J. DIFFERENTIAL GEOMETRY
8 (1973) 305-316

ISOTROPIC IMMERSIONS

TAKEHIRO ITOH & KOICHI OGIUE

1. Introduction

A Kaehler manifold of constant holomorphic sectional curvature is called a *complex space form*. A *Kaehler immersion* is an isometric immersion which is complex analytic. The second named author proved the following results.

Proposition 1 [2]. *Let M be an n-dimensional complex space form of constant holomorphic sectional curvature c, and $\tilde{M}$ be an $(n + p)$-dimensional complex space form of constant holomorphic sectional curvature $\tilde{c}$. If M is a Kaehler submanifold of $\tilde{M}$ with parallel second fundamental form, then either $c = \tilde{c}$ (i.e., M is totally geodesic in $\tilde{M}$) or $c = \frac{1}{2}\tilde{c}$, the latter case arising only when $\tilde{c} > 0$. Moreover, the immersion is rigid.*

Proposition 2 [3]. *Let M be an n-dimensional complex space form of constant holomorphic sectional curvature c, and $\tilde{M}$ be an $(n + \frac{1}{2}n(n + 1))$-dimensional complex space form of constant holomorphic sectional curvature $\tilde{c}$. If M is a Kaehler submanifold of $\tilde{M}$, then either $c = \tilde{c}$ (i.e., M is totally geodesic in $\tilde{M}$) or $c = \frac{1}{2}\tilde{c}$, the latter case arising only when $\tilde{c} > 0$. Moreover, the immersion is rigid.*

In the present paper, we shall prove similar results for real manifolds. An *isotropic immersion* is an isometric immersion such that all its normal curvature vectors have the same length at each point. A Riemannian manifold of constant curvature is called a *space form*.

Theorem 1. *Let M be an n-dimensional space form of constant curvature c, and $\tilde{M}$ be an $(n + \frac{1}{2}n(n + 1) - 1)$-dimensional space form of constant curvature $\tilde{c}$. If $c < \tilde{c}$, and M is an isotropic submanifold of $\tilde{M}$ with parallel second fundamental form, then $c = \dfrac{n}{2(n+1)}\tilde{c}$, and the immersion is rigid.*

Theorem 2. *Let M be an n-dimensional space form of constant curvature c, and $\tilde{M}$ be an $(n + \frac{1}{2}n(n + 1) - 1)$-dimensional space form of constant curvature $\tilde{c}$. If $c < \tilde{c}$, and M is an isotropic submanifold of $\tilde{M}$, then $c = \dfrac{n}{2(n+1)}\tilde{c}$, and the immersion is rigid provided that $n \leq 4$.*

Remark. Theorems 1 and 2 give a (local) characterization of a *Veronese manifold*.

Received February 14, 1972. Partially supported by the Sakko-kai Foundation and the Matsunaga Science Foundation.

2. Preliminaries

Let M be an n-dimensional Riemannian manifold immersed isometrically in an $(n+p)$-dimensional space form $\tilde{M}$ of constant curvature $\tilde{c}$. We denote by ∇ (resp. $\tilde{\nabla}$) the covariant differentiation on M (resp. $\tilde{M}$). Then the second fundamental form σ of the immersion is given by

$$\sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y ,$$

and satisfies $\sigma(X, Y) = \sigma(X, Y)$.

We choose a local field of orthonormal frames $e_1, \cdots, e_n, e_{\bar{1}}, \cdots, e_{\bar{p}}$ in $\tilde{M}$ in such a way that, restricted to M, $e_1, \cdots, e_n$ are tangent to M. With respect to the frame field of $\tilde{M}$ chosen above, let $\omega^1, \cdots, \omega^n, \omega^{\bar{1}}, \cdots, \omega^{\bar{p}}$ be the field of dual frames. Then the structure equations of $\tilde{M}$ are given by[1]

$$d\omega^A = -\Sigma\omega^A_B \wedge \omega^B , \qquad \omega^A_B + \omega^B_A = 0 , \tag{2.1}$$

$$d\omega^A_B = -\Sigma\omega^A_C \wedge \omega^C_B + \tilde{c}\omega^A \wedge \omega^B . \tag{2.2}$$

Restricting these forms to M, we have the structure equations of the immersion:

$$\omega^\alpha = 0 , \tag{2.3}$$

$$\omega^\alpha_i = \Sigma h^\alpha_{ij}\omega^j , \qquad h^\alpha_{ij} = h^\alpha_{ji} , \tag{2.4}$$

$$d\omega^i = -\Sigma\omega^i_j \wedge \omega^j , \qquad \omega^i_j + \omega^j_i = 0 , \tag{2.5}$$

$$d\omega^i_j = -\Sigma\omega^i_k \wedge \omega^k_j + \Omega^i_j , \qquad \Omega^i_j = \tfrac{1}{2}\Sigma R^i_{jkl}\omega^k \wedge \omega^l , \tag{2.6}$$

$$R^i_{jkl} = \tilde{c}(\delta^i_k\delta_{jl} - \delta^i_l\delta_{jk}) + \Sigma(h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}) . \tag{2.7}$$

The second fundamental form σ can be written as

$$\sigma(e_i, e_j) = \Sigma h^\alpha_{ij}e_\alpha \quad \text{or} \quad \sigma = \Sigma h^\alpha_{ij}\omega^i\omega^j e_\alpha . \tag{2.8}$$

Define h^α_{ijk} by

$$\Sigma h^\alpha_{ijk}\omega^k = dh^\alpha_{ij} - \Sigma h^\alpha_{ik}\omega^k_j - \Sigma h^\alpha_{kj}\omega^k_i + \Sigma h^\beta_{ij}\omega^\alpha_\beta . \tag{2.9}$$

