

## ON CHERN'S KINEMATIC FORMULA IN INTEGRAL GEOMETRY

ALBERT NIJENHUIS

*Dedicated to S. S. Chern on his 60th birthday*

### 1. Introduction

In 1939 Hermann Weyl [1] derived a formula for the volume of the tube of radius  $\rho$  about a compact manifold (without boundary) imbedded in a Euclidean space. The expression for this volume, for a manifold  $X$  of dimension  $k$  imbedded in a Euclidean  $n$ -space  $E^n$  is a polynomial  $V(T_\rho^{(n)}(X))$  in  $\rho$ , valid for small  $\rho$ , when no self-intersections in the normal bundle occur. The coefficients of this polynomial are integrals over  $X$  of invariant polynomial functions of the Riemann-Christoffel curvature tensor. The polynomial expression for the volume is of the form

$$(1.1) \quad V(T_\rho^{(n)}(X)) = \sum \gamma_{n,k,e} \mu_e(X) \rho^{n-k-e},$$

where the summation extends over all even values of  $e$  such that  $0 \leq e \leq k$ . The  $\mu_e(X)$  are the integral invariants referred to, while the  $\gamma_{n,k,e}$  depend only on their subscripts and not on more subtle geometric properties of  $X$ . Thus  $\gamma$  and  $\mu$  are uniquely determined up to a factor which depends on  $k$  and  $e$ . In what follows we add a superscript (1) to  $\mu$  when quoting others.

In 1966 S. S. Chern [2] studied the same  $\mu$ 's from the point of view of the kinematic formula. Let  $M^p$  and  $M^q$  be compact manifolds of dimensions  $p$  and  $q$  imbedded in  $E^n$ , and let  $g$  be an element of the group of isometries in  $E^n$ . Then, for almost all  $g$ ,  $M^p \cap gM^q$  is again a manifold, and the  $\mu_e^{(1)}(M^p \cap gM^q)$  are meaningful quantities. The kinematic formula of Chern deals with the integral  $\int \mu_e^{(1)}(M^p \cap gM^q) d^{(1)}g$ , where the integration extends over the group of isometries, and  $d^{(1)}g$  is the Haar measure on this group, i.e., the product of the measure on  $E^n$  and that on the orthogonal group in  $n$  dimensions, the latter being a product of measures on spheres. This integral, according to Chern's theorem, is expressible as follows:

---

Communicated September 17, 1973. Research partially supported by an N.S.F. grant at the University of Pennsylvania.

$$(1.2) \quad \int \mu_e^{(1)}(M^p \cap gM^q) d^{(1)}g = \sum_{\substack{i+j=e \\ i,j \text{ even}}} c_{i,j,n,p,q} \mu_i^{(1)}(M^p) \mu_j^{(1)}(M^q) .$$

Again, the  $c$ 's depend only on their subscripts and some normalization. Some of the hardest work in Chern's paper is devoted to a detailed calculation from which the  $c$ 's can be determined.

The author's interest in Chern's work was stimulated by the fact that the right side of (1.2) depends bilinearly on the  $\mu_i^{(1)}(M^p)$  and  $\mu_j^{(1)}(M^q)$ , and (at least to him) the suggestion of an underlying algebra with the  $c$ 's as structure constants was inevitable. Simple formal properties of the integral in (1.2) support this initial impression: the interchange of  $p$  and  $q$  in (1.2) does not change the value (a simple change of variable in the integral shows this), so the "algebra" is commutative. Similarly, a formal manipulation of (1.2) shows that the "algebra" is associative.

A first step toward finding this "algebra" was a retracing of the  $c$ 's; a second step a re-normalization of the  $\mu$ 's which would give the  $c$ 's in (1.2) a simple form: we find the value 1 works.

