymbol{H}^k(S)\approx\frac{\text{closed }k\text{-forms}/\mathfrak{m}^k(S)}{\text{exact }k\text{-forms}/\mathfrak{m}^k(S)}\approx\frac{\text{closed }k\text{-forms}}{\text{exact }k\text{-forms}}\ .$$

Similarly, if S and R are arbitrary, Σ is paracompactifying for the pair (S,R) (cf. [3]), and $\mathscr{S} = \boldsymbol{R}$, then

$$\begin{aligned}\boldsymbol{H}^k_\Sigma(S,R) &:= \boldsymbol{H}^k_\Sigma(S,R;\boldsymbol{R})\\ &\approx \frac{\{\omega\in A^k(S)\,|\,\mathrm{supp}\,\omega\in\Sigma\cap(S\backslash R)\text{ and }\boldsymbol{d}\omega = 0\}}{\{\omega\in A^k(S)\,|\,\exists\zeta\in A^{k-1}(S)\text{ such that supp }\zeta\in\Sigma\cap(S\backslash R)\text{ and }\boldsymbol{d}\zeta=\omega\}}\\ &\approx \frac{\{\varphi\in F^k(S)\,|\,\mathrm{supp}\,\varphi\in\Sigma\cap(S\backslash R),\text{ and }0\in d\varphi\}}{\{\varphi\in F^k(S)\,|\,\exists\theta\in F^{k-1}(S)\text{ with }\varphi\in d\theta\text{ and supp }\theta\in\Sigma\cap(S\backslash R)\}}\ ,\end{aligned}$$

i.e., *k-forms closed relative to R modulo k-forms exact relative to R.*

Let $R', \mathscr{S}'$, and Σ' be alternate choices of $R, \mathscr{S}$ and Σ. Define

$$\begin{aligned}\wedge : (\mathscr{A}_S \otimes \mathscr{S}) \otimes (\mathscr{A}_S \otimes \mathscr{S}') \to \mathscr{A}_S \otimes \mathscr{S} \otimes \mathscr{S}' ;\\ (\alpha \otimes \sigma) \wedge (\alpha' \otimes \sigma') := \alpha \wedge \alpha' \otimes \sigma \otimes \sigma' .\end{aligned} \tag{1.6}$$

This product determines a product

$$\wedge : K_\Sigma(S, R; \mathscr{S}) \otimes K_{\Sigma'}(S, R'; \mathscr{S}') \to K_{\Sigma \cap \Sigma'}(S, R \cup R'; \mathscr{S} \otimes \mathscr{S}') . \tag{1.7}$$

Products of cocycles are cocycles, and the product of a cocycle and coboundary is a coboundary. Thus $\wedge$ induces a product

$$\bigcup : \boldsymbol{H}^k_\Sigma(S, R; \mathscr{S}) \otimes \boldsymbol{H}^m_{\Sigma'}(S, R'; \mathscr{S}') \to \boldsymbol{H}^{k+m}_{\Sigma \cap \Sigma'}(S, R \cup R'; \mathscr{S} \otimes \mathscr{S}'), k, m \in \boldsymbol{Z} .$$

2. The Eilenberg-Steenrod axioms

Let (S, R) and (S', R') be pairs of subcartesian spaces, Σ and Σ' families of supports on (S, R) and (S', R'), and let $\mathscr{S}$ and $\mathscr{S}'$ be sheaves of $\boldsymbol{R}$-vector spaces over S and S'. Let $f : (S, R) \to (S', R')$ be a C^∞-mapping of pairs, proper with respect to Σ and Σ' (i.e., $f^{-1}(F) \in \Sigma$ for each $F \in \Sigma'$). Let $g : \mathscr{S}' \to \mathscr{S}$ be an f-cohomomorphism. Then the f-cohomomorphism $f^* \otimes g : \mathscr{A}_{S'} \otimes \mathscr{S}' \to \mathscr{A}_S \otimes \mathscr{S}$ induces a homomorphism of complexes $K_{\Sigma'}(S', R'; \mathscr{S}') \to K_\Sigma(S, R; \mathscr{S})$ and hence homomorphisms

$$(f, g)^\#_k : \boldsymbol{H}^k_{\Sigma'}(S', R'; \mathscr{S}') \to \boldsymbol{H}^k_\Sigma(S, R; \mathscr{S}) , \qquad k \in \boldsymbol{Z} .$$

Let $\mathscr{Q}$ be the category whose objects are quadruples $(S, R, \mathscr{S}, \Sigma)$ and whose morphisms are pairs $(f, g) : (S, R, \mathscr{S}, \Sigma) \to (S', R', \mathscr{S}', \Sigma')$, where $f : (S, R) \to (S', R')$ is C^∞ and proper, and $g : \mathscr{S}' \to \mathscr{S}$ is an f-cohomomorphism. Then $(\boldsymbol{H}, \#)$ is a contravariant functor from $\mathscr{Q}$ to the category of graded $\boldsymbol{R}$-vector spaces. The induced homomorphism $(f, g)^\#$ is compatible with $\cup$-products.

If $u = (f, g) : (S, R, \mathscr{S}, \Sigma) \to (S', R', \mathscr{S}', \Sigma')$ is a morphism in $\mathscr{Q}$, we shall write u^* for $f^* \otimes g : \mathscr{A}_{S'} \otimes \mathscr{S}' \to \mathscr{A}_S \otimes \mathscr{S}$. By abuse of notation, we may also write u^* for $\Gamma u^* : \Gamma_{\Sigma'} \mathscr{A}_{S'} \otimes \mathscr{S}' \to \Gamma_\Sigma \mathscr{A}_S \otimes \mathscr{S}$. If $i : R \hookrightarrow S$ is an inclusion, then we shall write simply i^* for $i^* \otimes |_R$, and $i^\#$ for the homomorphism in cohomology induced by i^*.

