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TYPE NUMBERS IN METRIC DIFFERENTIAL GEOMETRY OF HIGHER ORDER

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Introduction

C. B. Allendoerfer [2] proved a number of theorems concerning the type of a Riemannian manifold R_n isometrically imbedded in a euclidean space E_{n+p}. The main results of his work, which generalize the well-known theorems of R. Beez and T. Y. Thomas for spaces of class one, are the following:

Theorem I. *If the first normal space of a simply connected R_n is of q $(\leq p)$ dimensions at every point, and R_n is of type ≥ 3 at every point, then R_n can be imbedded in an E_{n+q}, and the imbedding is unique to within a rigid motion.*

Theorem II. *If the type of R_n is ≥ 4 at every point, then the Codazzi equations are consequences of the Gauss equations.*

In this paper we shall generalize these results in two directions: First, we replace the space E_{n+p} by a Riemannian space with a constant sectional curvature. Secondly, extending the notion of type to normal spaces of orders $k = 1, 2, 3, \cdots$ of R_n we obtain theorems on the higher order metric properties of immersed manifolds.

In Chapter I we present some ideas and instruments developed in the author's papers [7], [8]. The concepts of *a graded Riemannian vector bundle* and of a *Riemannian geometry of genus r* play a leading part here. We also give a short summary of basic results concerning these abstract structures. Chapter II is devoted to some preparatory algebraic lemmas on generalized type numbers. In Chapter III we give proofs of the theorems announced. Combining them with the results stated in Chapter I we also derive an immersion theorem leading to "exotic" integrability conditions.

I. INTRODUCTION TO THE METRIC DIFFERENTIAL GEOMETRY OF HIGHER ORDER

In this Chapter we present some definitions and results included in [7], [8]. For simplicity, all manifolds considered will be connected and of class C^∞, and all maps will also be class C^∞, if not otherwise stated.

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1. Let M be an m-dimensional manifold, and N a Riemannian space with constant sectional curvature C. We consider an *immersion* $\varphi: M \to N$ and the induced vector bundle $\varphi_* T(N)$, where $T(N)$ denotes the tangent bundle of N. For any $x \in M$ and $k = 1, 2, \cdots$ the osculating space $S^k_{|x}$ of order k can be thought a subspace of $\varphi_* T(N)_{|x}$.

Suppose φ to be *regular of order* r in the following sense:

(i) *For* $k = 1, \cdots, r$, $\dim S^k_{|x}$ *is constant on* M.

Thus the subspaces $S^k = \bigcup S^k_{|x}$ $(x \in M)$ are vector subbundles of $\varphi_* T(N)$, and we can identify $S^1 \equiv T(M)$.

(ii) *We have* $\varphi_* T(N) \equiv S^r$.

The induced bundle $E = \varphi_* T(N)$ is canonically a *Riemannian vector bundle* over M with a linear connection ∇ preserving the inner product in E. There are Riemannian vector subbundles $E^1, \cdots, E^r$ in $\varphi_* T(N)$ such that for $k = 1, \cdots, r$, we have an orthogonal decomposition

$$S^k = E^1 \oplus \cdots \oplus E^k \qquad (\oplus \text{ denotes the Whitney sum}) .$$

Here $E^1 \equiv S^1 \equiv T(M)$, and $E = E^1 \oplus \cdots \oplus E^r$.

In any E^k, there is *a canonical linear connection* $\nabla^{(k)}$, namely, *the orthogonal projection* of ∇ into E^k. Moreover, each $\nabla^{(k)}$ preserves the inner product in E^k, $k = 1, \cdots, r$ (see [7, p. 680]). A connection with this property is said to be *semi-Riemannian*.

Finally, let us consider the Riemannian structure on M induced by the immersion $\varphi: M \to N$. Then $\nabla^{(1)}$ can be thought as the Levi-Civita connection on M.

Convention. It is well-known that any local section in a vector bundle E over M (of class C^∞) can be prolonged to the whole base. For simplicity (to avoid speaking about definition domains) we shall work with global sections throughout this paper (if not otherwise stated). $X^{(k)}$ always denotes a section of E^k, $k = 1, \cdots, r$.

2. Now we can state *a decomposition formula.*

There exist bundle morphisms

$$P_k: E^1 \otimes E^k \longrightarrow E^{k+1} \qquad (k = 1, \cdots, r-1) ,$$
$$L_k: E^1 \otimes E^k \longrightarrow E^{k-1} \qquad (k = 2, \cdots, r)$$

such that for any vector $t \in E^1$ *and any section* $X^{(k)}$ *we have the following orthogonal decomposition of the covariant derivative* $\nabla_t X^{(k)}$:

$$\nabla_t X^{(k)} = L_k(t \otimes X^{(k)}) + \nabla^{(k)}_t X^{(k)} + P_k(t \otimes X^{(k)}) . \tag{1}$$

(We put $P_r = L_1 = 0$ by definition). *All* P_k *are surjective, and* P_1 *is symmetric as a bilinear map.* (see [7, Proposition 7]).

The *duality formula* [7, Formula (15)] states that

$$\langle L_k(T \otimes X^{(k)}), Y^{(k-1)} \rangle = -\langle X^{(k)}, P_{k-1}(T \otimes Y^{(k-1)}) \rangle \tag{2}$$

for any sections $T, X^{(k)}, Y^{(k-1)}$ of E^1, E^k, E^{k-1} respectively, $k = 2, \cdots, r$.

We can also consider the *composed bundle morphisms*

$$P^k = P_{k-1} \circ \cdots \circ P_2 \circ P_1 \ , \qquad P^k \colon \otimes^k E^1 \longrightarrow E^k \ , \qquad \text{for } k = 2, \cdots, r \ ,$$

and put $P^1 = \mathrm{id}_{E^1}$ by definition. A basic property of the k-linear maps $P^k(X_1 \otimes \cdots \otimes X_k)$ is that *they are symmetric with respect to all arguments.* (The last property is closely connected with the fact that N is a space of constant curvature, whereas the formulas (1), (2) are true for an arbitrary N.)