Then from (2.2), (2.3) and (2.4) we have

$$h^\alpha_{ijk} = h^\alpha_{ikj} . \tag{2.10}$$

[1] We use the following convention on the ranges of indices unless otherwise stated:

$$A, B, C = 1, \cdots, n, \bar{1}, \cdots, \bar{p}; \qquad i, j, k, l = 1, \cdots, n;$$

$$a, b, c = 1, \cdots, n-1; \qquad \alpha, \beta = \bar{1}, \cdots, \bar{p} .$$

The second fundamental form σ is said to be *parallel* if $h^\alpha_{ijk} = 0$ for all α, i, j and k. It is known that if the immersion is minimal, then the second fundamental form σ satisfies a differential equation. In fact, we have

Lemma 2.1 [1].

$$\tfrac{1}{2}\Delta(\Sigma h^\alpha_{ij}h^\alpha_{ij}) = \Sigma h^\alpha_{ijk}h^\alpha_{ijk} - 2\Sigma(h^\alpha_{ij}h^\alpha_{jk}h^\beta_{kl}h^\beta_{li} - h^\alpha_{ij}h^\beta_{jk}h^\alpha_{kl}h^\beta_{li}) - \Sigma h^\alpha_{ij}h^\beta_{ij}h^\alpha_{kl}h^\beta_{kl} + n\tilde{c}\Sigma h^\alpha_{ij}h^\alpha_{ij} ,$$

where Δ denotes the Laplacian.

3. Isotropically immersed space forms

For a unit vector X, $\sigma(X, X)$ is called the *normal curvature vector* determined by X. An isometric immersion is said to be *isotropic* if every normal curvature vector has the same length at each point. B. O'Neill [4] proved the following

Lemma 3.1. *Let M be an n-dimensional space form of constant curvature c, and $\tilde{M}$ be an $(n + \frac{1}{2}n(n + 1) - 1)$-dimensional space form of constant curvature $\tilde{c}$. If $c < \tilde{c}$, and M is an isotropic submanifold of $\tilde{M}$, then*

(i) *M is a minimal submanifold of $\tilde{M}$,*

(ii) $\|\sigma(X, X)\|^2 = \dfrac{2(n-1)}{n+2}(\tilde{c} - c)$ *for every unit vector X,*

(iii) $\|\sigma(X, Y)\|^2 = \dfrac{n}{n+2}(\tilde{c} - c)$ *for every orthonormal pair X and Y,*

(iv) *the $\frac{1}{2}n(n-1)$ vectors $\sigma(e_i, e_j)$, $i < j$, are orthogonal,*

(v) *the angle between $\sigma(e_i, e_i)$ and $\sigma(e_j, e_j)$ is the same (say θ) for every pair i and j ($i \neq j$) and* $\cos\theta = -1/(n-1)$,

(vi) *$\{\sigma(e_i, e_i)\}_{1\le i\le n}$ is orthogonal to $\{\sigma(e_i, e_j)\}_{1\le i<j\le n}$,*

(vii) *the dimension of the vector space generated by $\{\sigma(e_i, e_i)\}_{1\le i\le n}$ and $\{\sigma(e_i, e_j)\}_{1\le i<j\le n}$ is $\frac{1}{2}n(n+1) - 1$.*

Let M be an n-dimensional space form of constant curvature c, and $\tilde{M}$ be an $(n + \frac{1}{2}n(n + 1) - 1)$-dimensional space form of constant curvature $\tilde{c}$. We assume that $c < \tilde{c}$, and that M is an isotropic submanifold of $\tilde{M}$.

Let $e_1, \cdots, e_n$ be a local field of orthonormal frames in M. From Lemma 3.1 we can see that the $\frac{1}{2}n(n + 1) - 1$ vectors $\sigma(e_a, e_a)$ and $\sigma(e_i, e_j)$, $1 \le a \le n - 1$, $1 \le i < j \le n$, form a basis of the normal space at each point of M. Using the Gram-Schmidt process, we can obtain an orthonormal basis of the normal space at each point of M. In fact, we have the following

Lemma 3.2. *The $\frac{1}{2}n(n + 1) - 1$ vectors*

$$\frac{\sqrt{n+2}}{\sqrt{2n(n-a)(n-a+1)(\tilde{c}-c)}}\left\{\sum_{b=1}^{a-1}\sigma(e_b, e_b) + (n-a+1)\sigma(e_a, e_a)\right\}$$

and $\dfrac{\sqrt{n+2}}{\sqrt{n(\tilde{c}-c)}}\sigma(e_i, e_j)$ *for* $1 \le a \le n-1$ *and* $1 \le i < j \le n$ *form an* *orthonormal system.*

We choose a local field of orthonormal frames $e_1, \cdots, e_n, e_{\bar{1}}, \cdots, e_{\bar{p}}$ ($p = \frac{1}{2}n(n+1) - 1$) in $\tilde{M}$ in such a way that, restricted to M, $e_1, \cdots, e_n$ are tangent to M, and

$$e_{\bar{a}} = \frac{\sqrt{n+2}}{\sqrt{2n(n-a)(n-a+1)(\tilde{c}-c)}}\left\{\sum_{b=1}^{a-1}\sigma(e_b, e_b) + (n-a+1)\sigma(e_a, e_a)\right\},$$

$$e_{\widetilde{(i,j)}} = \frac{\sqrt{n+2}}{\sqrt{n(\tilde{c}-c)}}\sigma(e_i, e_j),$$

where $(i, j) = \min\{i, j\} + \frac{1}{2}|i-j|(2n+1-|i-j|) - 1$. With respect to this frame field, using Lemma 3.1 we can obtain

