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ALMOST SUBMANIFOLD STRUCTURES

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1. Introduction

The purpose of this paper is the investigation of certain structures on an n-dimensional C^∞ manifold determined by a second order connection, which we call almost submanifold structures. The case of an almost submanifold structure satisfying a certain additional condition is called an almost hypersurface structure, and is studied in detail. An almost hypersurface structure on a manifold allows us to treat the manifold almost as if it were a hypersurface of a second manifold. For example we may discuss the mean curvature and directions of curvature on a manifold bearing an almost hypersurface structure. We show that an almost hypersurface structure is integrable (isometrically imbeddable in a Euclidean space) if and only if the curvature tensor of the structure vanishes.

Various conditions are obtained that there exists a submanifold whose geodesics are also second order geodesics of a second order connection, and the mean curvature vector of and almost submanifold structure is investigated.

2. Preliminary remarks

In this section we will outline the results of [1] and [2] utilized in the main body of this paper. The notation utilized is essentially that of [1] and [3] with the summation convention employed on lower case Latin indices.

If ${}^2M$ denotes the third term of the extended sequence [2] of a manifold $M\,(\equiv {}^0M)$:

$$(1)\qquad {}^0M \xleftarrow{{}^1_0\pi} {}^1M \xleftarrow{{}^2_1\pi} {}^2M \xleftarrow{{}^3_2\pi} \cdots,$$

then a second order connection [1] on M is a connection on the bundle ${}^2_0\pi\colon {}^2M \to M$, which naturally induces a connection on M (which we called the first order connection induced on M). If ${}^1_0\pi_*$ denotes the tangent map of ${}^1_0\pi\colon TM \to M\,({}^1M \equiv TM)$, and K is the connection map of the induced first order connection, then TTM and consequently ${}^2M$ may be given a vector bundle structure over M relative to these maps. If HM and VM denote the horizontal and vertical subbundles of ${}^2M$ respectively, then

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$$(2) \qquad {}^1_0\pi_*: HM_p \to TM_{{}^1_0\pi(p)}, \qquad K: VM_p \to TM_{{}^1_0\pi(p)}$$

are isomorphisms of each $p \in TM$; and if (U, ϕ) is a coordinate chart of M, then there are naturally determined bases $\{X_i^h\}$ and $\{X_i^v\}$ of the horizontal and vertical subbundles respectively, and coordinates of ${}^2M$ (called vector bundle coordinates) relative to (U, ϕ) and the first order connection on M.

A second order connection on M determines a covariant differentiation of a section A of ${}^2_0\pi: {}^2M \to M$ with respect to a vector field X on M (here and throughout the remainder of the paper we identify the horizontal subbundle of ${}^2M$ with TM) which has the local form

$$(3) \quad \tilde{\nabla}_X A = \xi^j\left(\frac{\partial A^{0i}}{\partial x^{0j}} + \Gamma^{0i}_{jk}A^{0k}\right)X_i^h + \xi^j\left(\frac{\partial A^{1i}}{\partial x^{0j}} + \Gamma^{1i}_{j0k}A^{0k} + \Gamma^{1i}_{j1k}A^{1k}\right)X_i^v,$$

where $X = \xi^j \partial/\partial x^j$, and $A = A^{0i}X_i^h + A^{1i}X_i^v$.

If X and Y are C^∞ vector fields on M, and ξ is a vertical vector field on M, i.e., a C^∞ map $\xi: M \to VM$, then decomposing $\tilde{\nabla}$ into horizontal and vertical components yields

$$(4) \qquad \tilde{\nabla}_X Y = \nabla_X Y + \alpha(X, Y), \qquad \tilde{\nabla}_X \xi = D_X \xi,$$

where the horizontal component $\nabla_X Y$ is the covariant derivative of the induced first order connection, the vertical component $\alpha(X, Y)$, which we call the second fundamental form of $\tilde{\nabla}$, is bilinear, and the vertical component $D_X\xi$ is a connection in the vertical bundle.

If γ is a C^∞ curve of M, then γ' will denote the canonical lift of γ to TM, and γ'' the canonical lift of γ' to ${}^2M \subset TTM$.

