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AN AFFINE CONNECTION IN AN ALMOST QUATERNION MANIFOLD

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0. Introduction

The theory of differential concomitants was initiated by Schouten [4] and developed by Frölicher [1] and Nijenhuis [1], [2]. It is now well known that for two given tensor fields F and G of type $(1, 1)$ in a differentiable manifold the expression

$$\begin{aligned}[F, G](X, Y) = {} & [FX, GY] - F[X, GY] - G[FX, Y] + [GX, FY] \\ & - G[X, FY] - F[GX, Y] + (FG + GF)[X, Y]\end{aligned} \tag{0.1}$$

defines a tensor field $[F, G]$ of type $(1, 2)$, and that the tensor field $[F, F]$ plays a very important role in the discussion of integrability conditions of an almost complex structure defined by F. We call $[F, G]$ defined by (0.1) the Nijenhuis tensor of F and G.

The present authors [9] proved that if the tensor fields F and G of type $(1, 1)$ satisfy $FG = GF$, then the expression

$$[FX, GY] - F[X, GY] - G[FX, Y] + FG[X, Y] \tag{0.2}$$

defines a tensor field of type $(1, 2)$. If F and G satisfy $FG + GF = 0$, then the expression (0.1) takes the form

$$\begin{aligned}[F, G](X, Y) = {} & [FX, GY] - F[X, GY] - G[FX, Y] \\ & + [GX, FY] - G[X, FY] - F[GX, Y] .\end{aligned} \tag{0.3}$$

All these tensor fields of type $(1, 2)$ contain F, G and partial derivatives of F, G of the first order.

On the other hand, Walker [6] found a tensor field of type $(1, 4)$ formed with an almost complex structure F, which contains F and partial derivatives of F of the first and the second orders (see also Willmore [7]). Ślebodziński [5] announced that he obtained a tensor field of type $(1, 3)$ formed with an almost complex structure F containing F and partial derivatives of F of the

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first and the second orders, but Willmore [8] proved that the tensor field found by Ślebodziński is identically zero.

In previous papers [10], [11], we have studied almost quaternion structures of the first and the second kinds, that is, a set of three tensor fields F, G and H of type $(1, 1)$ satisfying

$$
\begin{aligned}
&F^2 = -1\,, \quad G^2 = -1\,, \quad H^2 = -1\,, \\
&F = GH = -HG\,, \quad G = HF = -FH\,, \quad H = FG = -GF\,,
\end{aligned} \tag{0.4}
$$

and

$$
\begin{aligned}
&F^2 = -1\,, \quad G^2 = -1\,, \quad H^2 = 1\,, \\
&F = -GH = HG\,, \quad G = HF = -FH\,, \quad H = FG = -GF
\end{aligned} \tag{0.5}
$$

respectively, and proved the following Theorems A and B.

Theorem A. *In a differentiable manifold with an almost quaternion structure (F, G, H) of the first or the second kind, if two of six Nijenhuis tensors*

$$
[F, F], [G, G], [H, H], [G, H], [H, F], [F, G] \tag{0.6}
$$

vanish, then the others also vanish.

Theoerm B. *For the existence of a symmetric affine connection ∇ in a differentiable manifold with an almost quaternion structure (F, G, H) of the first or the second kind such that*

$$
\nabla F = 0\,, \quad \nabla G = 0\,, \quad \nabla H = 0\,,
$$

it is necessary and sufficient that two of Nijenhuis tensors (0.6) *vanish.*

If there exists a coordinate system in which the components of F, G and H are all constant, the almost quaternion structure is said to be *integrable*. Using a result of Obata [3], the present authors also proved

Theorem C. *A necessary and sufficient condition that an almost quaternion structure (F, G, H) of the first or the second kind be integrable is that two of Nijenhuis tensors* (0.6) *vanish and $R = 0$, where R is the curvature tensor of the affine connection ∇ in Theorem* B.

The main purpose of the present paper is first to show the existence of an affine connection in an almost quaternion manifold whose torsion tensor is closely related to the Nijenhuis tensor of F and G, then to use this affine connection to improve the above Theorems A, B, C, and finally to derive tensor fields of type $(1, 3)$ formed with almost quaternion structure tensors.

1. An affine connection in an almost quaternion manifold

We consider a differentiable manifold with an almost quaternion structure (F, G, H) of the first kind and prove

Theorem 1.1. *In a differentiable manifold with an almost quaternion structure (F, G, H) of the first kind, there exists an affine connection ∇ such that $\nabla_X Y$ is given by*

$$-\tfrac{1}{2}H\{(\mathscr{L}_{GY}F - G\mathscr{L}_Y F)X\} ,$$

where X and Y are arbitrary vector fields, and $\mathscr{L}_Y$ denotes the operator of Lie differentiation with respect to Y.

