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IDEAL DECOMPOSITIONS OF KILLING AND HOLOMORPHIC VECTOR FIELDS

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1. Introduction

Let (M, g) be a compact, connected, oriented Riemannian manifold. We show in § 3 that the Lie algebra of Killing vector fields on M can be decomposed into a direct sum of ideals according to the reducibility of the linear holonomy group of M. A decomposition of this type is already known for the case of a simply connected, complete Riemannian manifold. If in a addition M is assumed to be Kählerian, we show in § 4 that the Lie algebra of holomorphic vector fields on M can also be decomposed in this way. Our proofs make use of part of a theory of Chern in the form described briefly in § 2.

2. Preliminaries

Let $O(M)$ and $\Gamma_{O(M)}$ denote respectively the oriented orthonormal frame bundle over M with structure group $SO(n)$ and the Riemannian connection in $O(M)$. We assume $\Gamma_{O(M)}$ to be reducible to a connection Γ_P in a subbundle P of $O(M)$ with structure group $G \subset SO(n)$ and projection π. The case which will interest us most in the sequel is where P is the holonomy bundle through some point of $O(M)$.

Let $\{e_1, \cdots, e_n\}$ and $\{A_1, \cdots, A_n\}$ be respectively the canonical basis of R^n and a basis for the Lie algebra of G. Let $\theta = \sum_{i=1}^{n} \theta^i e_i$, $\omega = \sum_{\lambda=1}^{m} \omega^\lambda A_\lambda$, and $\Omega = \sum_{\lambda=1}^{m} \Omega^\lambda A_\lambda$ be respectively the canonical form of P, the connection form of Γ_P, and the curvature form of Γ_P. We have the formula $\Omega^\lambda = \frac{1}{2} \sum_{i,j=1}^{n} r^\lambda_{ij} \theta^i \wedge \theta^j$ with $r^\lambda_{ij} = -r^\lambda_{ji}$. Let $(A_\lambda)_{ij} = a^i_{j\lambda}$, $i, j = 1, \cdots, n$. Then there exist functions $s^{\lambda\mu}$ on P such that $r^\lambda_{kl} = \sum_\mu s^{\lambda\mu} a^k_{l\mu}$ with $s^{\lambda\mu} = s^{\mu\lambda}$. Let A^*_λ denote the fundamental vector field on P corresponding to A_λ, $\lambda = 1, \cdots, m$. Let X_i, $i = 1, \cdots, n$, be vector fields on P such that $\theta^i(X_j) = \delta^i_j$, $\omega^\lambda(X_j) = 0$. Define differential operators P^* and S^* on the space of smooth functions on P by $P^* = \sum_k X_k{}^2$ and $S^* = \sum_{\mu,\lambda} s^{\lambda\mu} A^*_\mu A^*_\lambda$. These differential operators commute with right translation by

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elements of G. Thus, if ρ is a representation of G in a vector space V, and f is a V-valued equivariant function on P, then P^*f and S^*f are also equivariant, where P^*f and S^*f are defined componentwise for V-valued functions after selecting a basis for V.

Let j denote the natural representation of G in R^n. Let $\mathfrak{F} =$ {equivariant smooth functions on P of type (j, R^n)}, $\mathfrak{D} =$ {smooth 1-forms on M}, and $\mathfrak{X} =$ {smooth vector fields on M}. Given η in $\mathfrak{D}$, we have $\pi^*\eta = \sum_{i=1}^{n} f_i\theta^i$. To η we associate $f = \sum_{i=1}^{n} f_i e^i$ in $\mathfrak{F}$. This defines a 1 : 1 correspondence between $\mathfrak{D}$ and $\mathfrak{F}$. Moreover, the Riemannian metric gives a 1 : 1 correspondence between $\mathfrak{D}$ and $\mathfrak{X}$. Hence we have also a 1 : 1 correspondence between $\mathfrak{X}$ and $\mathfrak{F}$.