A homomorphism $f : \mathscr{S} \to \mathscr{S}'$ of sheaves over S is nothing but an id_S-cohomomorphism. In this special case we shall denote the induced homomorphism $(\mathrm{id}_S, f)^\# : \boldsymbol{H}_\Sigma(S, R; \mathscr{S}) \to \boldsymbol{H}_\Sigma(S, R; \mathscr{S}')$ simply by $f_\#$. For each S, R and Σ the covariant functor $(\boldsymbol{H}_\Sigma(S, R;\ ), \#)$ is additive (in fact, strongly additive).

Theorem 2.1. *Let F^k denote the functor $(\boldsymbol{H}^k_\Sigma(S, R;\ ), \#)$, and Σ be a family of supports on S paracompactifying for the pair (S, R). Then for each short exact sequence of sheaves of $\boldsymbol{R}$-vector spaces over S*

$$0 \longrightarrow \mathscr{S}' \xrightarrow{\ f\ } \mathscr{S} \xrightarrow{\ g\ } \mathscr{S}'' \longrightarrow 0 \tag{2.2}$$

and each $k \in \boldsymbol{Z}$, there is a homomorphism $b^k : F^k(\mathscr{S}'') \to F^{k+1}(\mathscr{S}')$ such that the cohomology sequence

$$\cdots \longrightarrow F^k(\mathscr{S}') \xrightarrow{f_\#} F^k(\mathscr{S}) \xrightarrow{g_\#} F^k(\mathscr{S}'') \xrightarrow{b^k} F^{k+1}(\mathscr{S}') \longrightarrow \cdots$$

is exact. Moreover, each b^k is natural, i.e., short commutative ladder diagrams yield long commutative ladder diagrams.

Proof. Tensoring (2.2) with $\mathscr{A}_S \otimes_{\boldsymbol{R}}$ and applying Γ_Σ give the exact sequence of complexes

$$(2.3) \qquad 0 \to \Gamma_\Sigma \mathscr{A}_S \otimes \mathscr{S}' \to \Gamma_\Sigma \mathscr{A}_S \otimes \mathscr{S} \to \Gamma_\Sigma \mathscr{A}_S \otimes \mathscr{S}'' .$$

Since $\mathscr{A}_S \otimes \mathscr{S}'$ is Σ-soft, (2.3) remains exact when augmented on the right by zero. Similarly, the sequence

$$(2.4) \quad 0 \to \Gamma_{\Sigma \cap R} \mathscr{A}_R \otimes \mathscr{S}'|_R \to \Gamma_{\Sigma \cap R} \mathscr{A}_R \otimes \mathscr{S}|_R \to \Gamma_{\Sigma \cap R} \mathscr{A}_R \otimes \mathscr{S}''|_R \to 0$$

is exact. Applying the 3×3 lemma three times to the following diagram

$$\begin{array}{ccccccccc}
0 \to & K_\Sigma(S, R; \mathscr{S}') & \to & K_\Sigma(S, R; \mathscr{S}) & \to & K_\Sigma(S, R; \mathscr{S}'') & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & \Gamma_\Sigma \mathscr{A}_S \otimes \mathscr{S}' & \longrightarrow & \Gamma_\Sigma \mathscr{A}_S \otimes \mathscr{S} & \longrightarrow & \Gamma_\Sigma \mathscr{A}_S \otimes \mathscr{S}'' & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & \Gamma_{\Sigma \cap R} \mathscr{S}_R \otimes \mathscr{S}'|_R & \to & \Gamma_{\Sigma \cap R} \mathscr{A}_R \otimes \mathscr{S}|_R & \to & \Gamma_{\Sigma \cap R} \mathscr{A}_R \otimes \mathscr{S}''|_R & \to 0
\end{array}$$

yields the exactness of the top row. The theorem now follows from the usual diagram chase (Snake lemma).

Theorem 2.5. *Let $(S, R, \mathscr{S}, \Sigma) \in \mathscr{Q}$ with R closed and Σ paracompactifying. Then there exist homomorphisms*

$$\Delta^k : \boldsymbol{H}^k_{\Sigma \cap R}(R; \mathscr{S}|_R) \to \boldsymbol{H}^{k+1}_\Sigma(S, R; \mathscr{S})$$

making the cohomology sequence

$$\cdots \longrightarrow \boldsymbol{H}^k_\Sigma(S, R; \mathscr{S}) \longrightarrow \boldsymbol{H}^k_\Sigma(S; \mathscr{S}) \longrightarrow \boldsymbol{H}^k_{\Sigma \cap R}(R; \mathscr{S}|_R) \xrightarrow{\Delta^k} \boldsymbol{H}^{k+1}_\Sigma(S, R; \mathscr{S}) \longrightarrow \cdots$$

exact. Each Δ^k is natural, i.e., if $f : (S, R, \mathscr{S}, \Sigma) \to (S', R', \mathscr{S}', \Sigma')$ is a morphism in $\mathscr{Q}$, R' is closed, and Σ' is paracompactifying, then $\Delta^k \circ (f|_R)^\# = f^\# \circ \Delta^k$. Given sequence (2.2), then $\Delta \circ b = b \circ \Delta = 0$.

Proof. Because $\mathscr{A}_S$ is Σ-soft and $\mathscr{A}_{S'}$ is Σ'-soft, (1.3) remains exact when augmented on the right by zero. Thus we have the following commutative diagram of complexes with exact rows:

$$\begin{array}{ccccccccc} 0 \to & K_\Sigma(S,R;\mathscr{S}) & \to & \Gamma_\Sigma \mathscr{A}_S \otimes \mathscr{S} & \to & \Gamma_{\Sigma\cap R}\mathscr{A}_R \otimes \mathscr{S}|_R & \to 0 \\ & \uparrow & & \uparrow & & \uparrow & \\ 0 \to & K_{\Sigma'}(S',R';\mathscr{S}') & \to & \Gamma_{\Sigma'}\mathscr{A}_{S'} \otimes \mathscr{S}' & \to & \Gamma_{\Sigma'\cap R'}\mathscr{A}_{R'} \otimes \mathscr{S}'|_{R'} & \to 0 \end{array}$$

Existence and naturality of Δ^k now follow as usual.

