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THE EQUIVARIANT COVERING HOMOTOPY PROPERTY FOR DIFFERENTIABLE G-FIBRE BUNDLES

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Let G be a compact Lie group, and X a differentiable G-manifold. If $p: E \to X$ is a differentiable fibre bundle, and G acts differentiably on E so that each $g \in G$ operates as a bundle map, then we call p a differentiable G-fibre bundle. We show that if p is a differentiable G-fibre bundle with Lie structure group or compact fibre, then it has the equivariant covering homotopy property. This generalizes the fact that a differentiable family of actions of a compact Lie group on a compact differentiable manifold is locally trivial.

We give some basic definitions in § 1, and in § 2 show that if X is a G-manifold and $E \to X$ a differentiable fibre bundle with Lie structure group H and associated principal bundle $P \to X$, then differentiable actions of G on E as a group of bundle maps are in natural one-one correspondence with such actions on P. In § 3 we establish the equivariant covering homotopy property for differentiable G-fibre bundles with compact Lie structure group, and show that if $p: E \to X$ is a differentiable G-fibre bundle with connected semi-simple Lie structure group H, then p can be reduced to a compact subgroup of H so that G still operates as a group of bundle maps, and hence p also has the equivariant covering homotopy property. Then in § 4 we define a notion of equivariant local triviality for G-fibre bundles, which implies the equivariant covering homotopy property, and show that any differentiable G-fibre bundle with Lie structure group or compact fibre is G-locally trivial. We conclude with some remarks relating G-local triviality to the equivalence of nearby differentiable actions of a compact Lie group.

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1. Basic definitions

Let G be a topological group. A *G-space* is a Hausdorff space X together with a *continuous action* of G on X, i.e., a continuous map $(g, x) \to gx$ of $G \times X$ into X such that $g_1(g_2x) = (g_1g_2)x$ for all $g_1, g_2 \in G$, $x \in X$, and $1x = x$,

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where 1 is the identity element of G. If G is a Lie group, then a (*differentiable*) *G-manifold* is a differentiable (C^∞) manifold X together with a *differentiable action* of G on X. The action is *effective* if whenever $gx = x$ for some g and all x, then $g = 1$.

Let G be a compact Lie group, and X a differentiable G-manifold. Let G_x be the isotropy subgroup of a point $x \in X$. The map $G/G_x \to X$ defined by $gG_x \to gx$ is an equivariant embedding whose image is the orbit Gx. Let $V_x = TX_x/T(Gx)_x$ be the normal space to the orbit Gx at the point x. For $g \in G_x$, the differential of $g: X \to X$ induces an automorphism of V_x, so we have a representation $G_x \to GL(V_x)$, called a *slice representation*. The *slice bundle* $G \times_{G_x} V_x$ is the G-vector bundle constructed from the product $G \times V_x$ by identifying $(gh, h^{-1}v)$ with (g, v) for all $g \in G$, $h \in G_x$, $v \in V_x$; we let $[g, v]$ denote the image of (g, v) in $G \times_{G_x} V_x$ under the identification map. Using the identification $G/G_x \to Gx$, we can identify the slice bundle $G \times_{G_x} V_x$ with the normal bundle of Gx in X by the map $[g, v] \to gv$. Hence, by an equivariant version of the tubular neighborhood theorem, there is an equivariant diffeomorphism from $G \times_{G_x} V_x$ onto a G-invariant open neighborhood of Gx in X, mapping the zero section G/G_x canonically onto the orbit Gx. We call the image of V_x a *slice* at x.

A *fibre bundle* is a continuous map $p: E \to X$ of a Hausdorff space E onto a Hausdorff space X such that p is locally trivial. Let $p_i: E_i \to X_i$, $i = 1, 2$, be two fibre bundles. A *bundle map* from p_1 to p_2 is a continuous map $F: E_1 \to E_2$ which carries each fibre homeomorphically onto a fibre. The induced map $f: X_1 \to X_2$ is clearly continuous. If $X_1 = X_2$ and the induced map is the identity (so that F is a homeomorphism), then the bundle map is called an *equivalence*.