The main result of this paper may be stated as follows:

**Theorem I.** *There exist a normalization of the  $\mu$ 's and a normalization of the Haar measure of the isometry group of  $E^n$ , as given by (3.5), (3.6), (3.7), such that the  $c_{i,j,n,p,q}$  in (1.2) are equal to 1.*

Rephrased in terms of the "algebra" (which did not quite work out) the theorem says:

**Theorem I'.** *There exist a normalization of the  $\mu$ 's and a Haar measure  $dg$  on the isometry group of  $E^n$ , given by (3.5), (3.6), (3.7), so that the Chern curvature polynomials defined by*

$$(1.3) \quad \mu(X, \lambda) = \sum_e \mu_e(X) \lambda^e \quad (e \text{ even}, 0 \leq e \leq \dim X)$$

satisfy

$$(1.4) \quad \int \mu(M^p \cap gM^q, \lambda) dg \equiv \mu(M^p, \lambda) \mu(M^q, \lambda) \pmod{\lambda^{p+q-n+1}} .$$

This version of the kinematic formula shows that the left-hand integral is to some extent independent of  $n$ , a fact which was not previously apparent.

Returning to the Weyl expression, we introduce a somewhat different normalization (cf. (2.5))

$$(1.5) \quad \bar{\mu}_e(X) = \mathcal{O}_{k-e+1} \mu_e(X)$$

and corresponding *Weyl curvature polynomials*

$$(1.6) \quad \bar{\mu}(X, \lambda) = \sum_e \bar{\mu}_e(X) \lambda^e \quad (e \text{ even}, 0 \leq e \leq \dim X) .$$

Then we have

**Theorem II.** *The Weyl curvature polynomials (1.6) satisfy*

$$(1.7) \quad \bar{\mu}(X \times Y, \lambda) = \bar{\mu}(X, \lambda) \bar{\mu}(Y, \lambda) ,$$

$$(1.8) \quad V(T_\rho^{(n)}(X)) = \sum_{\substack{0 \leq e \leq k \\ e \text{ even}}} B_{n-k+e}(\rho) \bar{\mu}_e(X) ,$$

where  $B_m(R)$  is the volume of the  $R$ -ball in  $E^m$ , and  $k = \dim X$ .

**Note.** The numerical coefficients here do not agree with those in (10) of Chern [2]; an error must have slipped in somewhere. See § 4 for details.

### 2. Some of Chern's formulas

Let  $X$  be a  $k$ -dimensional Riemann manifold. Then following Chern [2] denote by  $\varphi_{\alpha\beta}$  the Levi-Civita connection forms ( $1 \leq \alpha, \beta \leq k$ ), alternating in  $\alpha$  and  $\beta$ , and by  $\varphi_\alpha$  ( $1 \leq \alpha \leq k$ ) an orthonormal coframe field. Thus

$$(2.1) \quad d\varphi_\alpha = \sum_\beta \varphi_\beta \wedge \varphi_{\beta\alpha} ,$$

$$(2.2) \quad d\varphi_{\alpha\beta} = \sum_\delta \varphi_{\alpha\delta} \wedge \varphi_{\delta\beta} + \Phi_{\alpha\beta} ,$$

where

$$(2.3) \quad \Phi_{\alpha\beta} = \frac{1}{2} \sum_{\gamma, \delta} S_{\alpha\beta\gamma\delta} \varphi_\gamma \wedge \varphi_\delta .$$

The  $S_{\alpha\beta\gamma\delta}$  are components of the curvature tensor, and have the usual properties with respect to the pairs of alternating subscripts  $(\alpha, \beta)$  and  $(\gamma, \delta)$ . Define for even  $e$ ,  $0 \leq e \leq k$ , the pointwise function on  $X$ :

$$(2.4) \quad I_e^{(1)} = \frac{(-1)^{e/2} (k - e)!}{2^{e/2} k!} \sum \delta \left( \begin{matrix} \alpha_1 \cdots \alpha_e \\ \beta_1 \cdots \beta_e \end{matrix} \right) S_{\alpha_1 \alpha_2 \beta_1 \beta_2} \cdots S_{\alpha_{e-1} \alpha_e \beta_{e-1} \beta_e} ,$$