$$\text{(3.1)} \qquad (h^{\bar{a}}_{ij}) = \begin{bmatrix} 0 & & & & 0 \\ & \ddots & \vdots & & \\ & & 0 & & \\ & \cdots & \lambda_a & \cdots & \\ & & & \mu_a & \\ & & \vdots & & \ddots \\ 0 & & & & \mu_a \end{bmatrix} a, \qquad (h^{\widetilde{(i,j)}}_{ij}) = \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & 0 & \cdots & \star & \cdots \\ & \vdots & & \vdots & \\ \cdots & \star & \cdots & 0 & \cdots \\ & \vdots & & \vdots & \end{bmatrix} \begin{matrix} i \\ \\ j \end{matrix},$$

where

$$\lambda_a = \frac{\sqrt{2n(n-a)(\tilde{c}-c)}}{\sqrt{(n+2)(n-a+1)}}, \qquad \mu_a = -\frac{\sqrt{2n(\tilde{c}-c)}}{\sqrt{(n+2)(n-a)(n-a+1)}},$$

$$\star = \sqrt{n(\tilde{c}-c)}/\sqrt{n+2}$$

Thus from (3.1), Lemma 2.1 and Lemma 3.1 we have

Lemma 3.3. *Let M be an n-dimensional space form of constant curvature c, and $\tilde{M}$ be an $(n + \frac{1}{2}n(n+1) - 1)$-dimensional space form of constant curvature $\tilde{c}$. If $c < \tilde{c}$, and M is an isotropic submanifold of $\tilde{M}$, then*

$$\Sigma h^{\alpha}_{ijk}h^{\alpha}_{ijk} = \frac{2n^2(n^2-1)}{n+2}(\tilde{c}-c)\left\{\frac{n}{2(n+1)}\tilde{c} - c\right\}.$$

We need as well the following

Lemma 3.4. *Let M be an n-dimensional space form of constant curvature c, and $\tilde{M}$ be an $(n + \frac{1}{2}n(n+1) - 1)$-dimensional space form of constant curvature $\tilde{c}$. Suppose $c < \tilde{c}$, and M is an isotropic submanifold of $\tilde{M}$. Then the second fundamental form σ is parallel if and only if the following hold:*

$$\omega_{\tilde{b}}^{\tilde{a}} = 0 \, , \quad \omega_{(\widetilde{a,b})}^{\tilde{a}} = \frac{\sqrt{2(n-a)}}{\sqrt{n-a+1}}\omega_b^a \quad (b < a) \, , \tag{3.2}$$

$$\omega_{(\widetilde{a,j})}^{\tilde{a}} = \frac{\sqrt{2(n-a+1)}}{\sqrt{n-a}}\omega_j^a \qquad (a < j) \, , \tag{3.3}$$

$$\omega_{(\widetilde{j,k})}^{\tilde{a}} = 0 \qquad (j, k > a \quad or \quad < a) \, , \tag{3.4}$$

$$\omega_{(\widetilde{j,k})}^{\tilde{a}} = \frac{\sqrt{2}}{\sqrt{(n-a)(u-a+1)}}\omega_k^j \qquad (j < a < k) \, , \tag{3.5}$$

$$\omega_{(\widetilde{i,k})}^{(\widetilde{i,j})} = \omega_k^j \, , \tag{3.6}$$

$$\omega_{(\widetilde{k,l})}^{(\widetilde{i,j})} = 0 \, , \tag{3.7}$$

where different indices indicate different numbers.

Proof. From (2.9) and (3.1) it follows that the second fundamental form σ is parallel if and only if (3.2), $\cdots$, (3.7) and the following equations hold:

$$\frac{\sqrt{n-b}}{\sqrt{n-b+1}}\omega_{\tilde{b}}^{\tilde{a}} = \sum_{c<b} \frac{1}{\sqrt{(n-c)(n-c+1)}}\omega_{\tilde{c}}^{\tilde{a}} \, , \tag{3.8}$$

$$\sum_{c<a} \frac{1}{\sqrt{(n-c)(n-c+1)}}\omega_{\tilde{c}}^{\tilde{a}} = 0 \, , \tag{3.9}$$

$$\frac{\sqrt{n-a}}{\sqrt{n-a+1}}\omega_{(\widetilde{a,j})}^{\tilde{a}} + \sum_{k<a} \frac{1}{\sqrt{(n-k)(n-k+1)}}\omega_{\tilde{k}}^{(\widetilde{a,j})} = \sqrt{2}\,\omega_j^a \, , \tag{3.10}$$

$$\frac{\sqrt{n-b}}{\sqrt{n-b+1}}\omega_{\tilde{b}}^{(\widetilde{i,j})} = \sum_{c<b} \frac{1}{\sqrt{(n-c)(n-c+1)}}\omega_{\tilde{c}}^{(\widetilde{i,j})} \, , \tag{3.11}$$

where different indices indicate different numbers. We can see inductively that (3.8) and (3.9) are equivalent to $\omega_{\tilde{b}}^{\tilde{a}} = 0$. Moreover, (3.2), $\cdots$, (3.5) imply (3.10) and (3.11). q.e.d.