We also consider fibre bundles with a specified structure group which acts effectively on the typical fibre. When a structure group is specified, bundle maps are understood to be induced by principal bundle maps between the associated principal bundles. Note that if $p_i: E_i \to X_i$, $i = 1, 2$, are fibre bundles with structure group the identity and fibre Y, then E_i is equivalent to $X_i \times Y$, $i = 1, 2$, and over each map $f: X_1 \to X_2$ of the bases there is only one bundle map, corresponding to $f \times \mathrm{id}: X_1 \times Y \to X_2 \times Y$.

We will mainly be concerned with differentiable (C^∞) fibre bundles. In this case the spaces are differentiable manifolds, the maps are C^∞, and a structure group is a Lie group acting differentiably (from the left) on the typical fibre (and so differentiably from the right on the total space of the associated principal bundle).

2. *G*-fibre bundles

In the remainder of this paper G will denote a compact Lie group.

Let X be a G-space, and $p: E \to X$ a fibre bundle over X. If there is a continuous action of G on E such that each $g \in G$ operates as a bundle map over the given map $g: X \to X$ (hence, in the case that p has a specified structure group, is induced by a principal bundle map), then we say that G *acts on* $p: E \to X$ *as a group of bundle maps* and that p is a *G-fibre bundle* (*differentiable* if p is a differentiable fibre bundle and X, E are G-manifolds). Note that p is equivariant. A *G-fibre bundle map* (resp. *G-fibre bundle equivalence*) is a map (resp. equivalence) of G-fibre bundles which is equivariant with respect to the actions of G.

Example 1. Let X, Y be G-spaces. G acts on $X \times Y$ by $g(x, y) = (gx, gy)$ for $g \in G$, $(x, y) \in X \times Y$. The projection $p: X \times Y \to X$ is equivariant, and G acts as a group of bundle maps if we consider p as a trivial fibre bundle with structure group G. We call p a *trivial* G-fibre bundle.

Example 2. A G-vector bundle is a G-fiber bundle with structure group a general linear group. The results in this paper are given by Segal [7] for G-vector bundles over compact spaces, and by Wasserman [8] for differentiable G-vector bundles over G-manifolds.

Proposition 2.1. *Let $p_i: E_i \to X_i$, $i = 1, 2$, be G-fibre bundles with the same structure group and fibre. If $f: X_1 \to X_2$ is equivariant, then the induced bundle f^*E_2 over X_1 is naturally a G-fibre bundle, and the induced map $f^*E_2 \to E_2$ is a G-fibre bundle map. If $F: E_1 \to E_2$ is a G-fibre bundle map over f, then E_1 is G-equivalent to f^*E_2, and F is the composition of a G-equivalence $E_1 \to f^*E_2$ and the induced map $f^*E_2 \to E_2$.*

The proof is clear.

Now let $P \to X$ be a differentiable principal bundle with structure group a Lie group H. Let Y be an effective H-manifold, and $E = P \times_H Y \to X$ the bundle with fibre Y associated to P. In other words E is obtained from the product $P \times Y$ by identifying (p, y) with $(ph, h^{-1}y)$ for all $p \in P$, $y \in Y$, $h \in H$, and the projection $E \to X$ is induced by the projection $P \to X$. Since H acts effectively on Y, there is a one-one correspondence between actions of G as a group of bundle maps of $E \to X$ and actions as a group of bundle maps of $P \to X$; we just take, for the operation of each element of G as a bundle map $E \to X$, the associated bundle map from $P \to X$ to itself, and vice-versa. If G acts differentiably as a group of bundle maps of $P \to X$, then the induced action on $E \to X$ is differentiable. Conversely, we have

Theorem 2.2. *If G acts differentiably as a group of bundle maps of $E \to X$, then the induced action on the associated principal bundle $P \to X$ is differentiable.*