3. Almost snbmanifold structures

Suppose that g is a fiber metric on ${}^2_0\pi: {}^2M \to M$, and that $\mathfrak{X}(M)$ and $\mathfrak{X}^v(M)$ denote the modules of C^∞ horizontal and vertical vector fields on M respectively. If $\tilde{\nabla}$ is a second order connection on M, then for all $X, Y \in \mathfrak{X}(M)$ and $\xi \in \mathfrak{X}^v(M)$ there is a unique field in $\mathfrak{X}(M)$, which we denote by $\mathscr{A}_\xi(X)$ such that

$$(5) \qquad g(\mathscr{A}_\xi(X), Y) = g(\alpha(X, Y), \xi).$$

Using $\mathscr{A}$ we define an operator ∇' such that

$$(6) \qquad \nabla'_X Y = \nabla_X Y + \alpha(X, Y), \qquad \nabla'_X \xi = -\mathscr{A}_\xi(X) + D_X\xi,$$

where ∇, α and D are as in (4). Such an operator satisfying the additional condition (5) will be called an almost submanifold structure or AS-structure on M. Thus by construction we see that to each pair consisting of a fiber metric on ${}^2_0\pi\colon {}^2M \to M$ and a second order connection on M there corresponds a unique AS-structure on M.

Suppose that for a given fiber metric g on ${}^2_0\pi\colon {}^2M \to M$ and second order connection $\tilde{\nabla}$, the first order connection induced on M by $\tilde{\nabla}$ is metric with respect to the metric on M obtained by restricting g to the horizontal subbundle of ${}^2M$, and that the connection on the vertical bundle induced by $\tilde{\nabla}$ is metric with respect to the metric obtained by restricting g to the vertical subbundle of ${}^2M$. If in addition the torsion $\widetilde{\mathrm{Tor}}\,(X, Y)$ of $\tilde{\nabla}$ with respect to any X, Y, given by

$$\widetilde{\mathrm{Tor}}\,(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] \,, \tag{7}$$

vanishes, we will say that $\tilde{\nabla}$ is Riemannian with respect to g.

Theorem 1. *If $\tilde{\nabla}$ is Riemannian with respect to a fiber metric g, with the additional property that vertical and horizontal vector are orthogonal at each point of M, then ∇' is Riemannian with respect to g.*

Proof. Letting $A = A^h + A^v$ where A^h and A^v are the horizontal and vertical components of A we see that since horizontal and vertical vectors are orthogonal and $\tilde{\nabla}$ Riemannian,

$$\begin{aligned} Xg(A,B) &= Xg(A^h, B^h) + Xg(A^v, B^v) \\ &= g(\nabla_X A^h, B^h) + g(A^h, \nabla_X B^h) + g(D_X A^v, B^v) + g(A^v, D_X B^v) \,. \end{aligned}$$

From (5) it follows that

$$\begin{aligned} g(\alpha(X, A^h), B^v) + g(\mathscr{A}_{B^v}(X), A^h) = 0 \,, \\ g(\alpha(X, B^h), A^v) + g(\mathscr{A}_{A^v}(X), B^h) = 0 \,, \end{aligned}$$

so that

$$Xg(A, B) = g(\nabla'_X A, B) + g(A, \nabla'_X B) \,.$$

Since

$$\mathrm{Tor}'\,(X, Y) = \nabla'_X Y - \nabla'_Y X - [XY] = \widetilde{\mathrm{Tor}}\,(X, Y) = 0 \,,$$

we see that ∇' is Riemannian with respect to g.

Theorem 2. *If the AS-structure ∇' is Riemannian with respect to the fiber metric g, then the first order connection induced by ∇' is Riemannian with respect to the metric induced by g, and $\alpha(X, Y) = \alpha(Y, X)$.*

Proof. Since ∇' is Riemannian, using (6) we have

$$\nabla'_X Y - \nabla'_Y X - [X, Y] = \nabla_X Y - \nabla_Y X + \alpha(X, Y) - \alpha(Y, X) = 0 \,,$$

hence

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0 , \qquad \alpha(X, Y) = \alpha(Y, X) .$$

We define the first vertical space $V_1(x)$ of an AS-structure at $x \in M$ by

$$V_1(x) = \operatorname{span}\{\alpha(X, Y) \mid X, Y \in M_x\} . \tag{8}$$

If $V_1(x)$ has maximum dimension l at any point $x \in M$, we call l the pseudocodimension of M.