If $\gamma \in \Gamma_{\Sigma\cap R}\mathscr{A}^k_R \otimes \mathscr{S}|_R$ satisfies $\delta\gamma = 0$, $[\gamma]_R$ denotes the cohomology class of γ in $\boldsymbol{H}^k_{\Sigma\cap R}(R;\mathscr{S}|_R)$, and $\gamma' \in \Gamma_\Sigma \mathscr{A}_S \otimes \mathscr{S}$ is any preimage of γ, then

(2.6a) $$\Delta^k[\gamma]_R = [\delta\gamma']_{(S,R)} .$$

Similarly, if $\gamma'' \in K^k_\Sigma(S,R;\mathscr{S}'')$ satisfies $\delta\gamma'' = 0$, $\gamma \in K^k_\Sigma(S,R;\mathscr{S})$ satisfies $\mathrm{Id}\otimes g(\gamma) = \gamma''$, and $\gamma' \in K^{k+1}_\Sigma(S,R;\mathscr{S}')$ satisfies $\mathrm{Id}\otimes f(\gamma') = \delta\gamma$, then

(2.6b) $$b^k[\gamma''] = [\gamma'] .$$

If follows that $b \circ \Delta = \Delta \circ b = 0$.

Theorem 2.7. *Let $\mathscr{S}\otimes\mathscr{S}'$ and $\mathscr{S}'\otimes\mathscr{S}$, $\mathscr{S}\otimes(\mathscr{S}'\otimes\mathscr{S}'')$ and $(\mathscr{S}\otimes\mathscr{S}')\otimes\mathscr{S}''$ be identified respectively via the usual natural isomorphisms. Then the cup product is associative, graded-anticommutative and $\boldsymbol{H}^0_{\mathrm{cls}}(S)$-bilinear. Moreover,*

$$\smile : \boldsymbol{H}^k_\Sigma(S,R,\mathscr{S}) \otimes \boldsymbol{H}^m_{\Sigma'}(S,R;\mathscr{S}') \to \boldsymbol{H}^{k+m}_{\Sigma\cap\Sigma'}(S,R\cup R' : \mathscr{S}\otimes\mathscr{S}')$$

is a natural transformation of functors on $\mathscr{Q}$ satisfying the following conditions:

(i) *Let $0 \longrightarrow \mathscr{T}' \xrightarrow{h} \mathscr{S} \xrightarrow{g} \mathscr{T}'' \longrightarrow 0$ be an exact sequence of sheaves of $\boldsymbol{R}$-vector spaces over S, and let Σ be paracompactifying for the pair (S,R). If $c \in \boldsymbol{H}^k_\Sigma(S,R;\mathscr{S})$ and $c' \in \boldsymbol{H}^m_{\Sigma'}(S,R';\mathscr{T}'')$ then $b^m(c')\cup c = b^{m+k}(c'\cup c)$.*

(ii) *Let $i: R \subsetneq S$ be closed, let Σ and Σ' be paracompactifying support families on S, and let $\mathscr{S}$ and $\mathscr{S}'$ be sheaves of $\boldsymbol{R}$-vector space on S. If $c \in \boldsymbol{H}^k_{\Sigma\cap R}(R;\mathscr{S}|_R)$ and $c' \in \boldsymbol{H}^m_{\Sigma'}(S;\mathscr{S}')$, then $\Delta c \cup c' = \Delta(c \cup i^\# c')$.*

Proof. Associativity, graded-commutativity and bilinearity hold at the chain level and hence in cohomology. Naturality of $\smile$ with respect to induced maps has already been mentioned and is clear.

To establish (i), let $\eta'' \in \Gamma_{\Sigma'}(\mathscr{A}^m_S \otimes \mathscr{T}'')$ and $\gamma \in \Gamma_\Sigma(\mathscr{A}^k_S \otimes \mathscr{S})$ be representatives of c' and c, respectively. Let $\eta \in \Gamma_{\Sigma'}(\mathscr{A}^m_S \otimes \mathscr{T})$ be a preimage of η'' under $\Gamma(\mathrm{id}\otimes g)$, and let $\eta' \in \Gamma_{\Sigma'}(\mathscr{A}^{m+1}_S \otimes \mathscr{T}')$ be a preimage of $\delta\eta$ under $\Gamma(\mathrm{id}\otimes h)$. Then $\Gamma((\mathrm{id}\otimes h)\otimes \mathrm{id})(\eta'\wedge\gamma) = \delta\eta\wedge\gamma$, and $\eta'\wedge\gamma$ is a representative of $b^m c' \cup c$. On the other hand, $\Gamma((\mathrm{id}\otimes g)\otimes\mathrm{id})(\eta\wedge\gamma) = \eta''\wedge\gamma$ is a representative of $c'\cup c$. Because $\delta\gamma = 0$, $\delta(\eta\wedge\gamma) = \delta\eta\wedge\gamma$. Thus $\eta'\wedge\gamma$ is also a representative of $b^{k+m}(c'\cup c)$. Therefore $b^m c'\cup c = b^{k+m}(c'\cup c)$.

To establish (ii), let η and γ be representatives of c and c', respectively. If η' is any preimage of η under the induced map $\Gamma\mathscr{A}_S \otimes \mathscr{S} \to \Gamma\mathscr{A}_R \otimes \mathscr{S}|_R$, then $\delta(\eta'\wedge\gamma)$ is a representative of $\Delta(c \cup i^\# c')$. Because $\delta\gamma = 0$, $\delta(\eta'\wedge\gamma) = \delta\eta'\wedge\gamma$, and $\delta\eta'\wedge\gamma$ is a representative of $\Delta c\cup c'$. It follows that $\Delta(c\cup i^\# c') = \Delta c \cup c'$.