Convention. In the following we shall write $L_k(U, X^{(k)})$, $P_k(U, X^{(k)})$, $P^k(X_1, \cdots, X_k)$ instead of $L_k(U \otimes X^{(k)})$, $P_k(U \otimes X^{(k)})$, $P^k(X_1 \otimes \cdots \otimes X_k)$ respectively.

3. Let us denote by $R^{(k)}$ *the curvature transformation* of the linear connection $\nabla^{(k)}$, $k = 1, \cdots, r$. We have two systems of *integrability conditions* in $E = E^1 \oplus \cdots \oplus E^r$:

$$\begin{aligned} &\nabla_U^{(k+1)} P_k(T, X^{(k)}) - \nabla_T^{(k+1)} P_k(U, X^{(k)}) + P_k(U, \nabla_T^{(k)} X^{(k)}) \\ &\qquad - P_k(T, \nabla_U^{(k)} X^{(k)}) - P_k([U, T], X^{(k)}) = 0 \ , \quad k = 1, \cdots, r-1 \ , \end{aligned} \tag{3}$$

(equations of Codazzi),

$$\begin{aligned} &R_{UT}^{(k)} X^{(k)} + P_{k-1}(U, L_k(T, X^{(k)})) - P_{k-1}(T, L_k(U, X^{(k)})) \\ &\qquad + L_{k+1}(U, P_k(T, X^{(k)})) - L_{k+1}(T, P_k(U, X^{(k)})) \\ &\quad = C\{\langle T, X^{(k)} \rangle U - \langle U, X^{(k)} \rangle T\} \qquad (k = 1, \cdots, r) \ , \end{aligned} \tag{4}$$

(equations of Gauss).

The right-hand side of (4) is nonzero only for $k = 1$. If we introduce new operators:

$$R^{(k)}(U, T, X^{(k)}, Y^{(k)}) = \langle R_{UT}^{(k)} X^{(k)}, Y^{(k)} \rangle \ , \tag{5}$$

$$\begin{aligned} P_k(U, T, X^{(k)}, Y^{(k)}) = &\langle P_k(U, Y^{(k)}), P_k(T, X^{(k)}) \rangle \\ &- \langle P_k(U, X^{(k)}), P_k(T, Y^{(k)}) \rangle \ , \end{aligned} \tag{6}$$

$$\begin{aligned} L_k(U, T, X^{(k)}, Y^{(k)}) = &\langle L_k(U, Y^{(k)}), L_k(T, X^{(k)}) \rangle \\ &- \langle L_k(U, X^{(k)}), L_k(T, Y^{(k)}) \rangle \ , \end{aligned} \tag{7}$$

$$\begin{aligned} Q_k(U, T, X^{(k)}, Y^{(k)}) = &-L_k(U, T, X^{(k)}, Y^{(k)}) + R^{(k)}(U, T, X^{(k)}, Y^{(k)}) \\ &+ C\{\langle U, X^{(k)} \rangle \langle T, Y^{(k)} \rangle - \langle U, Y^{(k)} \rangle \langle T, X^{(k)} \rangle\} \ , \end{aligned} \tag{8}$$

we can rewrite (4) in the form

$$(9)\quad \begin{aligned} &P_k(U, T, X^{(k)}, Y^{(k)}) = Q_k(U, T, X^{(k)}, Y^{(k)}) , \qquad k = 1, \cdots, r-1 , \\ &Q_r(U, T, X^{(r)}, Y^{(r)}) = 0 . \end{aligned}$$

4. Now we shall adopt a more abstract point of view. We start with

Definition 1. A graded Riemannian bundle $\{E^k, P_k\}^r$ over a manifold M is a Riemannian vector bundle $E \to M$, $\dim E \geq \dim M$, in which the following structure is given:

(i) a fixed bundle injection j: $T(M) \to E$,

(ii) an orthogonal splitting (graduation) $E = E^1 \oplus \cdots \oplus E^r$ such that $E^1 \equiv jT(M)$,

(iii) a system of bundle epimorphisms

$$P_k : E^1 \otimes E^k \longrightarrow E^{k+1} , \qquad k = 1, \cdots, r-1$$

such that the mappings

$$P^k(X_1, \cdots, X_k) = (P_{k-1} \circ \cdots \circ P_2 \circ P_1)(X_1 \otimes \cdots \otimes X_k)$$

are all symmetric.

Thus *each* P^k *induces a bundle epimorphism* $*P^k$: $S^k E^1 \to E^k$, where S^k denotes the k-th symmetric tensor power. Further, the Riemannian inner product on E^1 induces a Riemannian structure on M via j: $T(M) \to E^1$, and hence we always have a *canonical linear connection* $\nabla^{(1)}$ in E^1, determined by the Levi-Civita connection on M. We usually *identify* E^1 with $T(M)$.

We also define *dual homomorphisms* L_k: $E^1 \otimes E^k \to E^{k-1}$, $k = 2, \cdots, r$, by means of formula (2).

5. Let $\{E^k, P_k\}^r$ be a graded Riemannian bundle over M. For any $l \leq r$ we can consider a graded Riemannian bundle $\{E^k, P_k\}^l$ called a *graded subbundle* of $\{E^k, P_k\}^r$. On the other hand, $\{E^k, P_k\}^r$ is called a *prolongation* of each $\{E^k, P_k\}^l$, $l \leq r$. Finally, two graded Riemannian bundles $\{E^k, P_k\}^r$, $\{E'^k, P'_k\}^r$ (of the same "length" r) over the same manifold M are said to be *equivalent* and denoted by $\{E^k, P_k\}^r \cong \{E'^k, P'_k\}^r$ if there is a map Φ: $E \to E'$, called an *equivalence*, with the following properties:

(i) For any $x \in M$, Φ maps $E_{|x}$ isometrically onto $E'_{|x}$.

(ii) For any $k = 1, \cdots, r$, Φ maps E^k onto E'^k.