4. Almost hypersurface structures

We first consider the case where the pseudocodimension of the AS-structure is 1. Let

$$\xi_x = \begin{cases} \dfrac{\alpha_x(X, Y)}{\|\alpha_x(X, Y)\|} & \text{if } \alpha_x(X, Y) \neq 0 \text{ for some } X, Y , \\ 0 & \text{if } \alpha_x \equiv 0 . \end{cases} \tag{9}$$

If $h(X, Y) = \|\alpha(X, Y)\|$, then $\alpha(X, Y) = h(X, Y)\xi$. If we take $\mathscr{A}(X) = \mathscr{A}_\xi(X)$ for the ξ defined in (9), the AS-structure becomes

$$\nabla'_X Y = \nabla_X Y + \alpha(X, Y) , \qquad \nabla_X \xi = -\mathscr{A}(X) + D_X \xi , \tag{10}$$

where $(D_X\xi)_x = 0$ if $\xi_x = 0$, and we restrict ourselves to the case where $\mathscr{A}$ is C^∞ henceforth.

On the basis of (10) we may define various notions analogous to those of a hypersurface. At a point $x \in M$ the mean curvature $H(x)$ is the trace of $\mathscr{A}_x$, and the total curvature $K(x)$ is the determinant of A_x. If $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of $\mathscr{A}_x$, they are the principal curvatures at x, and the corresponding eigenvectors are the directions of curvature at x. If two vectors at x have the property that $g(\mathscr{A}(X), Y) = 0$, then they are conjugate; and if $g(\mathscr{A}(X), X) = 0$, then X is asymptotic. If $\mathscr{A} = \lambda\,\mathrm{Id}$, then x is umbilical, etc.

We define the curvature tensor of the AS-structure ∇' in the usual manner as follows:

$$R'(X, Y)A = \nabla'_X \nabla'_Y A - \nabla'_Y \nabla'_X A - \nabla'_{[X,Y]} A , \tag{11}$$

and note that a standard calculation shows that the horizontal component of $R'(X, Y)Z$ is equal to

$$R(X, Y)Z + h(X, Z)\mathscr{A}(Y) - h(Y, Z)\mathscr{A}(X) , \tag{12}$$

where $X, Y, Z \in \mathfrak{X}(M)$, and R is the curvature tensor of the induced first order connection. The vertical component is equal to

$$(\nabla_X h)(Y, Z)\xi - (\nabla_Y h)(X, Z)\xi + h(Y, Z)D_X\xi - h(X, Z)D_Y\xi . \tag{13}$$

Suppose that $R' \equiv 0$. Then from the horizontal component we have

$$R(X,Y)Z = h(Y,Z)\mathscr{A}(X) - h(X,Z)\mathscr{A}(Y) \tag{14}$$

or equivalently

$$R(X,Y)Z = g(\mathscr{A}(Y),Z)\mathscr{A}(X) - g(\mathscr{A}(X),Z)\mathscr{A}(Y) . \tag{15}$$

From the vertical component of $R'(X,Y)Z$ it follows that at those $x \in M$ where ξ is not continuous, $D\xi = 0$ (by definition) whence

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z) .$$

At those $x \in M$ where ξ is C^∞, we have $g(\xi,\xi)$ constant and thus

$$Xg(\xi,\xi) = 2g(D_X\xi,\xi) = 0 .$$

Hence $D_X\xi$ and ξ are orthogonal, which together with (13) written in the form

$$(\nabla_X h)(Y,Z)\xi - \nabla_Y h(X,Z)\xi = h(X,Z)D_Y\xi - h(Y,Z)D_X\xi$$

implies that in either case

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z) , \tag{16}$$

or alternately

$$(\nabla_X \mathscr{A})(Y) = (\nabla_Y \mathscr{A})(X) . \tag{17}$$

Hence in the case of an AS-structure of pseudocodimension 1, (6) becomes

$$\nabla'_X Y = \nabla_X Y + \alpha(X,Y) , \qquad \nabla'_X \xi = -\mathscr{A}(X) . \tag{18}$$

Theorem 3. *On a manifold bearing a Riemannian AS-structure of pseudocodimension* 1 *such that for* $X,Y,Z \in \mathfrak{X}(M)$ *the horizontal component of* $R'(X,Y)Z$ *vanishes, the Ricci tensor is given by*

$$\mathrm{Ri}\,(X,Y) = g(\mathscr{A}(X),Y)\,\mathrm{tr}\,\mathscr{A} - g(\mathscr{A}^2(X),Y) .$$

Proof. By definition

$$\mathrm{Ri}\,(XY) = \text{trace of the map } Z \to R(Z,X)Y$$

using (15) is a standard fashion we obtain the desired formula.

We will say that an AS-structure of pseudocodimension 1 on M is integrable if there is an isometric imbedding of M into R^{n+1} with second fundamental tensor $\mathscr{A}$ the tensor determined by the symmetric transformation of the AS-structure in (18).