Theorem 2.8 (*Excision*). *Let $U \subseteq S$ satisfy $U \subseteq$ Interior R, and let i denote the inclusion $(S\backslash U, R\backslash U) \subseteq (S, R)$. Then*

$$i^{\#}: \boldsymbol{H}_{\Sigma}(S, R; \mathscr{S}) \to \boldsymbol{H}_{\Sigma \cap (S\backslash U)}(S\backslash U, R\backslash U; \mathscr{S}|_{S\backslash U})$$

is an isomorphism.

Proof. We shall show that

$$i^*: K_{\Sigma}(S, R; \mathscr{S}) \to K_{\Sigma \cap (S\backslash U)}(S\backslash U, R\backslash U; \mathscr{S}|_{S\backslash U})$$

is an isomorphism. Injectivity is trivial. To show surjectivity, let $\gamma \in K_{\Sigma \cap (S\backslash U)}(S\backslash U, R\backslash U; \mathscr{S}|_{S\backslash U})$, and let $p \in \partial U$. There is a neighborhood V of p in S such that $V \subseteq R$. For each $q \in V \cap (S\backslash U)$, $\gamma_q = 0$. If follows that γ may be extended by 0 to all of S to give a section $\gamma' \in K_{\Sigma}(S, R; \mathscr{S})$. Then $i^*(\gamma') = \gamma$.

Theorem 2.9 (*Dimension*). *Let $S = P$ be a one-point space, and V an $\boldsymbol{R}$-vector space. Then*

$$\boldsymbol{H}^k(P; V) = \begin{cases} V, & \text{if } k = 0, \\ 0, & \text{if } k \neq 0. \end{cases}$$

The proof is trivial.

Definition 2.10. A homotopy $h = (f, g): (S, R, \mathscr{S}, \Sigma) \times I \to (S', R', \mathscr{S}', \Sigma')$ *in* $\mathscr{Q}$ is a C^∞-homotopy of pairs f with f proper relative to $\Sigma \times I$ and Σ', and an f-cohomomorphism $g: \mathscr{S}' \to \mathscr{S} \times I$.

Theorem 2.11 (*Homotopy invariance*). *Let $\mathscr{S}$ and $\mathscr{S}'$ be sheaves of $\boldsymbol{R}$-vector spaces over S and S', respectively, and let h be a homotopy in $\mathscr{Q}$ as above. Suppose R is closed in S. Then*

$$h_0^{\#} = h_1^{\#}: \boldsymbol{H}_{\Sigma'}^k(S', R'; \mathscr{S}') \to \boldsymbol{H}_{\Sigma}^k(S, R; \mathscr{S}), \quad k \in \boldsymbol{Z}.$$

Proof. For each $t \in I$ and $p \in S$ let $j_t: S \to S \times I$; $p \mapsto (p, t)$, and $j^p: I \to S \times I$; $t \mapsto (p, t)$. Let $Z \in \mathscr{X}(S \times I)$ be the vector field $(p, t) \mapsto j_*^p(\partial/\partial t)(t)$. For each $\phi \in F^k(S \times I)$, define

$$\mathscr{I}\phi_p = \int_0^1 (j_t^*\phi)_p dt .$$

Since $t \mapsto (j_t^*\phi)_p \in \bigwedge^k(T_pS)^*$ is continuous, the integral converges. Using the compactness of I, it is easy to show that for every $p \in S$ and $\varphi \in \mathfrak{A}_S$ with $p \in U_\varphi$ there are a neighborhood V of φp and a local representative θ of ϕ relative to $\varphi \times \mathrm{Id}$ defined in $V \times I$. Then $\mathscr{I}\theta \in F^k(V)$ is a local representative of $\mathscr{I}\phi$ relative to φ, i.e., $\mathscr{I}\phi \in F^k(S)$. Since $\mathscr{I}d\theta = d\mathscr{I}\theta$, it follows that $\mathscr{I}\mathfrak{m}_{R\times I} \subseteq \mathfrak{m}_S$. Thus $\mathscr{I}$ induces a linear map $A^k(S \times I) \to A^k(S)$, also denoted $\mathscr{I}$, which commutes with $\boldsymbol{d}$. Define $M: A^k(S \times I) \to A^{k-1}(S)$, $k \in \boldsymbol{Z}$, by

(2.12) $$M\omega = \mathscr{I} i_Z \omega \ .$$

For each $p \in S$, $k \in \boldsymbol{Z}$,

$$(j_1^*\omega - j_0^*\omega)_p = \int_I \frac{\partial}{\partial t}((j_t^*\omega)_p)dt$$

Because Z has local flows on $S \times (0, 1)$ (cf. [7]), we have

(2.13) $$\int_I \frac{\partial}{\partial t}((j_t^*\omega)_p)dt = \int_I (j_t^*\mathscr{L}_Z\omega)_p dt = \int_I (j_t^*(\boldsymbol{d}i_Z\omega + i_Z\boldsymbol{d}\omega))_p dt \ ,$$

i.e.,

(2.14) $$j_1^*\omega - j_0^*\omega = \mathscr{I}\boldsymbol{d}i_Z\omega + \mathscr{I}i_Z\boldsymbol{d}\omega = \boldsymbol{d}M\omega + M\boldsymbol{d}\omega \ .$$

If supp $\omega \subseteq U \times I$ for $U \subseteq S$, then supp $M\omega \subseteq U$. We define a graded presheaf (P, ρ) of $\boldsymbol{R}$-vector spaces on S by setting

$$P^k(U) = \Gamma(U \times I, \mathscr{K}^k(U \times I, (R \cap U) \times I))$$

and $\rho_{U'}^U = \iota^*$, where $\iota \colon U' \hookrightarrow U$. Then $M \circ \rho_{U'}^U = \rho_{U'}^U \circ M$. We may thus consider M as a homomorphism from P^k to the presheaf of local sections of $\mathscr{K}^{k-1}(S, R)$.

