J. DIFFERFNTIAL GEOMETRY
8 (1972) 331-334

AN INTEGRAL FORMULA

HARLEY FLANDERS

The following results generalize in several directions a recent formula of Richard Kraft [3, Lemma 1] used in a problem of geometrical optics.

Theorem 1. *Let M be a compact n-manifold imbedded in a Euclidean $(n+1)$-space $\boldsymbol{E}^{n+1}$, where $n > 1$. For each $\boldsymbol{x} \in M$, let $\boldsymbol{e} = \boldsymbol{e}(\boldsymbol{x})$ be the outward unit normal, $r = |\boldsymbol{x}|$, and $p = p(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{e}$, the support function. Also let σ denote the element of n-volume. Then*

$$\frac{1}{V_n} \int_M \frac{p\boldsymbol{x}}{r^{n+2}} \sigma = \begin{cases} \boldsymbol{0} & \text{if } \boldsymbol{0} \notin M , \\ -\boldsymbol{e}(\boldsymbol{0}) & \text{if } \boldsymbol{0} \in M , \end{cases}$$

where $V_n = \pi^{n/2}/\Gamma(\frac{1}{2}n + 1)$ is the volume of the unit n-ball.

Our proof will be based on two formal lemmas. We shall denote by $[\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n]$ the cross (vector) product of n vectors in $\boldsymbol{E}^{n+1}$, assumed oriented. As usual, we extend this alternating multilinear function to vectors with differential form coefficients by

$$[\alpha_1 \boldsymbol{v}_1, \cdots, \alpha_n \boldsymbol{v}_n] = (\alpha_1 \wedge \cdots \wedge \alpha_n)[\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n] .$$

We refer to Flanders [1, pp. 43, 149] and [2] for this formalism.

Lemma 1. *On M we have*

$$n(\boldsymbol{x} \cdot d\boldsymbol{x}) \wedge [\boldsymbol{x}, d\boldsymbol{x}, \cdots, d\boldsymbol{x}] = r^2[d\boldsymbol{x}, \cdots, d\boldsymbol{x}] - n!\, p\boldsymbol{x}\sigma .$$

Proof. We shall give more detail than is really necessary, because the probability of an error in sign is high in calculations of this type.

Let $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n$ be a moving orthonormal frame on M, so

$$\boldsymbol{x} = p_i \boldsymbol{e}_i + p\boldsymbol{e} , \qquad \mathrm{d}\boldsymbol{x} = \sigma_i \boldsymbol{e}_i ,$$

where the σ_i are one-forms, and repeated indices are summed. Note that $\sigma_1 \wedge \cdots \wedge \sigma_n = \sigma$ is the volume element on M. We take the $\boldsymbol{e}_i$ so that $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n, \boldsymbol{e}$ is a right-handed frame for $\boldsymbol{E}^{n+1}$. Then $[\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n] = \boldsymbol{e}$. We also note for future reference that

$$[\boldsymbol{e}, \boldsymbol{e}_1, \cdots, \hat{\boldsymbol{e}}_i, \cdots, \boldsymbol{e}_n] = (-1)^i \boldsymbol{e}_i ,$$

Received May 3, 1972.

because it requires $(n-i)+n$ transpositions to pass from $\boldsymbol{e}, \boldsymbol{e}_1, \cdots, \hat{\boldsymbol{e}}_i, \cdots, \boldsymbol{e}_n, \boldsymbol{e}_i$ to $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n, \boldsymbol{e}$. (The circumflex denotes omission.)

We have

$$(\boldsymbol{x}\cdot d\boldsymbol{x}) \wedge [\boldsymbol{x}, d\boldsymbol{x}, \cdots, d\boldsymbol{x}]$$
$$= (p_i\sigma_i) \wedge \{[p_i\boldsymbol{e}_i, \sigma_j\boldsymbol{e}_j, \cdots, \sigma_k\boldsymbol{e}_k] + p[\boldsymbol{e}, \sigma_j\boldsymbol{e}_j, \cdots, \sigma_k\boldsymbol{e}_k]\} .$$

Now

$$[p_i\boldsymbol{e}_i, \sigma_j\boldsymbol{e}_j, \cdots, \sigma_k\boldsymbol{e}_k]$$
$$= p_i(\sigma_j \wedge \cdots \wedge \sigma_k)[\boldsymbol{e}_i, \boldsymbol{e}_j, \cdots, \boldsymbol{e}_k]$$
$$= (n-1)! \sum_i p_i(\sigma_1 \wedge \cdots \wedge \hat{\sigma}_i \wedge \cdots \wedge \sigma_n)[\boldsymbol{e}_i, \boldsymbol{e}_1, \cdots \hat{\boldsymbol{e}}_i, \cdots, \boldsymbol{e}_n]$$
$$= (n-1)! \left(\sum_i (-1)^{i-1} p_i\sigma_1 \wedge \cdots \wedge \hat{\sigma}_i \wedge \cdots \wedge \sigma_n\right)\boldsymbol{e} ,$$

hence

$$(p_i\sigma_i) \wedge [p_i\boldsymbol{e}_i, \sigma_j\boldsymbol{e}_j, \cdots, \sigma_k\boldsymbol{e}_k] = (n-1)!(\sum p_i^2)\sigma\boldsymbol{e} .$$