Theorem 4. *A Riemannian AS-structure of pseudocodimension* 1 *on a connected simply connected manifold is integrable if and only if* $R' \equiv 0$.

Proof. Suppose that $R' \equiv 0$. Then from (15) and (17) we see that the symmetric linear transformation $\mathscr{A}$ satisfies the Gauss-Codazzi equations. The integrability of the AS-structure on M then follows from the fundamental theorem for hypersurfaees [3]. On the other hand suppose that the AS-structure on M is integrable. Then $\mathscr{A}$ satisfies the Gauss-Codazzi equations, and equations (15) and (17) imply that $R'(X, Y)Z = 0$ for arbitrary $X, Y, Z \in \mathfrak{X}(M)$. From (18) it follows that

$$\begin{aligned} R'(X, Y)\xi &= \nabla'_X \nabla'_Y \xi - \nabla'_Y \nabla'_X \xi - \nabla'_{[X,Y]}\xi \\ &= (\nabla_Y \mathscr{A})(X) - (\nabla_X \mathscr{A})(Y) - \mathscr{A}(\mathrm{Tor}(X, Y)) \\ &\quad - \alpha(X, \mathscr{A}(Y)) + \alpha(Y, \mathscr{A}(X)) \,. \end{aligned} \tag{19}$$

Since the Codazzi equation (17) is satisfied, $(\nabla_Y \mathscr{A})(X) - (\nabla_X \mathscr{A})(Y) = 0$; and since ∇' is Riemannian, $\mathscr{A}(\mathrm{Tor}(X, Y)) = 0$. Noting that $g(\alpha(X, Y), \xi) = g(\mathscr{A}(X), Y)$ we see that

$$\begin{aligned} g(\alpha(X, \mathscr{A}(Y)), \xi) &= g(\mathscr{A}(X), \mathscr{A}(Y)) \\ &= g(\mathscr{A}(Y), \mathscr{A}(X)) = g(\alpha(Y, \mathscr{A}(X)), \xi) \,, \end{aligned}$$

so that $R'(X, Y)\xi = 0$.

Remark. Although an imbedding of M, to within an isometry of R^{n+1}, is determined by an integrable AS-structure, an imbedded submanifold of R^{n+1} may be determined, within an isometry of R^{n+1}, by several AS-structures. Suppose that M admits a global nonvanishing C^∞ vector field and is imbedded in R^{n+1} with second fundamental form h. If ξ and ξ' are C^∞ unit vector fields on M (these exist globaly since M admits a nonvanishing C^∞ vector field and a connection map K) take $\alpha(X, Y) = h(X, Y)\xi$ and $\alpha'(X, Y) = h(X, Y)\xi'$, then the AS-structures so obtained yield the same imbedding of M into R^{n+1}, to within an isometry of R^{n+1}.

Theorem 5. *Suppose that M bears a Riemannian AS-structure of pseudocodimension* 1 *having the properties that the vertical component of R' vanishes, and that the type number $t(x)$ of $\mathscr{A}$ at each point $x \in M$ is constantly l on an open neighborhood U of M. Then through each $x \in U$ there passes a maximal submanifold S of dimension $n - l$ having the property that each geodesic of S is also a second order geodesic of S (in the sense that ∇' and ∇ agree on S).*

Proof. Let $\mathscr{D}_x =$ kernel $\mathscr{A}_x$ for each x of the open submanifold U of M. If $X_x \in \mathscr{D}_x$ and $Y_x \in M_x$, then from

$$g(\mathscr{A}(X_x), Y_x) = g(X_x, \mathscr{A}(Y_x)) = 0 \tag{20}$$

we see that $\mathscr{A}(M_x) = \mathscr{D}_x^\perp$ (in the horizontal subbundle). Suppose that $X^1, \cdots, X^n$ are C^∞ vector fields which form a basis of M_p at each p in a neighborhood of x. Then from the set

$$\{\mathscr{A}(X^1), \cdots, \mathscr{A}(X^n)\} \tag{21}$$

we may select a minimal subset which spans $\mathscr{D}_x^\perp$. Since these are C^∞ and linearly independent at x, they are linearly independent and thus span $\mathscr{D}_p^\perp$ at each p in some neighborhood of x. Consequently $\mathscr{D}^\perp$ is a C^∞ distribution of dimension l on U, and hence $(\mathscr{D}^\perp)^\perp = \mathscr{D}$ is a C^∞ $(n-l)$-dimensional C^∞ distribution on U. Suppose that $X, Y \in \mathscr{D}$. Then

$$\begin{aligned}(\nabla_X\mathscr{A})(Y) &= \nabla_X\mathscr{A}(Y) - \mathscr{A}(\nabla_X Y) ,\\ (\nabla_Y\mathscr{A})(X) &= \nabla_Y\mathscr{A}(X) - \mathscr{A}(\nabla_Y X) .\end{aligned} \tag{22}$$