where  $\delta( \ )$  is a generalized Kronecker delta equal to  $\pm 1$  as the  $\beta$ 's are an even or odd permutation of the  $\alpha$ 's, and zero otherwise; summation is over all  $\alpha$ 's and  $\beta$ 's independently. The numerical factor which precedes  $\sum$  in (2.4) was chosen so that  $I_e^{(1)} = 1$  when  $X$  is the unit sphere  $S^k$  in  $E^{k+1}$ . (To effect this normalization of Chern's it is necessary to replace the factor  $2^{k/2}$  in his (7) by  $2^{e/2}$  as in our formula.) Following Chern we define the  $\mu$ 's (we add superscript (1)) as volume integrals:

$$\mu_e^{(1)}(X) = \int_X I_e^{(1)} dv .$$

With this normalization of the  $\mu$ 's, the  $c$ 's are obtained from Chern's calculation, as follows (he wrote  $c_i$  for our  $c_{i,j,n,p,q}$ ). Denote by  $\mathcal{O}_m$  the  $(m - 1)$ -dimensional volume of the unit  $(m - 1)$ -sphere in  $E^m$ 's, so

$$(2.5) \quad \mathcal{O}_m = 2\pi^{m/2} / \Gamma(m/2) ,$$

and then

$$(2.6) \quad c_{e-i} = \frac{\mathcal{O}_{n+1} \cdots \mathcal{O}_2 \mathcal{O}_{p+q-n+3} \mathcal{O}_{q+2-i} \mathcal{O}_{q+2-e+i}}{\mathcal{O}_{p+2} \mathcal{O}_{p+1} \mathcal{O}_{q+2} \mathcal{O}_{q+1} \mathcal{O}_{p+q-n+3-i} \mathcal{O}_{p+2-n+3-e+i}} b_{e,p+q-n+1-i} ,$$

where the  $b$ 's are given through an expression denoted by  $B_e$  which leads to the formula (Chern's (73) and an integral 13 lines below)

$$(2.7) \quad \begin{aligned} & \frac{\mathcal{O}_{m-1} \mathcal{O}_m}{2^{m-e}} \int_{-2R}^{2R} (t + 1 + R^2)^{e/2} (4R^2 - t^2)^{(m-e-2)/2} dt \\ & = b_{e,m-e-1} R^{m-e-1} + \cdots + b_{e,m-1} R^{m-1} , \end{aligned} \quad \text{where } m = p + q - n + 2 .$$

At the end of the paper appears a formula (81)

$$(2.8) \quad \begin{aligned} & \int \mu_e^{(1)}(M^p \cap E^q) d^{(1)}E^q \\ & = \frac{\mathcal{O}_{n+1} \cdots \mathcal{O}_{n-q+1}}{\mathcal{O}_{q+1} \cdots \mathcal{O}_1} \frac{\mathcal{O}_{p+q-n+2} \mathcal{O}_{p+q-n+1}}{\mathcal{O}_{p+q-n+2-e}} \frac{\mathcal{O}_{p+2-e}}{\mathcal{O}_{p+1} \mathcal{O}_{p+2}} \mu_e^{(1)}(M_p) , \end{aligned}$$

which refers to the case when  $gM^q$  is replaced by the planes  $E^q$  of dimension  $q$ . Since  $E^q$  is not compact, the integration is extended over the space of all  $q$ -planes;  $d^{(1)}E^q$  is an invariant measure on this space.

### 3. Calculation of the $c$ 's

In his § 7 Chern gave a formula for the  $b$ 's, hence by implication, for the  $c$ 's; but the expression is in the form of a *sum*, which is hard to manipulate. Instead, we aim for a *product* expression.

**Lemma 1.** *Let  $e \geq 0$  be even and  $r > e - 2$ . Denote by  $\alpha_i$  the coefficient of  $x^i$  in*

$$(3.1) \quad \int_{-1}^1 (x^2 + 2ux + 1)^{e/2} (1 - u^2)^{(r-e)/2} du ;$$

then

$$(3.2) \quad c_{i,j,n,p,q} = \frac{\mathcal{O}_{n+1} \cdots \mathcal{O}_2 \mathcal{O}_{r+1} \mathcal{O}_{r+2} \mathcal{O}_{r+3} \mathcal{O}_{q+2-e+i} \mathcal{O}_{p+2-i}}{\mathcal{O}_{p+1} \mathcal{O}_{p+2} \mathcal{O}_{q+1} \mathcal{O}_{q+2} \mathcal{O}_{r+3-e+i} \mathcal{O}_{r+3-i}} \alpha_{e-i} ,$$

where  $r = p + q - n$ .