Let Δ and S be respectively the Laplacian on M and the Ricci tensor interpreted as an endomorphism of the cotangent space at each point. By the 1 : 1 correspondence between $\mathfrak{D}$ and $\mathfrak{F}$, we may regard the maps $\Delta, S\colon \mathfrak{D} \to \mathfrak{D}$ as maps $\Delta, S\colon \mathfrak{F} \to \mathfrak{F}$. In terms of the operators P^* and S^*, we then have the expressions $\Delta = -P^* - S^*$ and $S = S^*$.

Suppose c is a linear transformation of R^n which commutes with the action of G. Let η be an element of $\mathfrak{F}$. Define $(c\cdot\eta)(u) = c(\eta(u))$ for each u in P. Then $c\cdot\eta$ is in $\mathfrak{F}$, and we have a map $\mathfrak{F} \to \mathfrak{F}$ also denoted by c. By a theorem of Chern the diagram

$$\begin{array}{ccc} \mathfrak{F} & \xrightarrow{c} & \mathfrak{F} \\ {\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle T} \\ \mathfrak{F} & \xrightarrow[c]{} & \mathfrak{F} \end{array}$$

commutes when T is P^* or S^*.

3. Decomposition of Killing vector fields

Our decomposition theorem for Killing vector fields is:

Theorem 1. *Let M be a compact, connected, oriented Riemannian manifold with oriented orthonormal frame bundle $O(M)$. Let P be any holonomy bundle of $O(M)$ with structure group Φ. Suppose $R^n = V_1 \oplus \cdots \oplus V_r$, where the V_i are mutually orthogonal subspaces of R^n with respect to the usual inner product and each V_i is invariant under Φ. Let $\mathfrak{G}$ be the Lie algebra of Killing vector fields on M. Then $\mathfrak{G} = \mathfrak{G}_1 \oplus \cdots \oplus \mathfrak{G}_r$, where $\mathfrak{G}_i = \{X \mid X \in \mathfrak{G}$ and the equivariant R^n-valued function on P corresponding to X is V_i-valued$\}$. Moreover, each $\mathfrak{G}_i$ is an ideal in $\mathfrak{G}$.*

Proof. We can find an element g in $SO(n)$ such that $g^{-1}V_i = R^{p_i}$, $i = 1, \cdots, r$, where R^{p_i} is the subspace of R^n spanned by $\{e_1, \cdots, e_{p_1}\}$, R^{p_2} is the subspace spanned by $\{e_{p_1+1}, \cdots, e_{p_1+p_2}\}$, etc. Let P' be the holonomy bundle

R_gP with structure group $g^{-1}\Phi g = \Phi'$. Then Φ' leaves each R^{p_i} invariant, and $R^n = R^{p_1} \oplus \cdots \oplus R^{p_r}$. Let $\mathfrak{G}'_i = \{X \mid X \in \mathfrak{G}$ and the equivariant R^n-valued function on P' corresponding to X is R^{p_i}-valued$\}$. It is easy to compute that $\mathfrak{G}_i = \mathfrak{G}'_i$. Thus we may consider V_i to be R^{p_i}, $i = 1, \cdots, r$. Furthermore, we restrict ourselves to the case $r = 2$ for simplicity and write $R^{p_1} = R^p$, $R^{p_2} = R^{n-p}$.

Let X be a Killing vector field on M with corresponding 1-form α. Let f be the R^n-valued function on P corresponding to X. Let ρ_1, ρ_2 be the projection maps $R^n \to R^p$ and $R^n \to R^{n-p}$ respectively. Then $f = f_1 + f_2$, where $f_i = \rho_i f$, $i = 1, 2$. Let X_1, X_2 be the vector fields on M corresponding to f_1, f_2 respectively. Then $X = X_1 + X_2$. If we show X_1 and X_2 are Killing, it follows easily that $\mathfrak{G} = \mathfrak{G}_1 \oplus \mathfrak{G}_2$. Let α_1, α_2 be the 1-forms on M corresponding to X_1, X_2 respectively. Applying the Chern theorem, we see that $\Delta\alpha_i = 2S\alpha_i$, $i = 1, 2$. To show X_1 and X_2 are Killing, it suffices now to prove that $\delta\alpha_i = 0$, $i = 1, 2$. In order to do this, we first make a felicitous choice of moving frame:

Let v be an element of P with $\pi(v) = x$. Write $v = (Y_{1x}, \cdots, Y_{nx})$, where $Y_{ix} \in T_xM$, $i = 1, \cdots, n$. Let $\{x^1, \cdots, x^n\}$ be the normal coordinate system on a neighborhood U of x determined by the frame v. Then $\partial/\partial x^i|_x = Y_{ix}$, $i = 1, \cdots, n$. Translate v parallelly along geodesics emanating from x to obtain a moving orthonormal frame $\sigma = [Y_1, \cdots, Y_n]$ over U. By the definition of P, σ is a section of P over U. It is called the adapted frame on a neighborhood U of x determined by v. Define two distributions V and W on U by $V_y =$ subspace of T_yM spanned by $\{Y_{1y}, \cdots, Y_{py}\}$ and $W_y =$ subspace spanned by $\{Y_{p+1y}, \cdots, Y_{ny}\}$, $y \in U$. We have that $T_yM = V_y \oplus W_y$ and $V_y \perp W_y$ for each y in U.

Lemma 1. *V_y and W_y are invariant under $\Phi(y)$, where $\Phi(y)$ is the linear holonomy group of $\Gamma_{O(M)}$ with reference point y regarded as a linear group acting on T_yM.*

Proof. We recall that the action of $\Phi(y)$ on T_yM may be described as follows: Let $w = (Y_{1y}, \cdots, Y_{ny})$, and suppose τ is an element of $\Phi(y)$. Regarding the linear frames w and $\tau(w)$ as maps from R^n to T_yM in the usual way, we associate to τ the linear transformation $\tau(w) \circ w^{-1}$ of T_yM. We need to show that $\tau(w) \circ w^{-1}$ leaves V_y and W_y invariant. Let $\Phi(w)$ denote the holonomy group of $\Gamma_{O(M)}$ with reference point w. Since $w \in P$, $\Phi(w) = \Phi$. Thus there is an element g in Φ such that $\tau(w)$ is the linear frame $w \circ g \colon R^n \to T_yM$. But g leaves R^p and R^{n-p} invariant. Moreover $w(R^p) = V_y$ and $w(R^{n-p}) = W_y$. Hence the result.

Next we need to establish some properties of the Christoffel symbols Γ^i_{jk} with respect to the frame σ. We do this in the next four lemmas, omitting most of the details.

Lemma 2. *$\Gamma^k_{ij} = 0$ if $k > p$ and $i, j \le p$, and $\Gamma^k_{ij} = 0$ if $k \le p$ and $i, j > p$, everywhere on U.*

Proof. Using Lemma 1 we can show that if X and Z are vector fields on

U belonging to the distribution V (respectively W), then $\nabla_Z X$ and $\nabla_X Z$ also belong to V (respectively W). The result then follows at once.

Lemma 3. *$\Gamma^j_{mi} = -\Gamma^i_{mj}$ $\forall$ i, m, j everywhere on U.*

Proof. This follows from the fact that $\nabla g = 0$ and the fact that the moving frame $[Y_1, \cdots, Y_n]$ is orthonormal.

Lemma 4. *$\Gamma^k_{ij} = \Gamma^k_{ji}$ at x $\forall$ i, j, k.*

Proof. Let $Y = \sum_{i=1}^{n} y^i \partial/\partial x^i$ be the vector field on U determined by parallelly translating a vector Y_x at x along geodesics emanating from x, where again $\{x^1, \cdots, x^n\}$ are normal coordinates on U determined by v. Using the differential equations for parallel translation, we can show that $\partial y^k/\partial x^l = 0$ at x $\forall$ k, l. We then apply this fact to show that $[Y_i, Y_j](x) = 0$ $\forall$ i, j. Since the torsion is zero, we then have the result.

Lemma 5. *$\Gamma^k_{im} = 0$ at x unless k, i, m are all $\leq p$ or all $> p$.*

Proof. Immediate from Lemmas 2, 3, 4.