(iii) We have $\Phi \circ P_k = P'_k \circ \Phi$ for $k = 1, \cdots, r-1$.

(iv) If j: $T(M) \to E$, j': $T(M) \to E'$ are the corresponding canonical injections, then $j' = \Phi \circ j$.

We can easily see that the only equivalence of $\{E^k, P_k\}^r$ onto itself is the identity map. Hence, *if an equivalence* Φ: $\{E^k, P_k\}^r \to \{E'^k, P'_k\}^r$ *exists, it is unique.*

6. If a graded Riemannian bundle $\{E^k, P_k\}^r$ is induced by a regular immersion φ: $M \to N$, then we have a system of canonical connections $\nabla^{(1)}, \cdots,$

$\nabla^{(r)}$ in $E^1, \cdots, E^r$ respectively. Here $\nabla^{(1)}$ is a Levi-Civita connection, and $\nabla^{(2)}$, $\nabla^{(3)}, \cdots$ can be considered successively as "solutions" of the Codazzi equations (3). This leads us to the following definition.

Definition 2. Let $\{E^k, P_k\}^r$ be a graded Riemannian bundle over M. A sequence of canonical connections in $\{E^k, P_k\}^r$ is a sequence of semi-Riemannian connections $\nabla^{(1)}, \cdots, \nabla^{(r)}$ in the vector bundles $E^1, \cdots, E^r$ respectively such that

(i) $\nabla^{(1)}$ is the canonical Levi-Civita connection in $E^1 \equiv T(M)$.

(ii) The Codazzi equation (3) holds for $k = 1, \cdots, r-1$.

We have the following uniqueness theorem.

Theorem A. *Let $\{E^k, P_k\}^r$ be a graded Riemannian bundle over a manifold M. If a sequence $\nabla^{(1)}, \cdots, \nabla^{(r)}$ of canonical connections exists in $\{E^k, P_k\}^r$, then it is unique.*

In fact, if a canonical sequence $\nabla^{(1)}, \cdots, \nabla^{(l-1)}$ *is given* in a graded subbundle $\{E^k, P_k\}^{(l-1)}$, $l \leq r$, there is an operator $S_l(Z | X_1, \cdots, X_l | Y_1, \cdots, Y_l)$ acting on sections of $T(M) \equiv E^1$ such that for any canonical $\nabla^{(l)}$ in E^l we have

$$\begin{aligned} 2\langle \nabla_z^{(l)} P^l(X_1, \cdots, X_l), P^l(Y_1, \cdots, Y_l)\rangle \\ = S_l(Z | X_1, \cdots, X_l | Y_1, \cdots, Y_l) . \end{aligned} \tag{10}$$

The expression S_l involves only P_{l-1}, P^l, $\nabla^{(l-1)}$ and the inner product in E^l. Hence $\nabla^{(l)}$ is uniquely determined by $\nabla^{(1)}, \cdots, \nabla^{(l-1)}$. (See [7, Proposition 12] for details.)

There is a principal question *under which conditions a canonical sequence $\nabla^{(1)}, \cdots, \nabla^{(r)}$ exists.* A formal answer was given by C. B. Allendoerfer in [1], where the invariant equation (10) was replaced by a system of corresponding "coordinate" equations: Suppose that a sequence $\nabla^{(1)}, \cdots, \nabla^{(l-1)}$ is already constructed. Then $\nabla^{(l)}$ exists if and only if the augmented matrix of the system has the same rank as the matrix of the system, i.e., if the system is *formally* solvable.

Naturally we are interested in sufficient conditions which are not of such formal character. One result in this direction was also obtained by C. B. Allendoerfer in [2]. We shall prove a higher order generalization of his theorem (see Theorem II in the Introduction) in our Chapter III. At this place (§ 9) we shall present a different theorem obtained by the author in [7], but we must state beforehand a number of definitions.

7. Let $\{E^k, P_k\}^r \to M$ be a graded Riemannian bundle, and H a bundle homomorphism of the form $H\colon E^1 \otimes E^1 \otimes E^k \otimes E^k \to M \times \boldsymbol{R}$. Then for any sections U, T, $X^{(k)}$, $Y^{(k)}$ of E^1, E^k respectively we have a real function $H(U, T, X^{(k)}, Y^{(k)})$. Suppose that a sequence of cannonical connections exists up to order k. Then we can define a new function $\nabla_V H$ of the same type as H, called *the covariant derivative of H*, by the formula

$$
\begin{aligned}
(\nabla_V H)(U, T, X^{(k)}, Y^{(k)}) &= V\{H(U, T, X^{(k)}, Y^{(k)})\} \\
&\quad - H(\nabla_V^{(1)} U, T, X^{(k)}, Y^{(k)}) - H(U, \nabla_V^{(1)} T, X^{(k)}, Y^{(k)}) \\
&\quad - H(U, T, \nabla_V^{(k)} X^{(k)}, Y^{(k)}) - H(U, T, X^{(k)}, \nabla_V^{(k)} Y^{(k)}) \,.
\end{aligned} \tag{11}
$$

We say that H *satisfies the Bianchi identity* if

$$
\begin{aligned}
(\nabla_V H)(U, T, X^{(k)}, Y^{(k)}) &+ (\nabla_T H)(V, U, X^{(k)}, Y^{(k)}) \\
&+ (\nabla_U H)(T, V, X^{(k)}, Y^{(k)}) = 0 \,.
\end{aligned}
$$

The following is true: *if a sequence* $\nabla^{(1)}, \cdots, \nabla^{(l)}$ *of canonical connections exists in the subbundle* $\{E^k, P_k\}^l$, *then* (a) *the functions* $P_k(U, T, X^{(k)}, Y^{(k)})$ *satisfy the Bianchi identity for* $k = 1, \cdots, l-1$, (b) *the functions* $L_k(U, T, X^{(k)}, Y^{(k)})$, $R^{(k)}(U, T, X^{(k)}, Y^{(k)})$, $Q_k(U, T, X^{(k)}, Y^{(k)})$ *satisfy the Bianchi identity for* $k = 1, \cdots, l$. (Remark that $L_1(U, T, X, Y) = 0$ by definition). The result for P_k and L_k follows from Proposition 11, and for $R^{(k)}$ from [7, Proposition 6]. As for Q_k, we can see it easily from (8).