*Proof.* Immediate from (2.6), (2.7), and the change of variable  $t = 2uR$  in (2.7).

**Lemma 2.** *Let  $\alpha_i$  be defined as in Lemma 1. Then  $\alpha_i = 0$  for  $i$  odd, and for  $i$  even*

$$(3.3) \quad \alpha_i = \binom{e/2}{i/2} \frac{\mathcal{O}_{r-i+3} \mathcal{O}_{r-e+i+3}}{\mathcal{O}_{r+3} \mathcal{O}_{r-e+2}}.$$

*Proof.* One can show that

$$(3.4) \quad \alpha_i = \pi^{\frac{1}{2}} \binom{\frac{e}{2}}{\frac{i}{2}} \frac{\Gamma\left(\frac{r+3}{2}\right) \Gamma\left(\frac{r-e+2}{2}\right)}{\Gamma\left(\frac{r-i+3}{2}\right) \Gamma\left(\frac{r-e+i+3}{2}\right)};$$

then (3.3) follows by application of (2.5). Formula (3.4) is most easily derived from properties of hypergeometric functions, as was shown by J. van Lint and by J. Boersma. A less elegant method is obtained from

$$\begin{aligned} & \int_{-1}^1 (x^2 + 2ux + 1)^n (1 - u^2) du \\ &= (x^2 + 1) \int_{-1}^1 (x^2 + 2ux + 1)^{n-1} (1 - u^2)^s du \\ & \quad - \frac{x}{s+1} \int_{-1}^1 (x^2 + 2ux + 1)^{n-1} d(1 - u^2)^{s+1} \end{aligned}$$

by completing the started integration by parts and deducing a recurrence relation for  $\alpha_i = \alpha_{i,n,s}$ . Details on the three methods are found in [3].

**Remark.** The crucial part of (3.3) is the exact dependence of  $\alpha_i$  on  $r$ , without which certain vital cancellations could not have taken place. This is reflected in the wording of the problem in [3].

*Proof. of Theorem I.* The value of  $c_{i,j,n,p,q}$  is found from (3.2) and (3.3), and can be written as

$$c_{i,j,n,p,q} = \frac{\mathcal{O}_{n+1} \cdots \mathcal{O}_2 \left( \frac{\mathcal{O}_{r+1} \mathcal{O}_{r+2} (e/2)!}{\mathcal{O}_{r-e+2}} \right)}{\left( \frac{\mathcal{O}_{p+1} \mathcal{O}_{p+2} (i/2)!}{\mathcal{O}_{p-i+2}} \right) \left( \frac{\mathcal{O}_{q+1} \mathcal{O}_{q+2} (j/2)!}{\mathcal{O}_{q-j+2}} \right)},$$

where  $i + j = e$ ,  $r = p + q - n$ . Note that  $r = \dim(M^p \cap gM^q)$ . Hence the  $c$ 's become all equal to 1 if we change  $d^{(1)}g$  by a factor  $(\mathcal{O}_m \cdots \mathcal{O}_2)^{-1}$  and choose  $\mu_e(X)$  equal to  $\mathcal{O}_{k-e+2} / [\mathcal{O}_{k+1} \mathcal{O}_{k+2} (e/2)!]$  times  $\mu_e^{(1)}(X)$ ; in addition we

may introduce a factor  $a^{e/2}$ , where  $a$  is any universal constant. In view of  $(e/2)! = 2\pi^{1+e/2}/\mathcal{O}_{e+2}$  we choose  $a = \pi$ , hence we define