We also define a presheaf (V, r) of $\boldsymbol{R}$-vector spaces on S by setting

$$V(U) = \Gamma(U \times I, \mathscr{S} \times I)$$

and letting $r_{U'}^U$ be the ordinary restriction mapping. For each $v \in V(U)$, there is a unique $\sigma \in \Gamma(U, \mathscr{S})$ such that $v = \sigma \times \mathrm{Id}_I$. Thus we have an isomorphism of presheaves

$$\beta_U \colon V(U) \to \Gamma(U, \mathscr{S}); \qquad \sigma \times \mathrm{Id}_I \mapsto \sigma \ .$$

Now let $\zeta \in \Gamma(U \times I, \mathscr{K}^k(S, R; \mathscr{S}) \times I)$ for some open $U \subseteq S$, and let $p \in U$. For each $t \in I$ there exist $\varepsilon_t > 0$ and a neighborhood $U_t \subseteq U$ of p such that for $V_t := U_t \times (t - \varepsilon_t, t + \varepsilon_t)$

$$\zeta|_{V_t} \in \Gamma(V_t, \mathscr{K}^k(S, R) \times I) \otimes_{\boldsymbol{R}} \Gamma(V_t, \mathscr{S} \times I) \ .$$

Using the compactness of I we can find a neighborhood $W \subseteq U$ of p and a finite partition $t_0, \cdots, t_{n+1}$ of I such that the cover $\{V_i := W \times [t_{i-1}, t_{i+1}] \mid i = 1, \cdots, n\}$ is a refinement of $\{V_t \mid t \in I\}$. Let $\{w\alpha\}$ be a basis of the $\boldsymbol{R}$-vector space $\Gamma(W, \mathscr{S})$. Then for each α and each i there is a uniqe $\omega^{\alpha,i} \in \Gamma(V_i, \mathscr{K}^k(S, R))$ such that

$$\zeta|_{V_i} = \sum_\alpha \omega^{\alpha,i} \otimes (w_\alpha \times \mathrm{Id}_I) \ .$$

It follows that $\omega^{\alpha,i-1}$ and $\omega^{\alpha,i}$ agree on their common domain, thus giving rise to sections $\omega^\alpha \in P^k(W)$ such that

$$\zeta|_{W\times I} = \sum_\alpha \omega^\alpha \otimes (w_\alpha \times \mathrm{Id}_I) . \tag{2.15}$$

We have thus shown that for each such ζ and p there exists a $W \ni p$ such that $\zeta|_{W\times I} \in P^k(W) \otimes V(W)$.

We now define

$$\kappa = (M \otimes \beta) \circ h^* : \Gamma(S', \mathscr{K}^k(S', R'; \mathscr{S}')) \to \Gamma(S, \mathscr{K}^{k-1}(S, R; \mathscr{S})) . \tag{2.16}$$

Because f is proper with respect to $\Sigma \times I$ and Σ', κ is proper with respect to Σ and Σ'. From (2.14), (2.15) and (2.16) it follows that

$$\kappa \circ \delta + \delta \circ \kappa = j_1^* \circ h^* - j_0^* \circ h^* = h_1^* - h_0^* .$$

Therefore we have

$$h_0^\# = h_1^\# : \boldsymbol{H}^k_{\Sigma'}(S', R'; \mathscr{S}') \to \boldsymbol{H}^k_\Sigma(S, R; \mathscr{S}) , \quad k \in \boldsymbol{Z} . \qquad \text{q.e.d.}$$

Let $\mathscr{T}$ be the category whose objects are triples (S, R, Σ), S a C^∞-subcartesian space, $R \subseteq S$ closed, and Σ paracompactifying, and whose morphisms are proper C^∞-mappings of pairs. Then $\mathscr{T}$ is an admissible category in the sense of Eilenberg-Steenrod. Let V be an $\boldsymbol{R}$-vector space. From the results of this section it follows that the functors

$$F^k(V) : (S, R, \Sigma) \mapsto \boldsymbol{H}^k_\Sigma(S, R; S \times V) , \qquad k \in \boldsymbol{Z} ,$$

form a cohomology theory on $\mathscr{T}$ in the sense of Eilenberg-Steenrod. Moreover, if $\mathscr{T}'$ is an admissible subcategory of $\mathscr{T}$ (e.g., the full subcategory of locally compact pairs and compact supports) then $\{F^k(V)|_{\mathscr{T}'} | \, k \in \boldsymbol{Z}\}$ also satisfies the Eilenberg-Steenrod axioms. We therefore have for each of these cohomology theories the well-known series of theorems valid for Eilenberg-Steenrod cohomology theories on admissible categories, including the standard theorems on triads and triples, and the Mayer-Vietories theorems (cf. [4, Chapter 1]).

3. Comparison of the de Rham and sheaf cohomology theories

We begin with an example showing that the de Rham and sheaf cohomology theories ([5] or [3]) are distinct even on the category of finite dimensional compact spaces. We denote the sheaf cohomology functors by H^m.

Example 3.1. Let $S = \{1/n \,|\, n \in \boldsymbol{N}\} \cup \{0\}$ have the C^∞-structure induced from $\boldsymbol{R}$. Set $\boldsymbol{R} = \emptyset$, $\Sigma = \text{cls}$, and let $\mathscr{S}$ be the constant sheaf of real numbers. Then $H^0(S)$ is the direct sum of countably many copies of $\boldsymbol{R}$. To compute $\boldsymbol{H}^0(S)$, note that $\dim T_{1/n}S = 0$, $n \in \boldsymbol{N}$, and $\dim T_0 S = 1$. Then $f \in A^0(S)$ is a zero-cocycle if and only if $df = 0$, or equivalently, if and only if f has a C^∞-extension F near 0 satisfying $F'(0) = 0$. If $C^\infty[-1,1]$ has the C^∞-topology, then $\mathfrak{n} := \{f \in C^\infty[-1,1] | f|_S = 0\}$ is a closed subspace, and the projection $C^\infty[-1,1] \to C^\infty[-1,1]/\mathfrak{n}$ is continuous. Clearly $A^0(S) \approx C^\infty[-1,1]/\mathfrak{n}$, and because $C^\infty[-1,1]$ has a complete linear metric, so does $A^0(S)$. The map $C^\infty[-1,1] \to \boldsymbol{R}$; $F \mapsto F'(0)$ is continuous and annihilates $\mathfrak{n}$. Thus $\{F + \mathfrak{n} | F'(0) = 0\} \cong \boldsymbol{H}^0(S)$ is a closed subspace of $C^\infty[-1,1]/\mathfrak{n}$, and hence carries a complete linear metric. On the other hand, the Baire category theorem implies that $H^0(S)$ cannot carry a complete linear metric. Thus $\boldsymbol{H}^0(S)$ and $H^0(S)$ are not isomorphic. This example further shows that $\boldsymbol{H}$ is not continuous in the sense that $\varinjlim \boldsymbol{H}(S, S_m) \not\cong \boldsymbol{H}(S)$, where $S_m = \{1/n | n \geq m\}$.