Since the vertical component of R' vanishes, the Codazzi equation $(\nabla_X\mathscr{A})(Y) = (\nabla_Y\mathscr{A})(X)$ holds, and since $\mathscr{A}(X) = \mathscr{A}(Y) = 0$ we have

$$\mathscr{A}(\nabla_X Y - \nabla_Y X) = 0 . \tag{23}$$

However, since the AS-structure is Riemannian, $\nabla_X Y - \nabla_Y X = [X, Y]$, and thus

$$\mathscr{A}([X, Y]) = 0 . \tag{24}$$

Thus $\mathscr{D}$ is an involutive distribution on U, and consequently through each $x \in U$ there passes a maximal integral $(n-l)$-dimensional manifold S of $\mathscr{D}$. Since $\mathscr{A}$ vanishes on tangent vectors to S and

$$g(\mathscr{A}(X), Y) = h(X, Y) , \qquad \alpha(X, Y) = h(X, Y)\xi , \tag{25}$$

we see that the second fundamental form α of the AS-structure vanishes on tangent vectors to S, and hence that each geodesic of S is also a second order geodesic of S.

Remark. If we define in the usual manner

$$R'(W, X, Y, Z) = g(R'(Y, Z)X, W) \tag{26}$$

for $W, X, Y, Z \in \mathfrak{X}(M)$, and the AS-structure is Riemannian, we see that if

$$\frac{R'(X, Y, X, Y)}{g(X, X)g(Y, Y) - g^2(X, Y)} = k \tag{27}$$

for all $X, Y \in \mathfrak{X}(M)$, then

$$R'(X, Y)Z = k(g(Z, Y)X - g(Z, X)Y) , \tag{28}$$

and consequently the vertical component of $R'(X, Y)Z$ vanishes. Hence the condition on the vertical component of $R'(X, Y)Z$ in Theorem 5 may be replaced with the above "constant curvature" condition (27).

Theorem 6. *Suppose that M bears a Riemannian AS-structure of pseudocodimension* 1, *with the properties that $\mathscr{A}$ is parallel with respect to the induced first order connection on M, and that the type number $t(x)$ of $\mathscr{A}$ at each point of an open neighborhood U of M is constantly l. Then the conclusion of Theorem* 5 *holds.*

Proof. Suppose that $S, Y \in \mathscr{D}$ where $\mathscr{D}$ is defined as in Theorem 5. If $\mathscr{A}$ is parallel with respect to the induced first order connection on M, then

$$\nabla_X \mathscr{A}(Y) = \mathscr{A}(\nabla_X Y) , \qquad \nabla_Y \mathscr{A}(X) = \mathscr{A}(\nabla_Y X) . \tag{29}$$

Since $X, Y \in \mathscr{D}$, it follows that $\mathscr{A}(\nabla_X Y) = \mathscr{A}(\nabla_Y X) = 0$, and hence that $\mathscr{A}(\nabla_X Y - \nabla_Y X) = \mathscr{A}([X, Y]) = 0$ due to the fact that the AS-structure is Riemannian. Thus $\mathscr{D}$ is an involutive C^∞ distribution on U (that $\mathscr{D}$ is C^∞ follows exactly as in Theorem 5) and the conclusion desired follows as in Theorem 5.

5. The mean curvature vector

The mean curvature vector of a manifold bearing an AS-structure is giving by

$$\eta = \operatorname{tr} \mathscr{A}_i \xi_i , \tag{30}$$

where $\operatorname{tr} \mathscr{A}_i$ denotes the trace of the map $X \to \mathscr{A}_i(X)$, ξ_i is an orthonormal basis of the vertical subbundle of ${}^2M$ (which exists locally at least), and $\mathscr{A}_i$ is defined by

$$g(\mathscr{A}_i(X), Y) = g(\alpha(X, Y), \xi_i) . \tag{31}$$

We first note that in the case of an AS-structure of pseudocodimension 1, α and η are linearly dependent.