8. We have already seen that we can join a system of bundle *epimorphisms*, $^*P^k\colon S^kE^1 \to E^k$, $k = 1, \cdots, r$, to any graded Riemannian bundle $\{E^k, P_k\}^r$. Now a bundle $\{E^k, P_k\}^r$ is said to be *maximal* if all $^*P^k$ are bundle *isomorphisms*. Then we have $\dim E^k = \dim S^kE^1 = C^k_{m+k-1}$ ($k = 1, \cdots, r$), where $m = \dim M$, and $C^k_s = s!/[k!(s-k)!]$ is a combinatorial number.

An immersion $\varphi\colon M \to N$, which is regular of order r, is said to be *maximal*, if the induced graded Riemannian bundle $\varphi_* T(N) \to M$ is maximal. The maximality property means here that all osculating spaces of the immersed manifold M are of maximal dimensions as possible. It requires that $\dim M = \sum_{k=1}^r C^k_{m+k-1}$. One can see that the maximal immersions are the "generic" ones in a certain sense. (Let us remark that our definition of maximality is closely connected with the assumption that N is a space of constant curvature.)

Each maximal graded Riemannian bundle $\{E^k, P_k\}^r$ is equivalent to a graded Riemannian bundle $\{S^kT(M), P^0_k\}^r$ ("normal form"), where the operators $P^0_1, \cdots, P^0_{r-1}$ are defined canonically as follows: if $X^{(k)} \in S^kT(M)_{|x}$, $X^{(k)} = \sum_i \lambda_i (X_{i1} \bigcirc \cdots \bigcirc X_{ik})$, $1 \le i \le s$, and $Z \in T(M)_{|x}$, then we put

$$
P^0_k(Z, X^{(k)}) = \sum_i \lambda_i (Z \bigcirc X_{i1} \bigcirc \cdots \bigcirc X_{ik}) \,, \qquad 1 \le i \le s \,.
$$

(Here $\bigcirc$ indicates a symmetric tensor product of a number of vectors.) Thus a normal form of order r over M is defined if there is given a Riemannian inner product H_k on each vector bundle $S^kT(M)$, $k = 1, \cdots, r$. Then we write $E = \{S^kT(M), H_k\}^r$ instead of $E = \{S^kT(M), P^0_k\}^r$.

9. Now we can present the promised theorem (cf. [7, Theorem 3]) and its consequences.

Theorem B. *Let* $\{E^k, P_k\}^r$ *be a maximal graded Riemannian bundle over* M, *and* $\nabla^{(1)}, \cdots, \nabla^{(l)}$ *be a given sequence of canonical connections in* $\{E^k, P_k\}^l$,

$l < r$. Then this sequence can be prolonged to a canonical sequence $\nabla^{(1)}, \cdots, \nabla^{(l)}, \nabla^{(l+1)}$ in $\{E^k, P_k\}^{l+1}$ if and only if the function $P_l(U, T, X^{(l)}, Y^{(l)})$ satisfies the Bianchi identity. Furthermore, if such prolongation exists, it is unique.

Corollary 1. *Under the assumptions of Theorem B, if the Gaussian equation $P_l(U, T, X^{(l)}, Y^{(l)}) = Q_l(U, T, X^{(l)}, Y^{(l)})$ holds, then we have the unique prolongation of $\nabla^{(1)}, \cdots, \nabla^{(l)}$ to a canonical sequence $\nabla^{(1)}, \cdots, \nabla^{(l)}, \nabla^{(l+1)}$ in $\{E^k, P_k\}^{l+1}$.*

Proof. The function $Q_l(U, T, X^{(l)}, Y^{(l)})$ satisfies the Bianchi identity, and hence so does $P_l(U, T, X^{(l)}, Y^{(l)})$.

Corollary 2. *Let $\{E^k, P_k\}^r$ be a maximal graded Riemannian bundle over a two-dimensional manifold M. Then a sequence $\nabla^{(1)}, \cdots, \nabla^{(r)}$ of canonical connections exists in $\{E^k, P_k\}^r$.*

Proof. Each function $P_l(U, T, X^{(l)}, Y^{(l)})$ trivially satisfies the Bianchi identity.

10. In this section we define an abstract model of an immersed manifold. (Cf. [8, Definition 3].)

Definition 3. A Riemannian geometry $G_{r,C}$ of genus r with the exterior curvature C on a manifold M is a graded Riemannian bundle $E = \{E^k, P_k\}^r$ over M such that

(i) a sequence $\nabla^{(1)}, \cdots, \nabla^{(r)}$ of canonical connections exists in E,

(ii) the Gaussian equations (9) hold for $k = 1, \cdots, r-1$ involving the parameter C.

A Riemannian geometry $G_{r,C}$ is said to be *integrable* if the r-th Gaussian equation $Q_r(U, T, X^{(r)}, Y^{(r)}) = 0$ also holds.

Theorem C. *If $\varphi\colon M \to N$ is a regular immersion of order r, and C is the constant sectional curvature of N, then the induced vector bundle $\varphi_* T(N)$ is an integrable Riemannian geometry $G_{r,C} = \{E^k, P_k\}^r$, and each of its graded subbundles $\{E^k, P_k\}^l$, $l \leq r$, is a Riemannian geometry $G_{l,C}$.*

Theorem D. *Let $G_{r,C} = \{E^k, P_k\}^r$ be an integrable Riemannian geometry over a simply connected manifold M, and N be a complete Riemannian space with the constant sectional curvature C such that* $\dim N = \dim G_{r,C}$ *and the isometry group of N acts transitively on the orthonormal frames. Then there is, exactly up to an isometry of N, a unique immersion $\varphi\colon M \to N$ such that $\varphi_* T(N) \cong G_{r,C}$.* (Cf. [7, Theorem 2].)