We now return to the proof that $\delta\alpha_i = 0$, $i = 1, 2$. Pick an arbitrary point x in M. Let v be an element of P such that $\pi(v) = x$. Let $[Y_1, \cdots, Y_n] = \sigma$ be the adapted frame on a neighborhood U of x determined by v. Let $[\beta_1, \cdots, \beta_n]$ be the moving coframe dual to σ. It is not difficult to check that we have representations of the form $\alpha_1 = \sum_{i=1}^{p} \eta_i \beta^i$ and $\alpha_2 = \sum_{i=p+1}^{p} \eta_i \beta^i$. Then $X_1 = \sum_{i=1}^{p} \eta^i Y_i$ and $X_2 = \sum_{i=p+1}^{n} \eta^i Y_i$, where $\eta^i = \eta_i$, $i = 1, \cdots, n$, because σ is an orthonormal frame. Since X is Killing, we have $\eta_{j;i} + \eta_{i;j} = 0$ $\forall$ i, j. Setting $i = j$, we have $\eta_{j;j} = 0$. But $\eta_{j;j} = Y_j(\eta_j) - \sum_i \eta_i \Gamma^i_{jj}$. Hence $Y_j(\eta_j) = \sum_i \eta_i \Gamma^i_{jj}$. By Lemma 5 we have

$$\text{(1)} \qquad \text{if} \quad j \leq p\,, \quad \text{then} \quad Y_j(\eta_j)(x) = \sum_{i \leq p} \eta_i(x) \Gamma^i_{jj}(x)\,,$$

$$\text{(2)} \qquad \text{if} \quad j > p\,, \quad \text{then} \quad Y_j(\eta_j)(x) = \sum_{i > p} \eta_i(x) \Gamma^i_{jj}(x)\,.$$

We recall that $\eta^j_{;j} = Y_j(\eta^j) + \sum_k \Gamma^j_{jk} \eta^k$. From this, using (1), (2), Lemmas 3 and 5, and the fact that $\eta^i = \eta_i$ it can be computed that $\eta^j_{;j}(x) = 0$ for each $j = 1, \cdots, n$. But $(\delta\alpha_1)(x) = -\sum_{j=1}^{p} \eta^j_{;j}(x)$ and $(\delta\alpha_2)(x) = -\sum_{j=p+1}^{n} \eta^j_{;j}(x)$. Hence $(\delta\alpha_1)(x) = (\delta\alpha_2)(x) = 0$ for an arbitrary point x in M. Thus X_1 and X_2 are Killing.

It remains only to show that $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are ideals in $\mathfrak{G}$. First we show $[\mathfrak{G}_1, \mathfrak{G}_1] \subset \mathfrak{G}_1$. Let X, Z be arbitrary elements of $\mathfrak{G}_1$, and v be an arbitrary point in P with $\pi(v) = x$. Let $\sigma = [Y_1, \cdots, Y_n]$ be the adapted frame on a neighborhood U of x determined by v. We see easily that X and Z are expressed as

$X = \sum_{i=1}^{n} \eta^i Y_i$, $Z = \sum_{i=1}^{n} \xi^i Y_i$, where $\eta^i = \xi^i \equiv 0$ on U for $i > p$. Now $[X, Z] = \sum_{m=1}^{n} [X, Z]^m Y_m$ where

$$[X, Z]^m = \sum_{i,j} (\xi^i Y_i(\eta^m) + \xi^i \eta^j \Gamma^m_{ij} - \eta^i Y_i(\xi_m) - \eta^j \xi^i \Gamma^m_{ji}) \ . \tag{3}$$

By Lemma 5 we obtain that $[X, Z]^m(x) = 0$ for $m > p$. Thus, if α_i, $i = 1, \cdots, n$, are the components of the 1-form corresponding to $[X, Z]$, then $\alpha_i(x) = 0$ for $i > p$. From this it follows that if $f = \sum_{i=1}^{n} f^i e_i$ is the R^n-valued function on P corresponding to $[X, Z]$, then $f^i(v) = 0$ for $i > p$. But v was an arbitrary point of P. Hence $[X, Z] \in \mathfrak{G}_1$. Similarly we prove that $[\mathfrak{G}_2, \mathfrak{G}_2] \in \mathfrak{G}_2$.