Although $\boldsymbol{H}$ and H are not isomorphic, there are spectral sequences relating them. To each $(S, R, \mathscr{S}, \Sigma) \in \mathscr{Q}$ there corresponds the first quadrant double complex

$$C^{k,m}(S, R, \mathscr{S}, \Sigma) := \Gamma_\Sigma \mathscr{C}^k(S; \mathscr{K}^m(S, R; \mathscr{S})) ,$$

where $(\mathscr{C}, d)$ is the canonical resolution of Godement. The first differential d' of the double complex is d, and the second differential d'' is that induced by

$$(-1)^k \delta : \mathscr{K}^m(S, R; \mathscr{S}) \to \mathscr{K}^{m+1}(S, R; \mathscr{S}) .$$

The exact sequence of sheaves

$$0 \to \mathscr{K}(S, R; \mathscr{S}) \to \mathscr{A}_S \otimes \mathscr{S} \to i(\mathscr{A}_R \otimes \mathscr{S}|_R) \tag{3.2}$$

induces the exact sequence of double complexes

$$0 \to C(S, R, \mathscr{S}, \Sigma) \to C(S, \emptyset, \mathscr{S}, \Sigma) \to C_\Sigma(S; i(\mathscr{A}_R \otimes \mathscr{S}|_R)) ,$$

and this exact sequence is natural with respect to morphisms of $\mathscr{Q}$. Composing with the restriction mapping

$$C_\Sigma(S; i(\mathscr{A}_R \otimes \mathscr{S}|_R)) \to C(R, \emptyset, \mathscr{S}|_R, \Sigma \cap R)$$

we obtain the (non-exact) sequence

$$C(S, R, \mathscr{S}, \Sigma) \rightarrowtail C(S, \emptyset, \mathscr{S}, \Sigma) \to C(R, \emptyset, \mathscr{S}|_R, \Sigma \cap R) . \tag{3.3}$$

With each pair $(S, R, \mathscr{S}, \Sigma)$, $(S', R', \mathscr{S}', \Sigma') \in \mathscr{Q}$ there are associated natural homomorphisms

$$C^{k,m}(S, R, \mathscr{S}, \Sigma) \otimes C^{l,n}(S, R', \mathscr{S}', \Sigma')$$

$$\to \Gamma_{\Sigma \cap \Sigma'} \mathscr{C}^{k+l}(S; \mathscr{K}^m(S, R; \mathscr{S}) \otimes \mathscr{K}^n(S, R'; \mathscr{S}')) \tag{3.4a}$$

$$\to C^{k+l, m+n}(S, R \cup R', \mathscr{S} \otimes \mathscr{S}', \Sigma \cap \Sigma') , \tag{3.4b}$$

where the first is the canonical product of Godement [5], and the second is that induced by (1.6).

Definition 3.5. Let (S, R) be a pair of C^∞-subcartesian spaces, and let $\mathscr{S}$ be a sheaf of $\boldsymbol{R}$-vector spaces over S. Define $\mathscr{H}^k(S, R; \mathscr{S})$ to be the k-th derived sheaf of $\mathscr{K}(S, R; \mathscr{S})$. Writing $\mathscr{H}(S, R; \boldsymbol{R}) = : \mathscr{H}(S, R)$ we note that $\mathscr{H}(S, R; \mathscr{S}) = \mathscr{H}(S, R) \otimes \mathscr{S}$ when $R \subseteq S$ is closed. We call $\mathscr{H}(S, R; \mathscr{S})$ (respectively $\mathscr{H}(S, R)$) *the de Rham sheaf of* $(S, R; \mathscr{S})$ (respectively (S, R)).

Let Tot be the total complex of C. Then there are the usual two spectral sequences $'E$ and $''E$ satisfying

$$\text{(3.6a)} \qquad 'E_2^{k,m} = H_\Sigma^k(S; \mathscr{H}^m\mathscr{K}(S, R; \mathscr{S})) \Rightarrow H^{k+m} \operatorname{Tot}(S, R, \mathscr{S}, \Sigma) ,$$

$$\text{(3.6b)} \qquad ''E_2^{k,m} = H^k H_\Sigma^m(S; \mathscr{K}(S, R; \mathscr{S})) \Rightarrow H^{k+m} \operatorname{Tot}(S, R, \mathscr{S}, \Sigma) .$$

The edge terms $'E_2^{k,0}$ and $''E_2^{k,0}$ are $H_\Sigma^k(S; \mathscr{H}^0(S, R; \mathscr{S}))$ and $\boldsymbol{H}_\Sigma^k(S, R; \mathscr{S})$, respectively. The edge homomorphisms

$$\text{(3.7)} \qquad 'E_2^{k,0} \xrightarrow{\alpha} H^k \operatorname{Tot}(S, R, \mathscr{S}, \Sigma) \xleftarrow{\beta} {''E_2^{k,0}}$$

are induced by the chain maps:

$$\text{(3.8)} \qquad \begin{array}{ccc} C_\Sigma(S; \mathscr{K}^0(S, R; \mathscr{S})) & & \\ \downarrow \cap & & \\ C(S, R, \mathscr{S}, \Sigma) & \longrightarrow & \operatorname{Tot}(S, R, \mathscr{S}, \Sigma) \\ \uparrow \cup & & \\ K_\Sigma(S, R; \mathscr{S}) & & \end{array}$$

It follows that the edge homomorphisms are natural with respect to morphisms of $\mathscr{Q}$, and they respect cup products.