From Corollary 1 of Theorem B we obtain the following result which is meaningful (for $\dim M > 2$) only if understood as an inductive process.

Theorem E. *Let $E = \{E^k, P_k\}^r$ be a maximal graded Riemannian bundle. Then the sufficient condition for E to be a Riemannian geometry* (*resp. an integrable Riemannian geometry*) *$G_{r,C}$ is that the Gaussian equations of orders $1, \cdots, r-1$* (*resp. of orders $1, \cdots, r$*) *hold in E, involving the parameter C.*

11. An important problem is the study of possible prolongations of a Riemannian geometry $G_{r,C}$. The most profound results in that line were obtained by V. V. Ryžkov in [9]. We shall present the contents of his immersion

Theorems 2 and 4, reformulated here as a prolongation theorem:

Theorem F. *Any maximal analytic Riemannian geometry $G_{r,C}$ over a real analytic manifold M can be prolonged, in a neighborhood of any point $x \in M$, to an integrable maximal analytic Riemannian geometry $G_{2r,C}$. In a given neighborhood U_x the set of all such prolongation geometries $G_{2r,C}$ depends on $\sum_{s=1}^{r} C_{m+2s-2}^{2s-1}$ arbitrary functions of m ($= \dim M$) variables.*

From this theorem one can derive

Theorem G. *Any maximal Riemannian geometry $G_{r,C}$ possesses (globally) a prolongation to a maximal Riemannian geometry $G_{r+1,C}$* (see [8, Theorem 7]).

It is not known to the author whether any analytic Riemannian geometry $G_{r,C}$ is locally prolongable to an integrable Riemannian geometry. There is another open problem: does any Riemannian geometry $G_{r,C}$ have a nontrivial prolongation $G_{r+1,C}$?

12. Let again $E = \{E^k, P_k\}^r$ be a graded Riemannian bundle. Then the tensor

$$h_k(X_1, \cdots, X_k \mid Y_1, \cdots, Y_k) = \langle P^k(X_1, \cdots, X_k), P^k(Y_1, \cdots, Y_k)\rangle$$

is called the *k-th fundamental form* (or the *k-th metric tensor*) *of* E, $k = 1, \cdots, r$. Further, the tensor

$$B_k(X_1, \cdots, X_k, X_{k+1}, \cdots, X_{2k}) = \frac{1}{(2k!)} \sum_{\mathfrak{S}} h_k(X_{\mathfrak{S}(1)}, \cdots, X_{\mathfrak{S}(k)} \mid X_{\mathfrak{S}(k+1)}, \cdots, X_{\mathfrak{S}(2k)}) ,$$

where $k = 1, \cdots, r$ and the summation extends over all permutations $\mathfrak{S}$ of the set $\{1, \cdots, 2k\}$, is called the *k-th Bompiani form of E*. Each B_k is a symmetric differential form of degree $2k$ on M.

We can also define the fundamental forms and Bompiani forms of an immersion $\varphi\colon M \to N$ as the corresponding forms of the induced bundle $\varphi_* T(N)$. In that case we have a classical interpretation for the Bompiani forms (see [9] for details and references). It is a classical result that a submanifold V_m in a euclidean space E^d (regular of order r) is uniquely determined to within a rigid motion by its Bompiani forms $B_1, \cdots, B_r$ so as by its fundamental forms $h_1, \cdots, h_r$.

The main result of [8] is the following existence theorem: any system of symmetric differential forms $B_1(X_1, X_2), \cdots, B_r(X_1, \cdots, X_{2r})$ on M satisfying certain positive definiteness conditions determines a maximal Riemannian geometry $G_{r,C}$ on M. This means simply that, *in the maximal case, our concept of a Riemannian geometry of genus r coincides with the classical one.* (Cf. E. Bompiani, *Géométrie riemanniennes d' espèce superieure*, Colloq. Géométrie Différentielle, Louvain, 1951, 125–156.) In order to make the last statement precise, we must provide us with several formulas.

Let $G_{r-1,C} = \{E^k, P_k\}^{r-1}$ be a Riemannian geometry of genus $r-1$, and $B_r(X_1, \cdots, X_{2r})$ a symmetric differential form on M. Define a tensor $*h_r(X_1, \cdots, X_r \mid Y_1, \cdots, Y_r)$ by the following formula:

$$\begin{aligned}
&*h_r(X_1, \cdots, X_r \mid Y_1, \cdots, Y_r) \\
(12)\qquad &= B_r(X_1, \cdots, X_r, Y_1, \cdots, Y_r) - \frac{1}{(2r)!}\sum_{\alpha=0}^{r-1}(C_r^\alpha)^2 \; {}^\alpha\!\!\!\!\sum_{(i\cup I, j\cup J)} \\
&\cdot S[i_1, \cdots, i_\alpha, I_1, \cdots, I_{r-\alpha}, j_1, \cdots, j_\alpha, J_1, \cdots, J_{r-\alpha}]\ ,
\end{aligned}$$

where the symbol ${}^\alpha\sum_{(i\cup I, j\cup J)}$ indicates the summation over all finite sequences $(i_1, \cdots, i_\alpha)$, $(I_1, \cdots, I_{r-\alpha})$, $(j_1, \cdots, j_\alpha)$, $(J_1, \cdots, J_{r-\alpha})$ selected from the index set $\{1, \cdots, r\}$ such that $\{i_1, \cdots, i_\alpha, I_1, \cdots, I_{r-\alpha}\} = \{j_1, \cdots, j_\alpha, J_1, \cdots, J_{r-\alpha}\} = \{1, \cdots, r\}$, and the expression $S[\cdots]$ denotes a tensor defined as follows:

$$\begin{aligned}
&S[i_1, \cdots, i_\alpha, I_1, \cdots, I_{r-\alpha}, j_1, \cdots, j_\alpha, J_1, \cdots, J_{r-\alpha}] \\
(13)\quad &= \sum_{\beta=1}^{r-\alpha} Q_{r-1}(X_{I_\beta}, Y_{J_\beta}, P^{r-1}(Y_{J_{\beta+1}}, \cdots, Y_{J_{r-\alpha}}, X_{i_1}, \cdots, X_{i_\alpha}, X_{I_1}, \cdots, X_{I_{\beta-1}}), \\
&\qquad P^{r-1}(X_{I_{\beta+1}}, \cdots, X_{I_{r-\alpha}}, Y_{j_1}, \cdots, Y_{j_\alpha}, Y_{J_1}, \cdots, Y_{J_{\beta-1}}))\ .
\end{aligned}$$