To complete the proof of Theorem 1, we need only to show $[\mathfrak{G}_1, \mathfrak{G}_2] = 0$. Let X, Z be arbitrary elements of $\mathfrak{G}_1, \mathfrak{G}_2$ respectively. Then $X = \sum_{i=1}^{n} \xi^i Y_i$ and $Z = \sum_{i=1}^{n} \eta^i Y_i$ with $\xi^i \equiv 0$ on U for $i > p$ and $\eta^i \equiv 0$ on U for $i \leq p$, where $\sigma = [Y_1, \cdots, Y_n]$ is an adapted frame on a neighborhood U of an arbitrary point x of M determined by some v in P such that $\pi(v) = x$. Again we have $[X, Z] = \sum_{m=1}^{n} [X, Z]^m Y_m$ where $[X, Z]^m$ is given by (3). We will show that $[X, Z]^m(x) = 0 \ \forall\, m$.

Case 1: $m > p$. Then $[X, Z]^m(x) = \sum_{i=1}^{p} \xi^i(x)(Y_i(\eta^m))(x)$ by Lemma 4 and the fact that $\xi^i \equiv 0$ on U for $i > p$. Since Z is Killing, we have $\eta_{m;i} + \eta_{i;m} = 0$. Since σ is orthonormal, $\eta^i = \eta_i$. Thus we obtain

$$Y_i(\eta^m) - \sum_k \eta_k \Gamma^k_{mi} + Y_m(\eta^i) - \sum_k \eta_k \Gamma^k_{im} = 0 \ .$$

For $m > p$ and $i \leq p$, it follows by Lemma 5 that $Y_i(\eta^m) = -Y_m(\eta^i)$ at x. Thus $[X, Z]^m(x) = -\sum_{i \leq p} \xi^i(x)(Y_m(\eta^i))(x) = 0$.

Case 2: $m \leq p$. The proof is similar.

4. Decomposition of holomorphic vector fields

We now assume our compact, connected, oriented Riemannian manifold M is Kählerian. The complex dimension is $n/2$, where n is now even. We denote the complex structure by J. Let $\mathfrak{A}$ be the real Lie algebra of infinitesimal automorphisms of M. $\mathfrak{A}$ is made into a complex Lie algebra using the almost complex structure J. Let $\mathfrak{H}$ be the complex Lie algebra of holomorphic vector fields on M. As complex Lie algebras, $\mathfrak{H} = \mathfrak{A}$. Thus we will deal with $\mathfrak{A}$ from now on.

The infinitesimal automorphisms of M are characterized by the property that their corresponding 1-forms satisfy $\Delta\alpha = 2S\alpha$. Therefore, if $\Gamma_{O(M)}$ is reducible to a connection Γ_P in a subbundle P of $O(M)$ with structure group $G \subset SO(n)$, and if $R^n = V_1 \oplus \cdots \oplus V_r$ where each V_i is invariant under the action of G, then by the theorem of Chern we have a corresponding decomposition of the real Lie algebra $\mathfrak{A}$ into a direct sum of vector subspaces: $\mathfrak{A} = \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_r$, where $\mathfrak{A}_i = \{X \mid X \in \mathfrak{A}$ and the equivariant R^n-valued function on P corresponding to X is V_i-valued$\}$.

Now we specialize P and G. Let x be an arbitrary point of M. Let $T_xM = V_{1x} \oplus \cdots \oplus V_{sx}$ be a direct sum decomposition of T_xM such that the V_{ix} are mutually orthogonal subspaces invariant under $\Phi(x)$ and J_x. There is an element u in $O(M)$, $\pi(u) = x$, which is of the form

$$u = (u_1, \cdots, u_{r_1}, Ju_1, \cdots, Ju_{r_1}, u_{r_1+1}, \cdots, u_{r_1+r_2}, Ju_{r_1+1}, \cdots, Ju_{r_1+r_2}, \cdots)$$

where the first $2r_1$ vectors in the frame u span V_{1x}, the next $2r_2$ vectors span V_{2x}, etc.