Proof. Let $g_0 \in G$, $x_0 \in X$. Choose neighborhoods U of x_0 in X, V of g_0x_0 in X, and W of g_0 in G such that the bundle $E \to X$ is trivial over U and V, and $W \cdot U \subseteq V$. With respect to trivializations $U \times Y$, $V \times Y$ of $E = P \times_H Y$ over U, V, the action of elements of G contained in W is given by a C^∞ map $W \times U \times Y \to V \times Y$, taking $(g, u, y) \in W \times U \times Y$ into $(gu, \alpha(u, g)y)$, where $\alpha(u, g) \in H$. We must show that the map $\alpha: U \times W \to H$ is C^∞.

Since H acts effectively on Y, there is a finite subset $\{y_1, \cdots, y_n\}$ of Y such that the Lie subgroup $\{h \in H \mid hy_i = y_i, i = 1, \cdots, n\}$ of H is zero-dimensional (see Gleason and Palais [2, Th. 8. 2, p. 646]). Let H act on $Z = Y \times \cdots \times Y$ (n copies) by the given action on each factor, and let $z = (y_1, \cdots, y_n) \in Z$. Then the isotropy subgroup H_z is zero-dimensional. Now the map $U \times W \to Z$ taking $(u, g) \in U \times W$ into $\alpha(u, g)z \in Z$ is C^∞, and the image of this map lies in the orbit Hz, which is diffeomorphic to H/H_z. In other words, the map $U \times W \to H/H_z$ taking (u, g) to $\alpha(u, g)H_z$ is C^∞. Since H_z is zero-dimensional, then $\alpha: U \times W \to H$ is C^∞.

3. The equivariant covering homotopy property for differentiable G-fibre bundles reducible to a compact Lie structure group

Using Theorem 2.2 we can prove the following theorem and corollary in the same way they are proved by Wasserman [8, Th. 2.4, Cor. 2.5, p. 134] in the case where the structure group is an orthogonal group. I denotes the interval $[0, 1]$, and G always acts trivially on I.

Theorem 3.1. *Let G be a compact Lie group, and $E \to X \times I$ a differentiable G-fibre bundle with structure group a compact Lie group H. Then there is a differentiable G-fibre bundle equivalence $E \to (E \mid X \times 0) \times I$.*

Corollary 3.2. *If $E \to X$ is a differentiable G-fibre bundle with compact Lie structure group, and $f_0, f_1: Y \to X$ are G-homotopic (resp. differentiably G-homotopic) G-maps from a differentiable G-manifold Y to X, then the induced bundles f_0^*E and f_1^*E are G-equivalent (resp. differentiably G-equivalent).*

Using Corollary 3.2, we easily deduce the following *equivariant covering homotopy property* for differentiable G-fibre bundles with compact Lie structure group.

Corollary 3.3. *Let $E_i \to X_i$, $i = 1, 2$, be differentiable G-fibre bundles having the same fibre and structure group, a compact Lie group. Let $F_0: E_1 \to E_2$ be a G-fibre bundle map over $f_0: X_1 \to X_2$, and $f: X_1 \times I \to X_2$ be a G-homotopy of f_0. Then there is a G-homotopy of F_0, which is a G-fibre bundle map $F: E_1 \times I \to E_2$ over f. Moreover, if F_0 is differentiable and f is a differentiable homotopy, then there is a differentiable covering homotopy F.*

The above results clearly hold as well for any differentiable G-fibre bundle whose structure group can be reduced to a compact Lie group so that G still acts as a group of bundle maps on the reduced bundle. The following theorem then shows that a differentiable G-fibre bundle whose structure group is a connected semi-simple Lie group has the equivariant covering homotopy property.

Theorem 3.4. *Let G be a compact Lie group, and $E \to X$ a differentiable G-fibre bundle with structure group a connected semi-simple Lie group H. Then the structure group of $E \to X$ can be reduced to a compact subgroup of H so that G still acts as a group of bundle maps on the reduced bundle.*

Remark. In the case that H is a general linear group, such a reduction to the orthogonal group can be given by a G-invariant Riemannian metric on the associated vector bundle.