(Cf. [8, (16), (13)].) Then the tensor $*h_r(X_1, \cdots, X_r \mid Y_1, \cdots, Y_r)$ is symmetric with respect to $X_1, \cdots X_r$ and $Y_1, \cdots, Y_r$. Also, we have $*h_r(Y_1, \cdots, Y_r \mid X_1, \cdots, X_r) = *h_r(X_1, \cdots, X_r \mid Y_1, \cdots, Y_r)$, and consequently, $*h_r$ determines a symmetric bilinear form on the vector bundle $S^rT(M)$. Finally, we have (cf. [8, Proposition 6])

$$\begin{aligned}
&\frac{1}{(2r)!}\sum_{\mathfrak{S}} *h_r(X_{\mathfrak{S}(1)}, \cdots, X_{\mathfrak{S}(r)} \mid X_{\mathfrak{S}(r+1)}, \cdots, X_{\mathfrak{S}(2r)}) \\
&\qquad = B_r(X_1, \cdots, X_r, X_{r+1}, \cdots, X_{2r})\ .
\end{aligned}$$

Now the form $B_r(X_1, \cdots, X_{2r})$ is said to be *relatively positive with respect to* $G_{r-1,C}$ if the bilinear form $*H_r$ on $S^rT(M)$ determined by $*h_r$ is positively definite at each $x \in M$. If $B_1, \cdots, B_{r-1}$ are the Bompiani forms of $G_{r-1,C}$, then B_r is also said to be *relatively positive with respect to the forms* $B_1, \cdots, B_{r-1}$, *involving the parameter* C. Let us remark that for $r = 1$, (12) implies $*h_1 = B_1$.

13. We start with a prolongation theorem.

Theorem H. *Let $G_{r-1,C} \to M$ be a maximal Riemannian geometry of genus $r-1$, and $B_r(X_1, \cdots, X_{2r})$ be a symmetric differential $2r$-form on M, which is relatively positive with respect to $G_{r-1,C}$. Then there is, exactly up to an equivalence, a unique maximal Riemannian geometry $G_{r,C}$ such that*

(i) *$G_{r,C}$ is a prolongation of $G_{r-1,C}$,*

(ii) *B_r is the r-th Bompiani form of $G_{r,C}$.*

Moreover, the tensor $*h_r$ defined by (12) *is the r-th fundamental form of $G_{r,C}$,* i.e., $*h_r = h_r$. (Cf. [8, Theorem 4].)

Remark. The most difficult part of the proof is to show that the $(r-1)$-th

Gaussian equation holds in the corresponding graded Riemannian bundle $\{E^k, P_k\}^r$:

$$
(14) \quad \begin{aligned} &{}^*h_r(X_1, \cdots, X_r \mid Y_1, \cdots, Y_r) - {}^*h_r(X_1, \cdots, X_{r-1}, Y_r \mid Y_1, \cdots Y_{r-1}, X_r) \\ &\quad = Q_{r-1}(Y_r, X_r, P^{r-1}(X_1, \cdots, X_{r-1}), P^{r-1}(Y_1, \cdots, Y_{r-1})) . \end{aligned}
$$

One can see that formula (14) itself does not depend on the assumption that B_r is relatively positive.

By induction, we can formulate the final result:

Theorem I. *Let $B_1(X_1, X_2), \cdots, B_r(X_1, \cdots, X_{2r})$ be symmetric differential forms of degrees $2, 4, \cdots, 2r$ respectively on a manifold M, and C be a real number. If, for $k = 1, \cdots, r$, B_k is relatively positive with respect to $B_1, \cdots, B_{k-1}$ involving C, then there is, exactly up to an equivalence, a unique maximal Riemannian geometry $G_{r,C}$ on M such that $B_1, \cdots, B_r$ are the Bompiani forms of $G_{r,C}$.*

Remark. We can represent the equivalence class of maximal Riemannian geometries determined above by a "normal form" $G_{r,C} = \{S^kT(M), H_k\}^r$, $H_k = {}^*H_k$ for $k = 1, \cdots, r$. Such representation was used throughout [8].

Theorem I has the following consequence:

Theorem J. *The only integrability condition for a system of relatively positive symmetric differential forms $B_1, \cdots, B_r$ of degrees $2, \cdots, 2r$ respectively to determine an integrable Riemannian geometry $G_{r,C}$ is that the r-th Gaussian equation $Q_r(U, T, X^{(r)}, Y^{(r)}) = 0$ hold.*

A corresponding *global immersion theorem* was stated in [8].

Remark. On the other hand, the Ryžkov's Theorem F says that the first r Bompiani forms $B_1, \cdots, B_r$ of an integrable maximal analytic Riemannian geometry $G_{2r,C}$ are completely independent, i.e., no integrability conditions are required at all for this partial system of forms.

II. ALGEBRAIC PROPERTIES OF TYPE NUMBERS

Let E, F, H be vector spaces over real numbers of dimensions e, f, h respectively, and $L: E \otimes F \to H$ be a linear map. L is said *to be of type t* if

(i) there is a subspace $E_0 \subset E$ of dimension t such that the restriction of L to $E_0 \otimes F$ is an injection,

(ii) t is the maximum number of that property.

E_0 is called a *distinguished subspace* of E. Obviously we have $t \leq e$, $t \leq [h/f]$ ($=$ the integral part of h/f). The following assertion is easy.