Proof. We put

$$f(X, Y) = -\tfrac{1}{2}H\{(\mathscr{L}_{GY}F - G\mathscr{L}_Y F)X\} , \tag{1.1}$$

and prove

(i) $$f(X_1 + X_2, Y) = f(X_1, Y) + f(X_2, Y) ,$$

(ii) $$f(X, Y_1 + Y_2) = f(X, Y_1) + f(X, Y_2) ,$$

(iii) $$f(\alpha X, Y) = \alpha f(X, Y) ,$$

(iv) $$f(X, \alpha Y) = \alpha f(X, Y) + (X\alpha)Y ,$$

for arbitrary vector fields X, X_1, X_2, Y, Y_1, Y_2 and function α.

Since (i), (ii) and (iii) are obvious from the definition (1.1) of $f(X, Y)$, we prove (iv). From

$$(\mathscr{L}_{GY}F)X = \mathscr{L}_{GY}(FX) - F\mathscr{L}_{GY}X = [GY, FX] - F[GY, X]$$

and

$$(\mathscr{L}_Y F)X = \mathscr{L}_Y(FX) - F\mathscr{L}_Y X = [Y, FX] - F[Y, X] ,$$

it follows that

$$(\mathscr{L}_{GY}F - G\mathscr{L}_Y F)X = [GY, FX] - F[GY, X] - G[Y, FX] - H[Y, X] ,$$

and consequently that

$$f(X, Y) = \tfrac{1}{2}H([FX, GY] - F[X, GY] - G[FX, Y]) + \tfrac{1}{2}[X, Y] . \tag{1.2}$$

Using αY for Y in (1.2) we obtain

$$\begin{aligned} f(X, \alpha Y) = {} & \tfrac{1}{2}\alpha H([FX, GY] - F[X, GY] - G[FX, Y]) + \tfrac{1}{2}\alpha[X, Y] \\ & + \tfrac{1}{2}H\{((FX)\alpha)GY - (X\alpha)HY - ((FX)\alpha)GY\} + \tfrac{1}{2}(X\alpha)Y \end{aligned}$$

and hence (iv).

Thus $f(X, Y)$ defines an affine connection ∇ such that $\nabla_X Y = f(X, Y)$, that is,

$$(1.3)\qquad \nabla_X Y = -\tfrac{1}{2}H\{(\mathscr{L}_{GY}F - G\mathscr{L}_Y F)X\}\ ,$$

and the theorem is proved.

Substituting $\nabla_X Y = f(X, Y)$ in (1.2) gives readily

$$(1.4)\qquad \nabla_X Y = \tfrac{1}{2}H([FX, GY] - F[X, GY] - G[FX, Y]) + \tfrac{1}{2}[X, Y]\ ,$$

which can be written as, in consequence of (0.3),

$$(1.5)\qquad \begin{aligned} \nabla_X Y &= \tfrac{1}{2}(H \barwedge [F, G])(X, Y) \\ &\quad - \tfrac{1}{2}H([GX, FY] - G[X, FY] - F[GX, Y]) + \tfrac{1}{2}[X, Y]\ , \end{aligned}$$

where

$$(H \barwedge [F, G])(X, Y) = H\{[F, G](X, Y)\}\ .$$

We next prove

Theorem 1.2. *The torsion tensor*

$$(1.6)\qquad T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

of the affine connection ∇ in Theorem 1.1 *is given by*

$$(1.7)\qquad T(X, Y) = \tfrac{1}{2}(H \barwedge [F, G])(X, Y)\ .$$

Proof. Theorem 1.2 follows immediately from (1.4), (1.6) and (0.3).

As an obvious consequence of Theorem 1.2, we have

Theorem 1.3. *The affine connection ∇ in Theorem* 1.1 *is symmetric if and only if*

$$[F, G] = 0\ .$$

2. Covariant derivatives of F and G

In this section, we first prove

Theorem 2.1. *The covariant derivative of F with respect to ∇ is given by*

$$(2.1)\qquad (\nabla_X F)Y = \tfrac{1}{2}(H \barwedge [F, G])(X, FY) + \tfrac{1}{2}(G \barwedge [F, G])(X, Y)$$

for arbitrary vector fields X and Y.

Proof. We have

$$(2.2)\qquad (\nabla_X F)Y = \nabla_X(FY) - F\nabla_X Y\ .$$

Replacing Y in (1.5) by FY, and noticing $F^2 = -1, H = -GF = FG$, $G^2 = -1$, we can easily obtain

$$\begin{aligned}\nabla_X(FY) = {} & \tfrac{1}{2}(H \barwedge [F,G])(X,FY) \\ & - \tfrac{1}{2}G(F[GX,Y] - H[X,Y] - [GX,FY] + G[X,FY]) \,.\end{aligned}$$

On the other hand, from (1.4), $FH = -G$ and $F = GH$ it follows that

$$F\nabla_X Y = -\tfrac{1}{2}G([FX,GY] - F[X,GY] - G[FX,Y] - H[X,Y]) \,.$$

Substitution of the above two equations in (2.2) and use of (0.3) thus give (2.1).