From (2.2), (2.3), (2.4) and (3.1) we have

$$\begin{aligned} &\left(\sum_{b<a} \frac{\sqrt{2}}{\sqrt{(n-b)(n-b+1)}}\omega_{\tilde{b}}^{\tilde{a}}\right) \wedge \omega^a \\ &+ \sum_{b<a} \left(\frac{\sqrt{2(n-a)}}{\sqrt{n-a+1}}\omega_b^a - \omega_{(\widetilde{a,b})}^{\tilde{a}}\right) \wedge \omega^b \\ &+ \sum_{a<b} \left(\frac{\sqrt{2(n-a+1)}}{\sqrt{n-a}}\omega_b^a - \omega_{(\widetilde{a,b})}^{\tilde{a}}\right) \wedge \omega^b = 0 \, , \end{aligned} \tag{3.12}$$

$$
\begin{aligned}
&\left(\frac{\sqrt{2(n-a)}}{\sqrt{n-a+1}}\omega_b^a - \omega^{\tilde a}_{(\widetilde{a,b})}\right) \wedge \omega^a \\
&\quad - \left(\frac{\sqrt{2(n-b)}}{\sqrt{n-b+1}}\omega^{\tilde a}_{\tilde b} - \sum_{c<b} \frac{\sqrt{2}}{\sqrt{(n-c)(n-c+1)}}\omega^{\tilde a}_{\tilde c}\right) \wedge \omega^b \\
&\quad - \sum_{c<a} \omega^{\tilde a}_{(\widetilde{b,c})} \wedge \omega^c + \sum_{a<k}\left(\frac{\sqrt{2}}{\sqrt{(n-a)(n-a+1)}}\omega^b_k - \omega^{\tilde a}_{(\widetilde{b,k})}\right) \wedge \omega^k \\
&= 0 \qquad (b < a)\ ,
\end{aligned} \tag{3.13}
$$

$$
\begin{aligned}
&\left(\frac{\sqrt{2(n-a+1)}}{\sqrt{n-a}}\omega_a^j - \omega^{\tilde a}_{(\widetilde{a,j})}\right) \wedge \omega^a \\
&\quad + \left(\frac{\sqrt{2(n-j)}}{\sqrt{n-j+1}}\omega^{\tilde a}_{\tilde j} - \sum_{k<j} \frac{\sqrt{2}}{\sqrt{(n-k)(n-k+1)}}\omega^{\tilde a}_{\tilde k}\right) \wedge \omega^j \\
&\quad + \sum_{c<a}\left(\frac{\sqrt{2}}{\sqrt{(n-a)(n-a+1)}}\omega^j_c + \omega^{\tilde a}_{(\widetilde{j,c})}\right) \wedge \omega^c + \sum_{a<k} \omega^{\tilde a}_{(\widetilde{j,c})} \wedge \omega^c \\
&= 0 \qquad (a < j)\ ,
\end{aligned} \tag{3.14}
$$

$$
\begin{aligned}
&\left(\frac{\sqrt{2(n-i)}}{\sqrt{n-i+1}}\omega_{\tilde i}^{(\widetilde{i,j})} - \sum_{k<i} \frac{\sqrt{2}}{\sqrt{(n-k)(n-k+1)}}\omega_{\tilde k}^{(\widetilde{i,j})} - 2\omega^j_i\right) \wedge \omega^i \\
&\quad + \sum \left(\omega^{(\widetilde{i,j})}_{(\widetilde{i,k})} - \omega^j_k\right) \wedge \omega^k = 0\ ,
\end{aligned} \tag{3.15}
$$

$$
\begin{aligned}
&\left(\omega^{(\widetilde{i,j})}_{(\widetilde{i,k})} - \omega^j_k\right) \wedge \omega^i + \left(\omega^{(\widetilde{i,j})}_{(\widetilde{k,j})} - \omega^i_k\right) \wedge \omega^j \\
&\quad + \left(\frac{\sqrt{2(n-k)}}{\sqrt{n-k+1}}\omega_{\tilde k}^{(\widetilde{i,j})} - \sum_{l<k} \frac{\sqrt{2}}{\sqrt{(n-l)(n-l+1)}}\omega_{\tilde l}^{(\widetilde{i,j})}\right) \wedge \omega^k \\
&\quad + \sum_{l \neq i,j,k} \omega^{(\widetilde{i,j})}_{(\widetilde{k,l})} \wedge \omega^l = 0\ .
\end{aligned} \tag{3.16}
$$

By (3.15) and Cartan's lemma we may write $\omega^{(\widetilde{i,j})}_{(\widetilde{i,k})} - \omega^j_k = \Sigma A^{ij}_{kl}\omega^l$, where $A^{ij}_{kl} = A^{ij}_{lk}$. Since $\omega^{(\widetilde{i,j})}_{(\widetilde{i,k})} - \omega^j_k + \omega^{(\widetilde{i,k})}_{(\widetilde{i,j})} - \omega^k_j = 0$ so that $A^{ij}_{kl} + A^{ik}_{jl} = 0$, we can see $A^{ij}_{kl} = 0$. Hence we have

$$
\omega^{(\widetilde{i,j})}_{(\widetilde{i,k})} = \omega^j_k\ . \tag{3.17}
$$

From (3.16) and (3.17) it follows that $\omega^{(\widetilde{i,j})}_{(\widetilde{k,l})}$ contain neither ω^i and ω^j nor ω^k and ω^l by symmetry. Therefore

(3.18) $\qquad \omega^{(\widetilde{i,j})}_{(\widetilde{k,l})}$ *do not contain* ω^i, ω^j, ω^k *and* ω^l .

4. Proofs of theorems

Theorem 1 follows immediately from Lemmas 3.3 and 3.4. We shall give here a proof of Theorem 2 for $n = 4$. The proof of Theorem 2 for $n = 2$ and $n = 3$ is quite similar to and easier than that for $n = 4$.