$$(3.5) \quad \mu_e(X) = \frac{\mathcal{O}_{k-e+2}\mathcal{O}_{e+2}}{2\pi\mathcal{O}_{k+1}\mathcal{O}_{k+2}}\mu_e^{(1)}(X) = \int_X I_e dv, \\ k = \dim X, \quad 0 \leq e \leq k, \quad e \text{ even,}$$

where

$$(3.6) \quad I_e = \frac{\mathcal{O}_{e+2}}{\mathcal{O}_{k-e+1}}(-1)^{e/2}2^{e/2-1}\pi^{-e/2} \sum \delta\left(\begin{matrix} \alpha_1 \cdots \alpha_e \\ \beta_1 \cdots \beta_e \end{matrix}\right) S_{\alpha_1\alpha_2\beta_1\beta_2} \cdots S_{\alpha_{e-1}\alpha_e\beta_{e-1}\beta_e},$$

$$(3.7) \quad dg = (\mathcal{O}_{n+1} \cdots \mathcal{O}_2)^{-1}d^{(1)}g.$$

This proves Theorem I. The re-normalization (3.5) also simplifies (2.8); particularly if the measure on the space of  $q$ -planes is also re-normalized as

$$(3.8) \quad dE^q = \frac{\mathcal{O}_{q+1} \cdots \mathcal{O}_1}{\mathcal{O}_{n+1} \cdots \mathcal{O}_{n-q+1}}d^{(1)}E^q,$$

then the formula becomes

$$(3.9) \quad \int \mu_e(M^p \cap E^q)dE^q = \mu_e(M^p), \quad e \leq p + q - n,$$

or

$$(3.10) \quad \int \mu(M^p \cap E^q, \lambda)dE^q \equiv \mu(M^p, \lambda) \pmod{\lambda^{p+q-n+1}}.$$

#### 4. The Weyl formula

The volume of an  $R$ -ball in  $E^m$  is

$$B_m(R) = \int_0^R \mathcal{O}_m r^{m-1} dr = \frac{\mathcal{O}_m}{m} R^m = \frac{\mathcal{O}_{m+2}}{2\pi} R^m.$$

The starting point of this section is (1.1), in which we assume the  $\mu$ 's are normalized as in (3.5), (3.6), i.e., by the property

$$(4.1) \quad \mu_e(S^k(R)) = \frac{\mathcal{O}_{k-e+2}\mathcal{O}_{e+2}}{2\pi\mathcal{O}_{k+2}} R^{k-e} = \frac{\mathcal{O}_{e+2}}{\mathcal{O}_{k+2}} B_{k-e}(R).$$

To find the numerical value of the  $\gamma$ 's we calculate the volume of the  $\rho$ -tube about  $S^k(R)$  imbedded in  $E^n$ . First  $n = k + 1$ :

$$(4.2) \quad B(T_\rho^{(k+1)}(S^k(R))) = \frac{\mathcal{O}_{k+1}}{k+1} ((R + \rho)^{k+1} - (R - \rho)^{k+1}) .$$

To calculate the volume for  $k + 1 < n$  we use the following theorem which is an obvious consequence of the possibility to build up  $\rho$ -tubes in product situations from products of thin layers of the tubes around the factors.

**Theorem III.** *Let  $X \subset E$  and  $Y \subset E^m$  be imbeddings, and  $X \times Y \subset E^{n+m}$  the corresponding imbedding of the product. Then*

$$(4.3) \quad V(T_\rho^{(n+m)}(X \times Y)) = \int_{\substack{\rho_1^2 + \rho_2^2 \leq \rho^2 \\ \rho_1, \rho_2 \geq 0}} dV(T_{\rho_1}^{(n)}(X)) \wedge dV(T_{\rho_2}^{(m)}(Y)) .$$

In particular, if  $X$  and  $Y$  are points, we have

$$(4.4) \quad B_{n+m}(R) = \int_{\substack{\rho_1^2 + \rho_2^2 \leq \rho^2 \\ \rho_1, \rho_2 \geq 0}} dB_n(\rho_1) \wedge dB_m(\rho_2) ,$$

or equivalently,

$$(4.5) \quad \int_{\substack{\rho_1^2 + \rho_2^2 \leq \rho^2 \\ \rho_1, \rho_2 \geq 0}} \rho_1^{n-1} \rho_2^{m-1} d\rho_1 d\rho_2 = \frac{\mathcal{O}_{n+m+2}}{2\pi \mathcal{O}_n \mathcal{O}_m} \rho^{n+m} .$$

**Note.** (4.5) is also easily derived analytically by changing variables:  $\rho_1 = r \cos \theta$ ,  $\rho_2 = r \sin \theta$  in the integral and evaluating.