Next,

$$[\boldsymbol{e}, \sigma_j\boldsymbol{e}_j, \cdots, \sigma_k\boldsymbol{e}_k]$$
$$= (\sigma_j \wedge \cdots \wedge \sigma_k)[\boldsymbol{e}, \boldsymbol{e}_j, \cdots, \boldsymbol{e}_k]$$
$$= (n-1)! \sum_i (\sigma_1 \wedge \cdots \wedge \hat{\sigma}_i \wedge \cdots \wedge \sigma_n)[\boldsymbol{e}, \boldsymbol{e}_1, \cdots, \hat{\boldsymbol{e}}_i, \cdots, \boldsymbol{e}_n]$$
$$= (n-1)! \sum_i (-1)^i(\sigma_1 \wedge \cdots \wedge \hat{\sigma}_i \wedge \cdots \wedge \sigma_n)\boldsymbol{e}_i ,$$

hence

$$(p_i\sigma_i) \wedge [\boldsymbol{e}, \sigma_j\boldsymbol{e}_j, \cdots, \sigma_k\boldsymbol{e}_k] = -(n-1)!\, \sigma(p_i\boldsymbol{e}_i) .$$

Consequently

$$(\boldsymbol{x}\cdot d\boldsymbol{x}) \wedge [\boldsymbol{x}, d\boldsymbol{x}, \cdots, d\boldsymbol{x}] = (n-1)!\, [(\sum p_i^2)\boldsymbol{e} - p(p_i\boldsymbol{e}_i)]\sigma$$
$$= (n-1)!\, (r^2\boldsymbol{e} - p\boldsymbol{x})\sigma .$$

Since $[d\boldsymbol{x}, \cdots, d\boldsymbol{x}] = n!\, \sigma\boldsymbol{e}$, the lemma follows.

Lemma 2. *On M we have*

$$n!\, r^{-(n+2)} p\boldsymbol{x}\sigma = d(r^{-n}[\boldsymbol{x}, d\boldsymbol{x}, \cdots, d\boldsymbol{x}]) .$$

Proof. Applying d and using Lemma 1 we obtain

$$\begin{aligned} & d(r^{-n}[x, dx, \cdots, dx]) \\ & \quad = -nr^{-(n+2)}(x \cdot dx) \wedge [x, dx, \cdots, dx] + r^{-n}[dx, \cdots, dx] \\ & \quad = -r^{-n}[dx, \cdots, dx] + n!\, r^{-(n+2)} px\sigma + r^{-n}[dx, \cdots, dx] \\ & \quad = n!\, r^{-(n+2)} px\sigma \;. \end{aligned}$$

Proof of Theorem 1. If $\mathbf{0} \notin M$, then the integrand is exact on M by Lemma 2, so the integral is zero. If $\mathbf{0} \in M$, the integrand is singular at $\mathbf{0}$. We choose ε so small that $\{r = \varepsilon\} \cap M$ is an $(n-1)$-sphere and set $M_\varepsilon = M \backslash \{r < \varepsilon\}$. By the lemma and two applications for Stokes's theorem,

$$\begin{aligned} \int_{M_\varepsilon} \frac{px}{r^{n+2}} \sigma &= \frac{1}{n!} \int_{M_\varepsilon} d\left(\frac{1}{r^n}[x, dx, \cdots, dx]\right) \\ &= \frac{1}{n!} \int_{\partial M_\varepsilon} \frac{1}{r^n}[x, dx, \cdots, dx] = -\frac{1}{n!} \int_{r=\varepsilon} \frac{1}{r^n}[x, dx, \cdots, dx] \\ &= \frac{-1}{n!\, \varepsilon^n} \int_{r=\varepsilon} [x, dx, \cdots, dx] = \frac{-1}{n!\, \varepsilon^n} \int_{r \le \varepsilon} [dx, dx, \cdots, dx] \\ &= \frac{-1}{n!\, \varepsilon^n} \int_{r \le \varepsilon} n!\, e\sigma \approx \frac{-1}{\varepsilon^n} e(\mathbf{0}) \int_{r \le \varepsilon} \sigma \approx \frac{-1}{\varepsilon^n} (\varepsilon^n V_n) e(\mathbf{0}) \to -V_n e(\mathbf{0}) \;. \end{aligned}$$

It is clear that the convergence is absolute as $\varepsilon \to 0$ so that $M_\varepsilon \to M$. Therefore any other family M'_ε converging to M would yield the same value for the singular integral.

Corollary. *If* $x_0 \in M$, *then*

$$\frac{1}{V_n} \int_M \frac{(x - x_0) \cdot e}{|x - x_0|^{n+2}} x d\sigma = -e(x_0) \;.$$

These results can be extended without difficulty to immersed rather than imbedded orientable hypersurfaces. The case of a closed curve is special.

Theorem 2. *Let* C *be a simple closed smooth counter-clockwise curve in* E^2 *with Frenet frame* t, n *at* x. *Set* $p = -x \cdot n$, $r = |x|$, *and* J *the* $90°$ *rotation. Then*

$$\frac{1}{2} \int_C \frac{p}{r^3} J(x) ds = \begin{cases} \mathbf{0} & \text{if } \mathbf{0} \notin C \;, \\ -t(\mathbf{0}) & \text{if } \mathbf{0} \in C \;. \end{cases}$$

Proof. Write $x = at - pn$. Then $dx = tds$, $r^2 = a^2 + p^2$, $rdr = x \cdot dx = ads$, and

$$\begin{aligned} d(r^{-1}x) &= -ar^{-3}dsx + r^{-1}dst = r^{-3}[-a(at - pn) + (a^2 + p^2)t]ds \\ &= pr^{-3}(pt + an)ds = pr^{-3}J(x)ds \;. \end{aligned}$$

The theorem follows easily.

References

[1] H. Flanders, *Differential forms with applications to the physical sciences,* Academic Press, New York, 1963.

[2] ——, *The Steiner point of a closed hypersurface,* Mathematika **13** (1966) 181–186.

[3] R. Kraft, *Uniqueness and existence for the integral equation of interreflections,* SIAM J. Math. Anal., to appear.

TEL AVIV UUIVERSITY