Proof of Theorem 3.4. Let $\pi: P \to X$ be the principal bundle associated with $E \to X$, and K a maximal compact subgroup of H (assume H is not compact). It suffices to find a G-equivariant section of the bundle $P/K = P \times_H (H/K) \to X$.

The homogeneous space H/K with any H-invariant Riemannian metric is a complete simply-connected Riemannian manifold of negative curvature, so that for each $h \in H$ the exponential map at $hK \in H/K$ is a diffeomorphism from the tangent space $T(H/K)_{hK}$ onto H/K (Helgason [3, Chap. I, Th. 13.3]). Now $P \times_H T(H/K)$ is a G-vector bundle over P/K, called the tangent bundle along the fibres, and the exponential map for $T(H/K)$ induces a G-equivariant map $P \times_H T(H/K) \to P/K$, taking $[ph, h^{-1}v]$ (where $p \in P$, $h \in H$, $v \in T(H/K)$) into $[ph, \exp(h^{-1}v)] = [ph, h^{-1} \exp v]$.

For each $x \in X$ the isotropy subgroup G_x acts on the fibre H/K of P/K over x via a homomorphism $G_x \to H$. Since all maximal compact subgroups of H are conjugate (Helgason [3, Chap. VI, Th. 2.2]), the image of this homomorphism is contained in hKh^{-1} for some $h \in H$, so that hK is a fixed point for the action of G_x on H/K. Since π induces a submersion of P/K onto X, there is a G_x-equivariant section σ of P/K defined on some slice V_x for X at x, with $\sigma(x) = hK$. We have then a pull-back diagram:

$$\begin{array}{ccc} \sigma^*(P \times_H T(H/K)) & \longrightarrow & P \times_H T(H/K) \\ \downarrow & & \downarrow \\ V_x & \longrightarrow & P/K \end{array}$$

so that $\sigma^*(P \times_H T(H/K))$ is a G_x-vector bundle over V_x, and the exponential map $\sigma^*(P \times_H T(H/K)) \to (P/K)\,|\,V_x$ is a G_x-equivariant fibre-preserving diffeomorphism.

We now construct a C^∞ equivariant section of $P/K \to X$. For each $x \in X$, shrink V_x equivariantly to U_x, $\mathrm{Cl}\,(U_x) \subset V_x$($\mathrm{Cl}$ = closure), and choose a countable number of points $x(1), x(2), \cdots$ such that the slice neighborhoods $G \cdot U_{x(i)}$ of the orbits $Gx(i)$ cover X. Set $A_0 = \emptyset$, and define A_n inductively by $A_n = G \cdot \mathrm{Cl}\,(U_{x(n)}) \cup A_{n-1}$. Suppose C^∞ equivariant sections s_i of $P/K \to X$ are defined on A_i for $i < n$, such that $s_i\,|\,A_{i-1} = s_{i-1}$. Since there is a $G_{x(n)}$-equivariant fibre-preserving diffeomorphism from $(P/K)\,|\,V_{x(n)}$ to a $G_{x(n)}$-vector bundle over $V_{x(n)}$, s_{n-1} extends to a C^∞ equivariant section s_n over A_n. Define s by $s(x) = s_n(x)$ for $x \in A_n - A_{n-1}$. Since X is the union of the interiors of the A_n, we see s is a C^∞ equivariant section $X \to P/K$.

4. *G*-local triviality and the equivariant covering homotopy property

Let $p: E \to X$ be a differentiable G-fibre bundle. We say p is *G-locally trivial* if for each $x \in X$ there is a G_x-invariant neighborhood U_x of x such that $p \mid U_x$ is differentiably G_x-equivalent to the trivial G_x-fibre bundle $U_x \times p^{-1}(x)$.

By the equivariant covering homotopy property (Corollary 3.3), a differentiable G-fibre bundle with structure group a compact or semi-simple Lie group is G-locally trivial. On the other hand, the following theorem implies that if $p: E \rightarrow X$ is a differentiable G-fibre bundle which is G-locally trivial, then p has the equivariant covering homotopy property.