Theorem 3.9. *If Σ is paracompactifying, then β is an isomorphism. If Σ is paracompactifying for the pair (S, R), then $\beta^{-1} \circ \alpha$ is natural with respect to all connecting maps b^k (cf. Theorem 2.1). If R is closed, then $\beta^{-1} \circ \alpha$ is natural with respect to the connecting maps Δ^k.*

Proof. If Σ is paracompactifying, then Lemma 1.2 implies $H_\Sigma^m(S; \mathscr{K}(S, R; \mathscr{S})) = 0$ for $m \neq 0$. Thus $''E_2^{k,m} = 0$ for $m \neq 0$, and it follows that β is an isomorphism (cf. [3, Chapter IV]). A short exact sequence of coefficient sheaves induces a short exact sequence of the corresponding double complexes, i.e., a short exact sequence of diagrams (3.8). If Σ is paracompactifying for the pair (S, R), then these give long exact sequences in cohomology with α and β being natural with respect to connecting maps. If R is closed, then (3.3) is exact and remains exact when augmented on the right by zero. It then follows as usual that α and β are natural with respect to the connecting maps. q.e.d.

Thus when Σ is paracompactifying, (3.6a) gives

$$(3.13)\quad {}'E_2^{k,m} = H_\Sigma^k(S;\, \mathscr{K}^m(S,R;\,\mathscr{S})) \Rightarrow H^{k+m}H_\Sigma^0(S;\,\mathscr{K}(S,R;\,\mathscr{S})) \approx H_\Sigma^{k+m}(S,R;\,\mathscr{S}) .$$

Theorem 3.11. *Suppose $\mathscr{H}^k(S,R;\,\mathscr{S}) = 0$ for $k > 0$. Then α is an isomorphism, and there results the natural isomorphism of cohomology algebras*

$$(3.12a)\quad \boldsymbol{H}_\Sigma(S,R;\,\mathscr{S}) \approx H_\Sigma(S;\,\mathscr{H}^0(S,R;\,\mathscr{S})) .$$

If Σ is paracompactifying for the pair (S,R), then (3.12a) *is natural with respect to the connecting maps b^k. If R is closed, then*

$$(3.12b)\quad \boldsymbol{H}_\Sigma(S,R;\,\mathscr{S}) \approx H_\Sigma(S,R;\,\mathscr{H}^0(S;\,\mathscr{S})) ,$$

and (3.12b) *is natural with respect to the connecting maps Δ^k.*

Proof. The proofs of (3.12a) and the naturality of the b^k are standard ([3, IV. 2] or [5, § 4]). If R is closed, then $\mathscr{H}^0(S,R;\,\mathscr{S}) = \mathscr{H}^0(S;\,\mathscr{S})_{S\setminus R}$ and $H_\Sigma(S,R;\,\mathscr{H}^0(S;\,\mathscr{S})) \approx H_\Sigma(S;\,\mathscr{H}^0(S,R;\,\mathscr{S}))$ [3, Proposition II. 12.2] from which (3.12b) and the naturality of the Δ^k follow.

Corollary 3.13 (*de Rham isomorphism*). *If Σ is paracompactifying, $(S,R,\mathscr{S},\Sigma)$ is locally contractible (in $\mathscr{Q}$), and $R \subseteq S$ is closed, then*

$$(3.14)\quad \boldsymbol{H}_\Sigma(S,R;\,\mathscr{S}) \approx H_\Sigma(S,R;\,\mathscr{S}) .$$

This isomorpahism is natural with respect to morphisms of $\mathscr{Q}$, and is also natural with respect to connecting maps.

Proof. Theorems 2.9 and 2.11 imply

$$\mathscr{H}^k(S,R;\,\mathscr{S})_p \approx \begin{cases} \mathscr{S}_p , & \text{for } p \notin R \text{ and } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\boldsymbol{H}_\Sigma(S,R;\,\mathscr{S}) \approx H_\Sigma(S;\,\mathscr{H}^0(S,R;\,\mathscr{S})) \approx H_\Sigma(S,R;\,\mathscr{S}) .$$

Example 3.15. Let $f\colon \boldsymbol{R} \to \boldsymbol{R}$ be continuous but nowhere differentiable, and define $f_1(t) = \int_0^t f$. Let S be the graph of f_1 equipped with the structure induced from $\boldsymbol{R}^2$. For each $p \in S$, $\dim T_pS = 2$. S is nowhere locally C^∞-contractible. Let $g \in F^0(S)$ satisfy $dg = 0$. Then for $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2) \in S$, $g(p_2) - g(p_1) = \int_{x_1}^{x_2} (g \circ f_1)' dt = 0$. Thus $\mathscr{H}^0 := \mathscr{H}^0(S, \emptyset;\, \boldsymbol{R})$ is the constant sheaf of real numbers, and $\boldsymbol{H}^0(S) \cong \boldsymbol{R}$. Clearly $\boldsymbol{H}^m(S) = 0$ for $m > 2$. If $\omega \in F^2(S)$, and $\Omega = g dx \wedge dy$ is a local representative of ω, then Ω is defined in some open convex neighborhood U of S in $\boldsymbol{R}^2$. It follows from the classical Poincaré lemma that Ω is exact in U. Hence ω is exact, $\mathscr{H}^2 = 0$, and $\boldsymbol{H}^2(S) = 0$. To compute $\boldsymbol{H}^1(S)$, let $\omega \in F^1(S)$ be closed, and let $\Omega = g\,dx + h\,dy$ be a local representative of ω defined in U. Then $d\Omega(x, f_1(x)) = 0$ for each $x \in \boldsymbol{R}$, i.e.,

$$\frac{\partial g}{\partial y}\bigg|_S = \frac{\partial h}{\partial x}\bigg|_S . \tag{3.16}$$

For each $(x, y) \in U$ define

$$G(x, y) = g(x, f_1(x)) + \int_{f_1(x)}^{y} \frac{\partial h}{\partial x}(x, t)dt .$$

Then $\partial G/\partial y = \partial h/\partial x$ in U. Using (3.16) one easily shows that $G \in C^\infty(U)$. Thus $\Theta = Gdx + hdy \in F^1(U)$, and Θ is closed and hence exact. Finally, $\Theta|_S = \Omega|_S = \omega$. Thus ω is exact. We conclude that $\mathscr{H}^1 = 0$ and $\boldsymbol{H}^1(S) = 0$; hence $\boldsymbol{H}(S) \approx H(S)$.