In consideration of (3.17) and (3.18), equations (3.12), $\cdots$, (3.16) can be written as follows:

$$\begin{aligned}
&\left(\frac{2\sqrt{2}}{\sqrt{3}}\omega^1_2 - \omega^{\tilde 1}_{\tilde 4}\right) < \omega^2 + \left(\frac{2\sqrt{2}}{\sqrt{3}}\omega^1_3 - \omega^{\tilde 1}_{\tilde 7}\right) \wedge \omega^3 \\
&\qquad + \left(\frac{2\sqrt{2}}{\sqrt{3}}\omega^1_4 - \omega^{\tilde 1}_{\tilde 9}\right) \wedge \omega^4 = 0 , \\
&\left(\frac{2}{\sqrt{3}}\omega^2_1 - \omega^{\tilde 2}_{\tilde 4}\right) \wedge \omega^1 + \frac{1}{\sqrt{6}}\omega^{\tilde 2}_{\tilde 1} \wedge \omega^2 + (\sqrt{3}\,\omega^2_3 - \omega^{\tilde 2}_{\tilde 5}) \wedge \omega^3 \\
&\qquad + (\sqrt{3}\,\omega^2_4 - \omega^{\tilde 2}_{\tilde 8}) \wedge \omega^4 = 0 , \\
&(\omega^3_1 - \omega^{\tilde 3}_{\tilde 7}) \wedge \omega^1 + (\omega^3_2 - \omega^{\tilde 3}_{\tilde 5}) \wedge \omega^2 + \left(\frac{1}{\sqrt{6}}\omega^{\tilde 3}_{\tilde 1} + \frac{1}{\sqrt{3}}\omega^{\tilde 3}_{\tilde 2}\right) \wedge \omega^3 \\
&\qquad + (2\omega^3_4 - \omega^{\tilde 3}_{\tilde 6}) \wedge \omega^4 = 0;
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
&-\frac{\sqrt{3}}{\sqrt{2}}\omega^{\tilde 2}_{\tilde 1} \wedge \omega^1 + \left(\frac{\sqrt{2}}{\sqrt{3}}\omega^2_1 - \omega^{\tilde 2}_{\tilde 4}\right) \wedge \omega^2 + \left(\frac{1}{\sqrt{3}}\omega^1_3 - \omega^{\tilde 2}_{\tilde 7}\right) \wedge \omega^3 \\
&\qquad + \left(\frac{1}{\sqrt{3}}\omega^1_4 - \omega^{\tilde 2}_{\tilde 9}\right) \wedge \omega^4 = 0 , \\
&-\frac{\sqrt{3}}{\sqrt{2}}\omega^3_1 \wedge \omega^1 - \omega^{\tilde 3}_{\tilde 4} \wedge \omega^2 + (\omega^3_1 - \omega^{\tilde 3}_{\tilde 7}) \wedge \omega^3 \\
&\qquad + (\omega^1_4 - \omega^{\tilde 3}_{\tilde 9}) \wedge \omega^4 = 0 , \\
&-\omega^{\tilde 3}_{\tilde 4} \wedge \omega^1 - \left(\frac{\sqrt{2}}{\sqrt{3}}\omega^{\tilde 3}_{\tilde 2} - \frac{1}{\sqrt{6}}\omega^{\tilde 3}_{\tilde 1}\right) \wedge \omega^2 + (\omega^3_2 - \omega^{\tilde 3}_{\tilde 5}) \wedge \omega^3 \\
&\qquad + (\omega^2_4 - \omega^{\tilde 3}_{\tilde 8}) \wedge \omega^4 = 0;
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
&\left(\frac{2\sqrt{2}}{\sqrt{3}}\omega^2_1 + \omega^{\tilde 1}_{\tilde 4}\right) \wedge \omega^1 + \frac{2}{\sqrt{3}}\omega^{\tilde 1}_{\tilde 2} \wedge \omega^2 + \omega^{\tilde 1}_{\tilde 5} \wedge \omega^3 + \omega^{\tilde 1}_{\tilde 8} \wedge \omega^4 = 0 , \\
&\left(\frac{2\sqrt{2}}{\sqrt{3}}\omega^3_1 + \omega^{\tilde 1}_{\tilde 7}\right) \wedge \omega^1 + \omega^{\tilde 1}_{\tilde 5} \wedge \omega^2 + \left(\omega^{\tilde 1}_{\tilde 3} - \frac{1}{\sqrt{3}}\omega^{\tilde 1}_{\tilde 2}\right) \wedge \omega^3 \\
&\qquad + \omega^{\tilde 1}_{\tilde 6} \wedge \omega^4 = 0 , \\
&\left(\frac{2\sqrt{2}}{\sqrt{3}}\omega^4_1 + \omega^{\tilde 1}_{\tilde 9}\right) \wedge \omega^1 + \omega^{\tilde 1}_{\tilde 8} \wedge \omega^2 + \omega^{\tilde 1}_{\tilde 6} \wedge \omega^3 \\
&\qquad - \left(\frac{1}{\sqrt{3}}\omega^{\tilde 1}_{\tilde 2} + \omega^{\tilde 1}_{\tilde 3}\right) \wedge \omega^4 = 0 ,
\end{aligned}$$