*Proof of Theorem II.* By applying Theorem III to  $X = S^k(R) \subset E^{k+1}$  and  $Y$  a single point in  $E^{n-k-1}$ , we find

$$\begin{aligned} V(T_\rho^{(n)}(S^k(R))) &= \int_{\substack{\rho_1^2 + \rho_2^2 \leq \rho^2 \\ \rho_1, \rho_2 \geq 0}} d \frac{\mathcal{O}_{k+1}}{k+1} ((R + \rho_1)^{k+1} - (R - \rho_1)^{k+1}) \wedge dB_{n-k-1}(\rho_2) \\ &= \int 2\mathcal{O}_{k+1} \sum_{\substack{e \text{ even} \\ 0 \leq e \leq k}} \binom{k}{e} R^{k-e} \rho_1^e d\rho_1 \wedge dB_{n-k-1}(\rho_2) \\ &= 2\mathcal{O}_{k+1} \sum_e \binom{k}{e} R^{k-e} \frac{1}{\mathcal{O}_{e+1}} \int dB_{e+1}(\rho_1) \wedge dB_{n-k-1}(\rho_2) \\ &= \sum_e \binom{k}{e} \frac{2\mathcal{O}_{k+1}}{\mathcal{O}_{e+1}} B_{n-k+e}(\rho) R^{k-e} \\ &= \sum_e \binom{k}{e} \frac{2\mathcal{O}_{k+1}}{\mathcal{O}_{e+1}} \frac{2\pi \mathcal{O}_{k+2}}{\mathcal{O}_{e+2} \mathcal{O}_{k-e+2}} \mu_e(S^k(R)) B_{n-k+e}(\rho) \\ &= \sum_e \mathcal{O}_{k-e+1} \mu_e(S^k(R)) B_{n-k+e}(\rho) . \end{aligned}$$

In the last step we have used  $k! \mathcal{O}_{k+1} \mathcal{O}_{k+2} = 2^{-k+2} \pi^{k+1}$ , which is just the doubling formula for the  $\Gamma$ -function.

Thus assuming Weyl's basic form of (1.1) is correct we have verified (1.8), and the general formula (1.7) follows easily from Theorem III. In fact, we have

$$\begin{aligned} V(T_{\rho}^{(n+m)}(X \times Y)) &= \sum_e \bar{\mu}_e(X \times Y) B_{n+m-p-q+e}(\rho), \\ V(T_{\rho_1}^{(n)}(X)) &= \sum_i \bar{\mu}_i(X) B_{n-p+i}(\rho_1), \\ V(T_{\rho_2}^{(m)}(Y)) &= \sum_j \bar{\mu}_j(Y) B_{m-q+j}(\rho_2). \end{aligned}$$

Now (4.3) relates the left sides, while (4.4) relates the right sides. It follows that

$$\bar{\mu}_e(X \times Y) = \sum_{\substack{i+j=e \\ i,j \text{ even}}} \bar{\mu}_i(X) \bar{\mu}_j(Y),$$

which implies (1.7).

### References

- [ 1 ] H. Weyl, *On the volume of tubes*, Amer. J. Math. **61** (1939) 461–472.
- [ 2 ] S. S. Chern, *On the kinematic formula in integral geometry*, J. Math. Mech. **16** (1966) 101–118.
- [ 3 ] A. Nijenhuis, *Problem 204 (Solutions contributed by J. van Lint and J. Boersma)*, Nieuw Archief v. Wisk. III Ser. **17** (1969) 166–168.

UNIVERSITY OF PENNSYLVANIA