Theorem 4.1. *Let G be a compact Lie group, and $p: E \rightarrow X \times I$ a differentiable G-fibre bundle which is G-locally trivial (G acts trivially on I). Then there is a differentiable G-fibre bundle equivalence $E \rightarrow (E \mid X \times 0) \times I$ (the map is understood to be induced by a principal bundle map in the case that p is a G-fibre bundle with Lie structure group H).*

Proof. The proof is similar to that of the equivariant covering homotopy property for locally trivial fibre spaces given, for example, in Husemoller [4, pp. 49–51]. We choose a locally finite countable invariant covering $G \cdot U_i$ of X such that U_i is a slice at $x(i)$, $i = 1, 2, \cdots$, and there is a $G_{x(i)}$-equivalence $h_i: U_i \times I \times Y_i \rightarrow E \mid (U_i \times I)$, where $Y_i = p^{-1}(x(i))$ (when p has structure group H, $G_{x(i)}$ acts on the H-manifold Y_i by a homomorphism $G_{x(i)} \rightarrow H$).

There is a G-invariant C^∞ map $u_i: X \rightarrow [0, 1]$ such that $u_i^{-1}(0, 1] \subseteq G \cdot U_i$ and $\max_i u_i(x) = 1$ for all $x \in X$. Define G-fibre bundle equivalences

$$k_i: G \times_{G_{x(i)}} (U_i \times I \times Y_i) \rightarrow E \mid (G \cdot U_i \times I)$$

by $k_i[g, (u, t, y)] = gh_i(u, t, y)$ for $g \in G$, $u \in U_i$, $t \in I$, $y \in Y_i$, and define G-fibre bundle maps

$$\begin{array}{ccc} E & \xrightarrow{F_i} & E \\ \downarrow & & \downarrow \\ X \times I & \xrightarrow{f_i} & X \times I \end{array}$$

as follows:

$$\begin{gathered} f_i(x, t) = (x, t(1 - u_i(x))) , \qquad (x, t) \in X \times I ; \\ F_i = \text{id} \quad \text{outside} \quad p^{-1}(G \cdot U_i \times I) ; \\ F_i \circ k_i[g, (u, t, y)] = k_i[g, (u, t(1 - u_i(u)), y)] , \\ [g, (u, t, y)] \in G \times_{G_{x(i)}} (U_i \times I \times Y_i) . \end{gathered}$$

Then $F = \cdots \circ F_3 \circ F_2 \circ F_1$ is a G-fibre bundle map over $f = \cdots \circ f_3 \circ f_2 \circ f_1$ (F, f are well-defined since all but a finite number of terms in the infinite compositions are equal to the identity near any point).

By Proposition 2.1, E is G-equivalent to f^*E, which is G-equivalent to $(E \mid X \times 0) \times I$ by the definiton of f. This completes the proof.

Using Theorem 4.1, we can now deduce the equivariant covering homotopy property for a differentiable G-fibre bundle with structure group any Lie group:

Theorem 4.2. *Let G be a compact Lie group, and $E \to X$ a differentiable G-fibre bundle with structure group a Lie group H. Then E is G-locally trivial.*

Proof. Let $\pi: P \to X$ be the associated principal bundle, and let $x \in X$. The isotropy subgroup G_x acts on the fibre H of P over x via a homomorphism $\alpha: G_x \to H$. Consider the bundle $P/\alpha(G_x)$ with fibre $H/\alpha(G_x)$ associated with P. The point $1\alpha(G_x)$ in the fibre over x is a fixed point for the action of G_x. Since $P/\alpha(G_x) \to X$ is an equivariant submersion onto X, there is a G_x-equivariant section σ of $P/\alpha(G_x)$ defined on some G_x-invariant neighborhood U_x of x, which is G_x-contractible to x.