$$
\begin{aligned}
&\left(\frac{1}{\sqrt{3}}\omega_1^3 + \omega_{\tilde{7}}^{\tilde{2}}\right) \wedge \omega^1 + (\sqrt{3}\,\omega_2^3 + \omega_{\tilde{5}}^{\tilde{2}}) \wedge \omega^2 + \left(\omega_{\tilde{3}}^{\tilde{2}} - \frac{1}{\sqrt{6}}\omega_{\tilde{1}}^{\tilde{2}}\right) \wedge \omega^3 \\
&\qquad + \omega_{\tilde{6}}^{\tilde{2}} \wedge \omega^4 = 0\,, \\
&\left(\frac{1}{\sqrt{3}}\omega_1^4 + \omega_{\tilde{9}}^{\tilde{2}}\right) \wedge \omega^1 + (\sqrt{3}\,\omega_2^4 + \omega_{\tilde{8}}^{\tilde{2}}) \wedge \omega^2 + \omega_{\tilde{6}}^{\tilde{2}} \wedge \omega^3 \\
&\qquad - \left(\frac{1}{\sqrt{6}}\omega_{\tilde{1}}^{\tilde{2}} + \omega_{\tilde{3}}^{\tilde{2}}\right) \wedge \omega^4 = 0\,, \\
&(\omega_1^4 + \omega_{\tilde{9}}^{\tilde{3}}) \wedge \omega^1 + (\omega_2^4 + \omega_{\tilde{8}}^{\tilde{3}}) \wedge \omega^2 + (2\omega_3^4 + \omega_{\tilde{6}}^{\tilde{3}}) \wedge \omega^3 \\
&\qquad - \left(\frac{1}{\sqrt{6}}\omega_{\tilde{1}}^{\tilde{3}} + \frac{1}{\sqrt{3}}\omega_{\tilde{2}}^{\tilde{3}}\right) \wedge \omega^4 = 0;
\end{aligned} \tag{4.3}
$$

$$
\begin{aligned}
&\left(\frac{\sqrt{3}}{\sqrt{2}}\omega_{\tilde{1}}^{\tilde{4}} - 2\omega_1^2\right) \wedge \omega^1 = 0\,, \\
&\left(\frac{\sqrt{3}}{\sqrt{2}}\omega_{\tilde{1}}^{\tilde{7}} - 2\omega_1^3\right) \wedge \omega^1 = 0\,, \\
&\left(\frac{\sqrt{3}}{\sqrt{2}}\omega_{\tilde{1}}^{\tilde{9}} - 2\omega_1^4\right) \wedge \omega^1 = 0\,, \\
&\left(\frac{2}{\sqrt{3}}\omega_{\tilde{2}}^{\tilde{4}} - \frac{1}{\sqrt{6}}\omega_{\tilde{1}}^{\tilde{4}} - 2\omega_2^1\right) \wedge \omega^2 = 0\,, \\
&\left(\frac{2}{\sqrt{3}}\omega_{\tilde{2}}^{\tilde{5}} - \frac{1}{\sqrt{6}}\omega_{\tilde{1}}^{\tilde{5}} - 2\omega_2^3\right) \wedge \omega^2 = 0\,, \\
&\left(\frac{2}{\sqrt{3}}\omega_{\tilde{2}}^{\tilde{8}} - \frac{1}{\sqrt{6}}\omega_{\tilde{1}}^{\tilde{8}} - 2\omega_2^4\right) \wedge \omega^2 = 0\,, \\
&\left(\omega_{\tilde{3}}^{\tilde{7}} - \frac{1}{\sqrt{6}}\omega_{\tilde{1}}^{\tilde{7}} - \frac{1}{\sqrt{3}}\omega_{\tilde{2}}^{\tilde{7}} - 2\omega_3^1\right) \wedge \omega^3 = 0\,, \\
&\left(\omega_{\tilde{3}}^{\tilde{5}} - \frac{1}{\sqrt{6}}\omega_{\tilde{1}}^{\tilde{5}} - \frac{1}{\sqrt{3}}\omega_{\tilde{2}}^{\tilde{5}} - 2\omega_3^2\right) \wedge \omega^3 = 0\,, \\
&\left(\omega_{\tilde{3}}^{\tilde{6}} - \frac{1}{\sqrt{6}}\omega_{\tilde{1}}^{\tilde{6}} - \frac{1}{\sqrt{3}}\omega_{\tilde{2}}^{\tilde{6}} - 2\omega_3^4\right) \wedge \omega^3 = 0\,, \\
&\left(\frac{1}{\sqrt{6}}\omega_{\tilde{1}}^{\tilde{9}} + \frac{1}{\sqrt{3}}\omega_{\tilde{2}}^{\tilde{9}} + \omega_{\tilde{3}}^{\tilde{9}} + 2\omega_4^1\right) \wedge \omega^4 = 0\,, \\
&\left(\frac{1}{\sqrt{6}}\omega_{\tilde{1}}^{\tilde{8}} + \frac{1}{\sqrt{3}}\omega_{\tilde{2}}^{\tilde{8}} + \omega_{\tilde{3}}^{\tilde{8}} + 2\omega_4^2\right) \wedge \omega^4 = 0\,, \\
&\left(\frac{1}{\sqrt{6}}\omega_{\tilde{1}}^{\tilde{6}} + \frac{1}{\sqrt{3}}\omega_{\tilde{2}}^{\tilde{6}} + \omega_{\tilde{3}}^{\tilde{6}} + 2\omega_4^3\right) \wedge \omega^4 = 0;
\end{aligned} \tag{4.4}
$$

$$\left(\omega_{\tilde{3}}^{\tilde{4}} - \frac{1}{\sqrt{6}}\omega_{\tilde{1}}^{\tilde{4}} - \frac{1}{\sqrt{3}}\omega_{\tilde{2}}^{\tilde{4}}\right) \wedge \omega^3 = 0 ,$$

$$\left(\frac{1}{\sqrt{6}}\omega_{\tilde{1}}^{\tilde{4}} + \frac{1}{\sqrt{3}}\omega_{\tilde{2}}^{\tilde{4}} + \omega_{\tilde{3}}^{\tilde{4}}\right) \wedge \omega^4 = 0 ,$$

$$\left(\frac{2}{\sqrt{3}}\omega_{\tilde{2}}^{\tilde{7}} - \frac{1}{\sqrt{6}}\omega_{\tilde{1}}^{\tilde{7}}\right) \wedge \omega^2 = 0 ,$$

$$\left(\frac{1}{\sqrt{6}}\omega_{\tilde{1}}^{\tilde{7}} + \frac{1}{\sqrt{3}}\omega_{\tilde{2}}^{\tilde{7}} + \omega_{\tilde{3}}^{\tilde{7}}\right) \wedge \omega^4 = 0 ,$$