Lemma 1. *A linear map $L: E \otimes F \to H$ is of type t if and only if the following are true*:

(i) *there is a subspace $E_0 \subset E$ of dimension t such that for any linearly* *independent vectors* $x_1, \cdots, x_k \in E_0$, $k \leq t$, *and any nonzero vectors* $\xi^1, \cdots, \xi^k \in F$ *the vectors* $L(x_1 \otimes \xi^1), \cdots, L(x_k \otimes \xi^k)$ *are linearly independent,*

(ii) *t is the maximum number of that property.*

Convention. If $L\colon E \otimes F \to H$ is a linear map, we shall put $L(x, \xi) = L(x \otimes \xi)$ for $x \in E$, $\xi \in F$.

Lemma 2. *Let* $L\colon E \otimes F \to H$ *be a linear map of type* $t \geq 3$, *and* $Q\colon E \times E \to F$ *an arbitrary map. If* $L(x, Q(y, z)) + L(y, Q(z, x)) + L(z, Q(x, y)) = 0$ *for any* $x, y, z \in E$, *then* $Q(x, y) = 0$ *identically.*

Proof. Let $E_0 \subset E$ be a distinguished subspace of E, and first let x, y, z be linearly independent vectors of E_0. Then Lemma 1 implies easily that $Q(x, y) = Q(y, z) = Q(z, x) = 0$. Thus Q is zero on $E_0 \times E_0$. Now let $x, y \in E_0$ be linearly independent, and $z \in E$ arbitrary. Then $Q(x, y) = 0$ and $L(x, Q(y, z)) + L(y, Q(z, x)) = 0$. According to Lemma 1, $Q(y, z) = Q(z, x) = 0$, and consequently Q is zero on the set $(E_0 \times E) \cup (E \times E_0)$. Finally, let $x, y \in E$ be arbitrary, and $z \in E_0$ be nonzero. Then we obtain $L(z, Q(x, y)) = 0$ and hence $Q(x, y) = 0$. q.e.d.

Let $A^1, \cdots, A^d$ be linear maps of E into H. *The system of homomorphisms* $\{A^1, \cdots, A^d\}$ is said *to be of type t* if the induced homomorphism $A\colon E \otimes \boldsymbol{R}^d \to H$ determined by $A(x, \xi^i) = A^i x$, ($x \in E$, $\{\xi^1, \cdots, \xi^d\}$ = the canonical basis of $\boldsymbol{R}^d$), is of type t (cf. [6, Note 17]).

In the following we also consider the vector space $\bigwedge_H^2$ (the exterior product) and the corresponding operations with elements of H.

Lemma 3. *Let* $\{A^1, \cdots, A^d\}$ *and* $\{\bar{A}^1, \cdots, \bar{A}^d\}$ *be two systems of homomorphisms of E into H, and assume* $\{A^1, \cdots, A^d\}$ *to be of type* $t \geq 3$. *If* $\sum_{\alpha=1}^d A^\alpha(x) \wedge A^\alpha(y) = \sum_{\alpha=1}^d \bar{A}^\alpha(x) \wedge \bar{A}^\alpha(y)$ *for all* $x, y \in E$, *then there is an orthogonal matrix* $s = (s_{\alpha\beta})$ *of degree d such that* $\bar{A}^\alpha = \sum_{\beta=1}^d s_{\alpha\beta} A^\beta$, $\alpha = 1, \cdots, d$.

Proof. We get this statement and its proof by a slight modification of Theorem 1, [6, Note 17] and its proof. See also [3], [4].

Lemma 4. *Let* $\{A^1, \cdots, A^d\}$ *be a system of linear maps of E into H of type* $t \geq 4$, *and* $\{\Delta_1, \cdots, \Delta_d\}$ *be a system of anti-symmetric bilinear maps of* $E \times E$ *into H. If*

$$(1) \qquad \sum_{\alpha=1}^d \{\Delta_\alpha(x, y) \wedge A^\alpha z + \Delta_\alpha(y, z) \wedge A^\alpha x + \Delta_\alpha(z, x) \wedge A^\alpha y\} = 0$$

for any $x, y, z \in E$, *then there is a unique linear matrix form* $(w_{\alpha\beta})$ *of degree d on E such that* $w_{\alpha\beta} = -w_{\alpha\beta}$ *for* $\alpha, \beta = 1, \cdots, d$, *and*

$$(2) \qquad \Delta_\alpha(x, y) = \sum_{\beta=1}^d w_{\alpha\beta}(y) A^\beta x - \sum_{\beta=1}^d w_{\alpha\beta}(x) A^\beta y\,, \qquad (\alpha = 1, \cdots, d)$$

for any $x, y \in E$.

Proof. First let x, y, z be linearly independent vectors of a distinguished subspace $E_0 \subset E$ ($\dim E_0 \geq 4$). Then by the Cartan's lemma all Δ_α's are linear combinations of the linearly independent vectors $A^1z, \cdots, A^dz, A^1x, \cdots, A^dx, A^1y, \cdots, A^dy$, and the corresponding matrix of coefficients is symmetric. Moreover, if $v \in E_0$ is another vector linearly independent of x, y, z, then each $\Delta_\alpha(x, y)$, for instance, is also a linear combination of $A^1v, \cdots, A^dv, A^1x, \cdots, A^dx, A^1y, \cdots, A^dy$. Hence we get

$$(3) \qquad \Delta_\alpha(x, y) = \sum_{\beta=1}^{d} (u_{\alpha\beta}(x, y)A^\beta x + v_{\alpha\beta}(x, y)A^\beta y) , \qquad \alpha = 1, \cdots, d ,$$

where $u_{\alpha\beta}(x, y)$, $v_{\alpha\beta}(x, y)$ are well-defined real numbers. We obtain similar expressions for $\Delta_\alpha(y, z)$ and $\Delta_\alpha(z, x)$. From the symmetry of the coefficient matrix it follows $u_{\alpha\beta}(x, y) = v_{\beta\alpha}(y, z)$, $\alpha, \beta = 1, \cdots, d$, and the anti-symmetry condition $\Delta_\alpha(x, y) = -\Delta_\alpha(y, x)$ implies