Theorem 2. *Let $P = P(u)$ be the holonomy bundle of $O(M)$ through u with structure group $G = \Phi(u) = \Phi$. Let $R^n = R^{2r_1} \oplus \cdots \oplus R^{2r_s}$. Then each R^{2r_i}, $i = 1, \cdots, s$, is invariant under Φ. Moreover, if $\mathfrak{A} = \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_s$ is the decomposition of $\mathfrak{A}$ corresponding to this decomposition of R^n, then the $\mathfrak{A}_i$ are complex subspaces of $\mathfrak{A}$ and in fact ideals.*

Proof. For simplicity we assume $s = 2$ and write $R^{2r_1} = R^p$, $R^{2r_2} = R^{n-p}$, $V_{1x} = V_x$, and $V_{2x} = W_x$. To show R^p is invariant under Φ, let g be an arbitrary element of Φ. Then there is an element τ of $\Phi(x)$ such that $\tau(u)$ is the frame $u \circ g: R^n \to T_xM$. Since V_x is invariant under $\Phi(x)$, we have $\tau(u) \circ u^{-1}V_x = u \circ g \circ u^{-1}V_x = V_x$. But $u^{-1}V_x = R^p$. Therefore $u \circ gR^p = V_x$ and $gR^p = u^{-1}V_x = R^p$. Similarly R^{n-p} is invariant under Φ.

Now let v be an arbitrary element of P with $\pi(v) = y$. By the definition of $P = P(u)$ and the fact that $\nabla J = 0$ we see that v is of the form

$$v = (v_1, \cdots, v_{r_1}, Jv_1, \cdots, Jv_{r_1}, v_{r_1+1}, \cdots, v_{r_1+r_2}, Jv_{r_1+1}, \cdots, Jv_{r_1+r_2})$$

where $2r_1 = p$ and $2r_2 = n - p$. Thus, if V_y is the subspace of T_yM spanned by the first p vectors of this frame and W_y is the subspace spanned by the last $n - p$ vectors, then we have $J_yV_y = V_y$ and $J_yW_y = W_y$. It follows readily that $J\mathfrak{A}_i \subset \mathfrak{A}_i$, $i = 1, 2$. Thus $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are complex subspaces of $\mathfrak{A}$.

It remains to show that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are ideals in $\mathfrak{A}$. The proof that $[\mathfrak{A}_i, \mathfrak{A}_i] \subset \mathfrak{A}_i$, $i = 1, 2$, is exactly the same as the proof of the analogous fact in Theorem 1. We therefore have only to show $[\mathfrak{A}_1, \mathfrak{A}_2] = 0$. Let X, Z be arbitrary elements of $\mathfrak{A}_1, \mathfrak{A}_2$ respectively. Let y be an arbitrary point of M and let $\sigma = [Y_1, \cdots, Y_n]$ be an adapted frame on a neighborhood U of y determined by an element $v = (Y_{1y}, \cdots, Y_{ny})$ in P. Let the components of X, Z, and $[X, Z]$ be ξ^i, η^i and $[X, Z]^m$ respectively in the frame σ. Then $\xi^i \equiv 0$ for $i > p$ and $\eta^i \equiv 0$ for $i \leq p$. From Lemma 4 we obtain

$$[X, Z]^m(y) = \sum_{i=1}^{n} \xi^i(y)(Y_i(\eta^m))(y) - \eta^i(y)(Y_i(\xi^m))(y) \ .$$

Thus $[X, Z]^m(y) = -\sum_{i>p} \eta^i(y)(Y_i(\xi^m))(y)$ for $m \leq p$, and $[X, Z]^m(y) = \sum_{i \leq p} \xi^i(y)(Y_i(\eta^m))(y)$ for $m > p$. In order to show $[X, Z]^m(y) = 0 \,\forall\, m$, it suffices to show

$$(Y_i(\xi^m))(y) = 0 \quad \text{for } i > p \quad \text{and} \quad m \leq p \ , \quad \text{and}$$
$$(Y_i(\eta^m))(y) = 0 \quad \text{for } i \leq p \quad \text{and} \quad m > p \ .$$

Lemma 6. *$\Phi^0(y)$, the restricted linear holonomy group at y (which is the identity component of $\Phi(y)$) with respect to $\Gamma_{O(M)}$, is decomposed into the direct product of two normal subgroups $\Phi_1^0(y)$ and $\Phi_2^0(y)$ such that $\Phi_1^0(y)$ is trivial on W_y and $\Phi_2^0(y)$ is trivial on V_y.*