(4.5) $$\left(\frac{2}{\sqrt{3}}\omega_{\tilde{2}}^{\tilde{9}} - \frac{1}{\sqrt{6}}\omega_{\tilde{1}}^{\tilde{9}}\right) \wedge \omega^2 = 0 ,$$

$$\left(\omega_{\tilde{3}}^{\tilde{9}} - \frac{1}{\sqrt{6}}\omega_{\tilde{1}}^{\tilde{9}} - \frac{1}{\sqrt{3}}\omega_{\tilde{2}}^{\tilde{9}}\right) \wedge \omega^3 = 0 , \quad \omega_{\tilde{1}}^{\tilde{5}} \wedge \omega^1 = 0 ,$$

$$\left(\frac{1}{\sqrt{6}}\omega_{\tilde{1}}^{\tilde{5}} + \frac{1}{\sqrt{3}}\omega_{\tilde{2}}^{\tilde{5}} + \omega_{\tilde{3}}^{\tilde{5}}\right) \wedge \omega^4 = 0 , \quad \omega_{\tilde{1}}^{\tilde{8}} \wedge \omega^1 = 0 ,$$

$$\left(\omega_{\tilde{3}}^{\tilde{8}} - \frac{1}{\sqrt{6}}\omega_{\tilde{1}}^{\tilde{8}} - \frac{1}{\sqrt{3}}\omega_{\tilde{2}}^{\tilde{8}}\right) \wedge \omega^3 = 0 , \quad \omega_{\tilde{1}}^{\tilde{6}} \wedge \omega^1 = 0 ,$$

$$\left(\frac{2}{\sqrt{3}}\omega_{\tilde{2}}^{\tilde{6}} - \frac{1}{\sqrt{6}}\omega_{\tilde{1}}^{\tilde{6}}\right) \wedge \omega^2 = 0 .$$

From $(4.1)_1$, $(4.4)_1, \cdots, (4.4)_3$; $(4.3)_2$, $(4.5)_7$, (4.7); $(4.3)_3$, $(4.5)_{11}$, (4.8); $(4.3)_1$, $(4.5)_9$, (4.6); $(4.3)_1$, (4.6), (4.9), (4.10); $(4.3)_2$, (4.7), (4.9), (4.10); $(4.3)_3$, (4.8), (4.10), (4.11); (4.12), $\cdots$, (4.14); $(4.15)_{12}$, (4.10); $(4.4)_5$, (4.9); $(4.4)_6$, (4.11); $(4.3)_5$, (4.16), (4.18); $(4.4)_9$, $(4.4)_{12}$, (4.10), (4.19); $(4.1)_2$, (4.12), (4.17); $(4.1)_2$, (4.12), (4.18), (4.21); we obtain, respectively,

(4.6) $$\omega_{\tilde{4}}^{\tilde{1}} = \frac{2\sqrt{2}}{\sqrt{3}}\omega_2^1 ,$$

(4.7) $$\omega_{\tilde{7}}^{\tilde{1}} = \frac{2\sqrt{2}}{\sqrt{3}}\omega_3^1 ,$$

(4.8) $$\omega_{\tilde{9}}^{\tilde{1}} = \frac{2\sqrt{2}}{\sqrt{3}}\omega_4^1 ;$$

(4.9) $$\omega_{\tilde{5}}^{\tilde{1}} = 0 ;$$

(4.10) $$\omega_{\tilde{6}}^{\tilde{1}} = 0 ;$$

(4.11) $$\omega_{\tilde{8}}^{\tilde{1}} = 0 ;$$

(4.12) $$\omega_{\tilde{2}}^{\tilde{1}} \in \{\{\omega^2\}\} ,$$

" $\in \{\{\cdots\}\}$" indicating "is a linear combination of $\cdots$";

(4.13) $$\sqrt{3}\,\omega_{\tilde 3}^{\tilde 1} - \omega_{\tilde 2}^{\tilde 1} \in \{\{\omega^3\}\};$$

(4.14) $$\omega_{\tilde 2}^{\tilde 1} + \sqrt{3}\,\omega_{\tilde 3}^{\tilde 1} \in \{\{\omega^4\}\};$$

(4.15) $$\omega_{\tilde 2}^{\tilde 1} = 0\,, \qquad \omega_{\tilde 3}^{\tilde 1} = 0;$$

(4.16) $$\omega_{\tilde 6}^{\tilde 2} \in \{\{\omega^2\}\};$$

(4.17) $$\omega_{\tilde 5}^{\tilde 2} - \sqrt{3}\,\omega_3^2 \in \{\{\omega^2\}\};$$

(4.18) $$\omega_{\tilde 8}^{\tilde 2} - \sqrt{3}\,\omega_4^2 \in \{\{\omega^2\}\};$$