Theorem 7. *If a manifold M bears an AS-structure of pseudocodimension* 1, *then there exists a C^∞ map h: $\mathfrak{X}(M) \times \mathfrak{X}(M) \to R$ such that*

$$\alpha(X, Y) = h(X, Y)\eta/\|\eta\|$$

at each point of M except where $\eta = 0$ and $\alpha \neq 0$.

Proof. Suppose that $\xi_1, \cdots, \xi_n$ form an orthonormal basis of the vertical subbundle such that η and ξ_1 are linearly dependent. Since $g(\mathscr{A}_i(X), Y) = g(\alpha(X, Y), \xi_i)$, we see that $\mathscr{A}_i = 0$ and consequently $\operatorname{tr} \mathscr{A}_i = 0$, $i = 2, \cdots, n$. Thus

$$\eta = \operatorname{tr} \mathscr{A}_i \xi_i = \operatorname{tr} \mathscr{A}_1 \xi_1 . \tag{32}$$

On the other hand

$$\alpha(X, Y) = g(\alpha(X, Y), \xi_i)\xi_i = g(\mathscr{A}_i(X), Y)\xi_i = g(\mathscr{A}_1(X), Y)\xi_1 ,$$

and $\xi_1 = \eta/\mathrm{tr}\,\mathscr{A}_1 = n/\|\eta\|$ for $\eta \neq 0$. Thus taking $h(X, Y) = g(\mathscr{A}_1(X), Y)$ we have

$$\alpha(X, Y) = h(X, Y)\eta/\|\eta\|$$

at each point of M where $\eta \neq 0$. If $\alpha = 0$, then $\mathscr{A}_i = 0$, $i = 1, \cdots, h$ and $\eta = 0$; hence $h(X, Y) = 0$. If we define $\eta/\|\eta\| = 0$ when $\eta = 0$, we again obtain the desired formula.

Suppose that we define the mean curvature of a manifold bearing an *AS*-structure by

$$H(x) = \|\eta_x\| , \tag{33}$$

and define $\mathscr{A}$ by

$$g(\mathscr{A}(X), Y) = g(\alpha(X, Y), \eta/\|\eta\|) \tag{34}$$

for $\eta \neq 0$ and $\mathscr{A} = 0$ for $\eta = 0$. Then $\mathscr{A}$ is C^∞ except where $\eta = 0$ and $\alpha \neq 0$, as in Theorem 7 we have

$$g(\mathscr{A}(X), Y) = g(h(X, Y)\eta/\|\eta\|, \eta/\|\eta\|) = h(X, Y)$$

except when $\eta = 0$ and $\alpha \neq 0$.

Theorem 8. *If M is a manifold bearing an AS-structure, then*

$$H(x) = \mathrm{tr}\,\mathscr{A}_x \quad .$$

Proof. If x is a point of M such that $\eta_x \neq 0$, then

$$\begin{aligned} g(\mathscr{A}(X_x), Y_x) &= g(\alpha(X_x, Y_x), (\mathrm{tr}\,\mathscr{A}_i)\xi_i/\|\eta_x\|) \\ &= (\mathrm{tr}\,\mathscr{A}_i)g(\alpha(X_x, Y_x), \xi_i)/\|\eta_x\| \\ &= (\mathrm{tr}\,\mathscr{A}_i)g(\mathscr{A}_i(X_x), Y_x)/\|\eta_x\| . \end{aligned}$$

Thus

$$\mathscr{A} = (\mathrm{tr}\,\mathscr{A}_i)\mathscr{A}_i/\|\eta\| , \tag{35}$$

and hence that

$$\mathrm{tr}\,\mathscr{A} = (1/\|\eta_x\|) \textstyle\sum_{i=1}^n (\mathrm{tr}\,\mathscr{A}_i)^2 = \|\eta_x\| = H(x) . \tag{36}$$

If $\eta_x = 0$, then from (34) we see that $\mathscr{A}_x = 0$ and hence that once again $H(x) = \mathrm{tr}\,\mathscr{A}_x$.

In the case where there are no points of M such that $\eta = 0$ and $\alpha \neq 0$, we may endow M with an AS-structure of pseudocodimension 1 via the formulas

$$\nabla'_X Y = \nabla_X Y + g(\alpha(X, Y), \eta/\|\eta\|)\eta/\|\eta\| ,$$
$$\nabla'_X \eta/\|\eta\| = -\mathscr{A}(X) + D_X\eta/\|\eta\| ,$$

and the mean curvature is the same as that of the original AS-structure.

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