Using this connection ∇, we define $\tilde{\nabla}$ by

$$\tilde{\nabla}_X Y = \nabla_X Y + \tfrac{1}{2}(\nabla_X F)FY \,,$$

then we have $\tilde{\nabla}F = \tilde{\nabla}G = \tilde{\nabla}H = 0$ (cf. Obata [3]). The torsion tensor of $\tilde{\nabla}$ is given by

$$\tfrac{1}{4}G \barwedge [F,G] \barwedge F \,.$$

We next prove

Theorem 2.2. *The covariant derivative of G with respect to ∇ vanishes, that is,*

$$(\nabla_X G)Y = 0 \tag{2.3}$$

for arbitrary vector fields X and Y.

Proof. We have

$$(\nabla_X G)Y = \nabla_X(GY) - G\nabla_X Y \,. \tag{2.4}$$

Replacing Y in (1.4) by GY, and noticing $G^2 = -1$, $H^2 = -1$, $H = FG = -GF$, $GH = F$ we obtain

$$\nabla_X(GY) = -\tfrac{1}{2}F(G[FX,Y] + H[X,Y] - [FX,GY] + F[X,GY]) \,.$$

On the other hand, from (1.5), $GH = F$ and $FH = -G$ it follows that

$$\begin{aligned}G\nabla_X Y = {} & \tfrac{1}{2}(F \barwedge [F,G])(X,Y) \\ & - \tfrac{1}{2}F([GX,FY] - G[X,FY] - F[GX,Y] + H[X,Y]) \,.\end{aligned}$$

Substitution of the above two equations in (2.4) and use of (0.3) thus give (2.3).

Combining Theorems 2.1 and 2.2 we have

Theorem 2.3. *If $[F,G] = 0$, then the affine connection ∇ is symmetric and satisfies $\nabla F = 0$, $\nabla G = 0$, $\nabla H = 0$.*

We next prove

Theoerm 2.4. *If the Nijenhuis tensor* $[F, G]$ *vanishes, then the other Nijenhuis tensors in* (0.6) *also vanish.*

This theorem follows immediately from Theorem 2.3 and the following lemma.

Lemma. *If there exists a symmetric affine connection* ∇ *such that*

$$\nabla P = 0 , \qquad \nabla Q = 0 , \tag{2.5}$$

P and Q being tensor fields of type $(1, 1)$, *then the Nijenhuis tensor* $[P, Q]$ *formed with P and Q vanishes.*

Proof. Using the relations

$$[X, Y] = \nabla_X Y - \nabla_Y X , \qquad \nabla_{PX}(QY) = Q\nabla_{PX} Y + (\nabla_{PX} Q)Y$$

and (2.5) in (0.3) for $[P, Q](X, Y)$, we can easily prove the lemma by a straightforward computation.

3. The curvature tensor of ∇

We denote the curvature tensor of the connection ∇ defined in § 1 by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z , \tag{3.1}$$

X, Y, Z being arbitrary vector fields. Combination of Theorems C, 2.3 and 2.4 gives

Theorem 3.1. *A necessary and sufficient condition that an almost quaternion structure* (F, G, H) *be integrable is that* $[F, G] = 0$, $R = 0$.

If the curvature tensor R of the affine connection ∇ does not vanish identically, it gives a tensor field of type $(1, 3)$ containing F, G and their first and second partial derivatives. In a previous paper [9] we gave concomitants of various types formed with F and G satisfying $FG = GF$. The curvature tensor R above is of type $(1, 3)$ and formed with F and G satisfying $FG + GF = 0$. The problem of finding a tensor field of type $(1, 3)$ formed only with an almost complex structure F has not yet been solved.

Now for the curvature tensor R we have the Bianchi identity of the first kind:

$$\begin{aligned} &R(X, Y)Z + R(Y, Z)X + R(Z, X)Y \\ &\quad = (\nabla_X T)(Y, Z) + (\nabla_Y T)(Z, X) + (\nabla_Z T)(X, Y) \\ &\qquad + (T \barwedge T)(X, Y, Z) , \end{aligned} \tag{3.2}$$

where

$$(T \barwedge T)(X, Y, Z) = T(T(X, Y), Z) + T(T(Y, Z), X) + T(T(Z, X), Y) ,$$

for arbitrary vector fields X, Y and Z. Since ∇ and T are formed with F and G only, if the right hand side of (3.2) does not vanish, it gives a vector-valued 3-form constructed with F and G only.

Remark. Let (F, G, H) be an almost quaternion structure of the first kind, then $(G, iF, -iH)$ or $(G, -iF, iH)$ can be regarded as an almost quaternion structure of the second kind. But all the quantities discussed above are real when we replace (F, G, H) by $(G, iF, -iH)$ or $(G, -iF, iH)$, and consequently the results obtained above are also valid for an almost quaternion structure of the second kind.

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