(4.19) $$\omega_{\tilde 6}^{\tilde 2} = 0;$$

(4.20) $$\omega_{\tilde 6}^{\tilde 3} - 2\omega_4^3 = 0;$$

(4.21) $$\omega_{\tilde 5}^{\tilde 2} - \sqrt{3}\,\omega_3^2 = 0;$$

(4.22) $$\omega_{\tilde 8}^{\tilde 2} - \sqrt{3}\,\omega_4^2 = 0;$$

and hence

(4.23) $$\sqrt{3}\,\omega_{\tilde 4}^{\tilde 2} - 2\omega_1^2 \in \{\{\omega^1\}\}\,.$$

From $(4.2)_1$, (4.15), (4.23); $(4.3)_4$, $(4.3)_5$, (4.15), (4.17), $\cdots$, (4.19); $(4.2)_1$, $(4.3)_4$, (4.15), $\cdots$, (4.17), (4.23); $(4.2)_1$, $(4.3)_5$, (4.15), (4.16), (4.18), (4.23); $(4.4)_8$, (4.9), (4.21); $(4.4)_{11}$, (4.11), (4.22); $(4.2)_2$, $(4.2)_3$, (4.15), (4.28), (4.29); we have, respectively,

(4.24) $$\sqrt{3}\,\omega_{\tilde 4}^{\tilde 2} = 2\omega_1^2;$$

(4.25) $$\omega_{\tilde 3}^{\tilde 2} \in \{\{\omega^1\}\};$$

(4.26) $$\sqrt{3}\,\omega_{\tilde 7}^{\tilde 2} - \omega_3^1 \in \{\{\omega^3\}\};$$

(4.27) $$\sqrt{3}\,\omega_{\tilde 9}^{\tilde 2} - \omega_4^1 \in \{\{\omega^4\}\};$$

(4.28) $$\omega_{\tilde 5}^{\tilde 3} - \omega_2^3 \in \{\{\omega^3\}\};$$

(4.29) $$\omega_{\tilde 8}^{\tilde 3} - \omega_4^2 \in \{\{\omega^4\}\};$$

(4.30) $$\omega_{\tilde 4}^{\tilde 3} \in \{\{\omega^2\}\}\,.$$

From $(4.5)_1$ and $(4.5)_2$ it follows that $\omega_{\tilde 4}^{\tilde 3} \in \{\{\omega^3, \omega^4\}\}$ which, together with (4.30), implies

$$\omega_{\tilde{4}}^{\tilde{3}} = 0 . \tag{4.31}$$

From $(4.2)_3$, (4.15), (4.28), (4.29) and (4.31) we obtain $\omega_{\tilde{3}}^{\tilde{2}} \in \{\{\omega^2\}\}$ which, together with (4.25), implies

$$\omega_{\tilde{3}}^{\tilde{2}} = 0 . \tag{4.32}$$

From $(4.1)_3$, $(4.2)_2$, (4.15), (4.20), (4.30) and (4.32) we have

$$\omega_{\tilde{7}}^{\tilde{3}} - \omega_1^3 = 0 , \tag{4.33}$$

and hence

$$\omega_{\tilde{5}}^{\tilde{3}} - \omega_2^3 \in \{\{\omega^2\}\} , \tag{4.34}$$

$$\omega_{\tilde{9}}^{\tilde{3}} - \omega_4^1 \in \{\{\omega^4\}\} . \tag{4.35}$$

From (4.28), (4.34); $(4.3)_4$, (4.15), (4.16), (4.21), (4.32); we obtain, respectively,

$$\omega_{\tilde{5}}^{\tilde{3}} - \omega_2^3 = 0; \tag{4.36}$$

and $\sqrt{3}\,\omega_{\tilde{7}}^{\tilde{2}} - \omega_3^1 \in \{\{\omega^1\}\}$ which, together with (4.26), implies

$$\sqrt{3}\,\omega_{\tilde{7}}^{\tilde{2}} - \omega_3^1 = 0 . \tag{4.37}$$

From $(4.3)_5$, (4.15), (4.16), (4.18) and (4.32) we have $\sqrt{3}\,\omega_{\tilde{9}}^{\tilde{2}} - \omega_4^1 \in \{\{\omega^1\}\}$ which, together with (4.27), implies

$$\sqrt{3}\,\omega_{\tilde{9}}^{\tilde{2}} - \omega_4^1 = 0 . \tag{4.38}$$

From $(4.3)_6$, (4.15), (4.20), (4.29), (4.32) and (4.35) we obtain

$$\omega_{\tilde{9}}^{\tilde{3}} - \omega_4^1 = 0 , \tag{4.39}$$

$$\omega_{\tilde{8}}^{\tilde{3}} - \omega_4^2 = 0 . \tag{4.40}$$

Now it is easy to see that (3.17), (3.18), (4.6), $\cdots$, (4.11), (4.15), (4.19), $\cdots$, (4.22), (4.24), (4.31), $\cdots$, (4.33) and (4.36), $\cdots$, (4.40), together with Lemma 3.4, imply that the second fundamental form is parallel. This, combined with Theorem 1, thus gives Theorem 2.

Bibliography

[1] S. S. Chern, M. P. do Carmo & S. Kobayashi, *Minimal submanifolds of a sphere with second fundamental form of constant length,* Functional Analysis and Related Fields, Springer, Berlin, 1970, 59–75.
[2] K. Ogiue, *On Kaehler immersions,* Canad. J. Math. **24** (1972) 1178–1182.
[3] ——, n-*dimensional complex space forms immersed in* $\{n + \frac{1}{2}n(n+1)\}$*-dimensional complex space forms*, J. Math. Soc. Japan **24** (1972) 518-526.
[4] B. O'Neill, *Isotropic and Kaehler immersions,* Canad. J. Math. **17** (1965) 907–915.

TOKYO UNIVERSITY OF EDUCATION
TOKYO METROPOLITAN UNIVERSITY