We now compare $\boldsymbol{H}(S)$ with the cohomology of S as introduced by Smith [11]. Let $\mathscr{F} = F^0(S)$, and let $\mathfrak{H}(S, \mathscr{F})$ denote the Smith cohomology of the pair $(S, \mathscr{F})$. If $V \subseteq \boldsymbol{R}^n$ is open and $g: V \to S$ is continuous such that $f \circ g \in C^\infty(V)$ for every $f \in \mathscr{F}$, then certainly both $\pi_x \circ g$ and $\pi_y \circ g$ are of class C^∞, where $\pi_x: (x, y) \mapsto x$ and $\pi_y: (x, y) \mapsto y$. Since $f_1 \circ \pi_x \circ g = \pi_y \circ g \in C^\infty(V)$, $\pi_x \circ g$ and hence g are constant maps. The Smith completion $\mathscr{F}^*$ of $\mathscr{F}$ is then $C(S)$, all continuous $\boldsymbol{R}$-valued functions on S, and each is a 0-cocycle in the Smith theory. Thus $\mathfrak{H}^0(S, \mathscr{F}) = C(S)$.

4. Appendix: The C^k-de Rham cohomology theory

Throughout let $0 \leq k \leq l \leq \infty$. If S is a paracompact C^l-subcartesian space, then S admits C^k-partitions of unity. If S is of class C^{l+1}, then TS is of class C^l. If S is of class C^{l+2}, then the Lie product of two C^{k+1} vector fields is defined and is of class C^k.

Let $C^kF^m(S)$ denote the m-forms on S of class C^k, and let ${}^kF^m(S) \subseteq C^kF^m(S)$ denote the m-forms on S having a differential also of class C^k (e.g., closed m-forms). Let ${}^k\mathfrak{m}^n(S)$ be the differentials of 0 in ${}^kF^m(S)$, and define ${}^kA^m(S) = {}^kF^m(S)/{}^k\mathfrak{m}^m(S)$.

Let (S, R) be a pair of C^{l+2}-subcartesian spaces. Let ${}^{l+2}\mathscr{Q}$, ${}^{k+1}\mathscr{K}^m(S, R; \mathscr{S})$, and ${}^{k+1}K^m_\Sigma(S, R; \mathscr{S})$ be the obvious analogies of $\mathscr{Q}, \mathscr{K}^m(S, R; \mathscr{S})$ and $K^m_\Sigma(S, R; \mathscr{S})$. Let ${}^{k+1}\boldsymbol{H}_\Sigma(S, R; \mathscr{S})$ be the homology of the complex $({}^{k+1}K_\Sigma(S, R; \mathscr{S}), \delta)$. Then ${}^{k+1}\boldsymbol{H}$ is a connected family of functors on ${}^{l+2}\mathscr{Q}$ as before, satisfying the excision and dimension axioms.

To check the homotopy axiom, let $h: S \times I \to S'$ be a homotopy of class C^{k+2}. If $\omega \in {}^{k+1}F^m(S')$, then $h^*\omega \in {}^{k+1}F^m(S \times I)$. If ω is closed, then so is $h^*\omega$, and $\boldsymbol{d}i_Z h^*\omega = \mathscr{L}_Z h^*\omega$, where $\mathscr{L}_Z$ is as in Theorem 2.11. Thus

$$Mh^*\omega = \int_0^1 (j_t^* \mathscr{L}_Z h^*\omega)dt \ ,$$

and this is evidentally an element of $C^{k+1}F^m(S)$. Since

$$\boldsymbol{d} \circ M + M \circ \boldsymbol{d} = j_1^* - j_0^* \ ,$$

it follows that $\boldsymbol{d}Mh^*\omega \in C^{k+1}F(S)$. Thus $Mh^*\omega \in {}^{k+1}F(S)$. The homotopy axiom now follows as before.

The results of § 3 remain valid in the C^k-case. Thus if ${}^{k+1}\mathscr{H}^m(S, R; \mathscr{S}) = 0$ for $m > 0$, then

$${}^{k+1}\boldsymbol{H}_\Sigma(S, R; \mathscr{S}) \approx H_\Sigma(S; {}^{k+1}\mathscr{H}^0(S, R; \mathscr{S}))$$

when Σ is paracompactifying. In particular, if $(S, R, \mathscr{S}, \Sigma) \in {}^{l+2}\mathscr{Q}$ is C^{k+2}-locally contractible, then

$${}^{k+1}\boldsymbol{H}_\Sigma(S, R; \mathscr{S}) \approx H_\Sigma(S, R; \mathscr{S}) \ .$$

We end by showing that ${}^k\boldsymbol{H}$ is not a topological invariant for $k < \infty$.

Example. Let S be an arc in $\boldsymbol{R}^{k+2}$ for which there is a function $f \in C^{k+1}(\boldsymbol{R}^{k+2})$ with $df|_S = 0$ but $f|_S$ not constant (cf. [14]). Then ${}^{k+1}\boldsymbol{H}^0(S) \supsetneqq \boldsymbol{R}$. On the other hand, ${}^{k+1}\boldsymbol{H}^0(\boldsymbol{R}) = \boldsymbol{R}$.

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