Proof. This follows from the proof of Proposition 5.3, p. 183 in [2]. By Lemma 1 we have that $\Phi_1^0(y)V_y \subset V_y$ and $\Phi_2^0(y)W_y \subset W_y$. From this and Lemma 6 it follows that the holonomy algebra $\phi(y)$ at y splits into a direct sum of subalgebras $\phi_1(y)$ and $\phi_2(y)$, where, as linear endomorphisms of T_yM with respect to the basis $\{Y_{1y}, \cdots, Y_{ny}\}$, elements of $\phi_1(y)$ (respectively $\phi_2(y)$) are represented by matrices of the form

$$\begin{Vmatrix} a_{p\times p} & 0 \\ 0 & 0 \end{Vmatrix} \quad \left(\text{respectively} \begin{Vmatrix} 0 & 0 \\ 0 & b_{(n-p)\times(n-p)} \end{Vmatrix}\right) .$$

Now let η be the 1-form corresponding to the infinitesimal automorphism Z. By an easy computation we find that with respect to the moving frame σ, $(\nabla\eta)_{li} = Y_i(\eta^l) + \sum_j \eta^j \Gamma^l_{ij}$. In particular, $(\nabla\eta)_{li}(y) = (Y_i(\eta^l))(y)$ if $l > p$ and $i \leq p$. But an element of $\phi(y) + J_y\phi(y)$ is represented with respect to the basis $\{Y_{1y}, \cdots, Y_{ny}\}$ of T_yM by a matrix of the form

$$\begin{Vmatrix} A_{p\times p} & O_{p\times(n-p)} \\ O_{(n-p)\times p} & B_{(n-p)\times(n-p)} \end{Vmatrix} .$$

This follows from the above representations of $\phi_i(y)$, $i = 1, 2$, and from the fact that $J_yV_y = V_y$ and $J_yW_y = W_y$. By a theorem of Lichnerowicz [3, p. 151], $(\nabla\eta)_y \in \phi(y) + J_y\phi(y)$. This implies that $(\nabla\eta)_{li}(y) = 0$ if $l > p$ and $i \leq p$. Thus we must have $(Y_i(\eta^m))(y) = 0$ for $i \leq p$ and $m > p$. Similarly, we can prove that $(Y_i(\xi^m))(y) = 0$ for $i > p$ and $m \leq p$. This completes the proof of Theorem 2.

Finally, if we assume the Kähler manifold M is nondegenerate, we can obtain a decomposition of $\mathfrak{A}$ starting, as in the Killing case, from any holonomy bundle:

Theorem 3. *Let M be nondegenerate. Suppose P is any holonomy bundle of $O(M)$, and $G = \Phi$ is its structure group. Suppose $R^n = V_1 \oplus \cdots \oplus V_r$ is a decomposition of R^n into a direct sum of subspaces mutually orthogonal with respect to the usual inner product and invariant under Φ. Let $\mathfrak{A} = \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_r$ be the corresponding decomposition of $\mathfrak{A}$. Then the $\mathfrak{A}_i$ are complex subspaces of $\mathfrak{A}$ and in fact ideals.*

Proof. As in Theorem 1, it suffices to consider $V_1 \oplus \cdots \oplus V_r$ to be $R^{p_1} \oplus \cdots \oplus R^{p_r}$. For simplicity, we consider only the case $r=2$ and write $R^n = R^p \oplus R^{n-p}$. Let $u = (Y_{1x}, \cdots, Y_{nx})$ be an element of P with $\pi(u) = x$. Then by Lemma 1, V_x and W_x, the subspaces of T_xM spanned by $\{Y_{1x}, \cdots, Y_{px}\}$ and $\{Y_{p+1x}, \cdots, Y_{nx}\}$ respectively, are invariant under $\Phi(x)$. Since M is nondegenerate, we have $J_x \in \Phi(x)$ [3, p. 173]. Then, in particular, $JV_x \subset V_x$ and $JW_x \subset W_x$. The rest of the proof proceeds as the proof of Theorem 2.

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