Hence $E|U_x$ can be reduced to the compact subgroup $\alpha(G_x)$ of H so that G_x still acts as a group of bundle maps. The result now follows from the equivariant covering homotopy property for G-fibre bundles with compact Lie structure group (Corollary 3.3).

We conclude with some remarks relating G-local triviality to the equivalence of nearby differentiable actions of a compact Lie group.

If $p: E \to X$ is any differentiable G-fibre bundle with *compact* fibre, then we can obtain the equivariant covering homotopy property for p by proving an analogue of Theorem 3.1. Hence p is also G-locally trivial.

Definitions. Let G be a compact Lie group, and X, Y two differentiable manifolds. A *differentiable family of actions of G on Y parametrized by X* is a differentiable map $\Phi: X \times G \times Y \to Y$ such that for each $x \in X$ the map $\Phi_x: G \times Y \to Y$ taking $(g, y) \in G \times Y$ into $\Phi(x, g, y)$ is a differentiable action of G on Y. This family is said to be *locally trivial at $x_0 \in X$* if there are an open neighborhood U of x_0 in X and a differentiable map $\Psi: U \times Y \to Y$ such that:

1. for each $x \in U$ the map $\Psi_x: Y \to Y$ taking y into $\Psi(x, y)$ is a diffeomorphism of Y, and $\Psi_{x_0} = \mathrm{id}_Y$;
2. $\Phi(x, g, \Psi(x, y)) = \Psi(x, \Phi(x_0, g, y))$ for each $x \in U$, $g \in G$, and $y \in Y$.

A family of differentiable actions of G on Y is said to be *locally trivial* if it is locally trivial at each point x of the parameter space X.

Now if $p: E \to X$ is a product bundle $X \times Y \to X$ with compact fibre Y, and G acts on E as a group of bundle maps with the induced action on X trivial, then the G-local triviality of p is just a restatement of the fact that a differentiable family of actions of a compact Lie group on a compact manifold Y is locally trivial (Palais [5], Calabi [1]).

The conjecture of Calabi [1, p. 213] that such a family is locally trivial even when Y is not compact had already been shown to be false by Palais and Stewart [5], [6]. This shows that a differentiable G-fibre bundle $p: E \to X$ with noncompact fibre does not in general have the equivariant covering homotopy propery. (We note here also the observation of Palais and Stewart that a continuous family of actions of a compact Lie group on a compact space Y is not in general locally trivial. Hence, though Theorem 4.1, for example, is valid in the continuous case (when X is paracompact), we cannot hope for an equivariant covering homotopy property for a broad class of continuous G-fibre bundles.)

From the G-local triviality of G-fibre bundles with Lie structure group (Theorem 4.2), we deduce, however, the following result.

Theorem 4.3. *Let H be a Lie group, and Y an H-manifold. Let $\Phi: X \times G \times Y \to Y$ be a differentiable family of actions of a compact Lie group G on Y such that for each $x \in X$, there is a homomorphism $\varphi_x: G \to H$, and $\Phi(x, g, y) = \varphi_x(g)y$ for all $g \in G$, $y \in Y$. Then Φ is locally trivial.*

As a final remark we note a type of action of a compact Lie group G on a differentiable fibre bundle with Lie structure group which is more general than an action as a group of bundle maps and for which the equivariant covering homotopy property still holds. Let $\pi: P \to X$ be a differentiable principal bundle with Lie structure group H, on which G acts as a group of bundle maps, and let Y be an effective H-manifold. If G acts (on the left) on Y commuting with the the action of H, there is an induced action on the total space $E = P \times_H Y$ of the associated bundle with fibre Y given by $g[p, y] = [gp, gy]$, with respect to which the projection $E \to X$ is equivariant. The equivariant covering homotopy property for π clearly implies that for $E \to X$. This also gives a generalization of Theorem 4.3: with the same notation as in Theorem 4.3, if G acts on Y commuting with the action of H, and Φ is given by $\Phi(x, g, y) = \varphi_x(g)gy$, then Φ is locally trivial.

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