$$(4) \qquad v_{\alpha\beta}(y, x) = -u_{\alpha\beta}(x, y) .$$

Thus we get $u_{\alpha\beta}(x, y) = -u_{\beta\alpha}(z, y)$. If $v \in E_0$ is linearly independent of x, y, z, then obviously $u_{\alpha\beta}(x, y) = -u_{\beta\alpha}(v, y) = u_{\alpha\beta}(z, y)$, and *we conclude* that for a fixed $y \in E_0$, $y \neq 0$, the coefficients $u_{\alpha\beta}(x, y)$ are independent of $x \in E_0$ ($x \wedge y \neq 0$), and $u_{\alpha\beta} = -u_{\beta\alpha}$. In other words, *there is a matrix function* $(u_{\alpha\beta})$ *of degree d defined on the set* $E_0 - \{0\}$ *such that* $u_{\alpha\beta} = -u_{\beta\alpha}$ *and that for any two vectors* $x, y \in E_0$, $x \wedge y \neq 0$, *we have*

$$(5) \qquad \Delta_\alpha(x, y) = \sum_{\beta=1}^{\alpha} \{u_{\alpha\beta}(y)A^\beta x - u_{\alpha\beta}(x)A^\beta y\} , \qquad \alpha = 1, \cdots, d .$$

(Cf. (3) and (4).)

Now let $x, y \in E_0$ be linearly independent, and $z \in E$ arbitrary. Substituting (5) into (1) we obtain

$$\sum_{\alpha=1}^{d} \left\{ \left[\Delta_\alpha(y, z) - \sum_{\beta=1}^{d} u_{\beta\alpha}(y)A^\beta z \right] \wedge A^\alpha x \right.$$
$$\left. + \left[\Delta_\alpha(z, x) + \sum_{\beta=1}^{d} u_{\beta\alpha}(x)A^\beta z \right] \wedge A^\alpha y \right\} = 0 .$$

Use of the Cartan's lemma and an auxiliary vector $v \in E_0$ independent of x, y thus gives

$$\Delta_\alpha(y, z) - \sum_{\beta=1}^{d} u_{\beta\alpha}(x)A^\beta z = \sum_{\beta=1}^{d} w_{\alpha\beta}(z, y)A^\beta y ,$$
$$\Delta_\alpha(z, x) + \sum_{\beta=1}^{d} u_{\beta\alpha}(x)A^\beta z = -\sum_{\beta=1}^{d} w_{\alpha\beta}(z, x)A^\beta x .$$

From the symmetry of the coefficient matrix it follows that $w_{\alpha\beta}(z, y) = -w_{\beta\alpha}(z, x)$, $\alpha, \beta = 1, \cdots, d$. Thus $w_{\alpha\beta}(z, y) = -w_{\beta\alpha}(z, v) = w_{\alpha\beta}(z, x)$, i.e., the functions $w_{\alpha\beta}(z,x)$ defined on $E \times (E_0 - \{0\})$ are independent of x, and moreover, $w_{\beta\alpha} = -w_{\alpha\beta}$.

We can summarize: *There is a unique matrix function* $(w_{\alpha\beta})$ *of degree* d *defined on* E *such that* $w_{\alpha\beta} = -w_{\beta\alpha}$, $\alpha, \beta = 1, \cdots, d$, *and that for any* $x \in E_0$, $x \neq 0$, $z \in E$, *we have*

$$(6) \qquad \Delta_\alpha(z,x) = \sum_{\beta=1}^{d} u_{\alpha\beta}(x)A^\beta z - \sum_{\beta=1}^{d} w_{\alpha\beta}(z)A^\beta x.$$

It is obvious that the functions $w_{\alpha\beta}$ are *linear*, and that $w_{\alpha\beta} = u_{\alpha\beta}$ on $E_0 - \{0\}$.

Finally, let $x, y \in E$ be arbitrary, and $z \in E_0$ be nonzero. Replacing the terms $\Delta_\alpha(y,z)$, $\Delta_\alpha(z,x)$ in (1) by the corresponding expressions of the form (6), after a simple rearrangement we get $\sum_{\alpha=1}^{d} B_\alpha(x,y) \wedge A^\alpha z = 0$, where

$$B_\alpha(x,y) = \Delta_\alpha(x,y) - \sum_{\beta=1}^{d} w_{\beta\alpha}(x)A^\beta y + \sum_{\beta=1}^{d} w_{\beta\alpha}(y)A^\beta x\,, \qquad \alpha = 1, \cdots, d\,.$$

By the Cartan's lemma we see that $B_\alpha(x,y)$ are linear combinations of $A^1z, \cdots, A^d z$ for any $z \in E_0$, $z \neq 0$, and hence $B_\alpha(x,y) = 0$, $\alpha = 1, \cdots, d$, which proves (2).

III. GEOMETRIC CONSEQUENCES

Let $\{E^k, P_k\}^r$ be a graded Riemannian vector bundle, and consider the bundle morphism $L_r\colon E^1 \otimes E^r \to E^{r-1}$ (cf. Chapter I, (2)). *The type of* $\{E^k, P_r\}^r$ is a function $t\colon M \to \{0, 1, \cdots, m\}$ defined as follows: for any $x \in M$, $t(x)$ is the type of the linear map $L_{r,x}\colon E^1_{|x} \otimes E^r_{|x} \to E^{r-1}_{|x}$. $\{E^k, P_k\}^r$ is said *to be of type* t. The inequality $t \geq k$ $(k = 0, 1, 2, \cdots)$ means that $t(x) \geq k$ for any $x \in M$.

Theorem 1. *Any Riemannian geometry* $G_{r,C}$ *of type* $t \geq 3$ *is integrable.*

Proof. Put $G_{r,C} = \{E^k, P_k\}^r$. According to [8, Proposition 2], we have the identity

$$(1) \qquad \begin{aligned} &Q_r(U, T, P_{r-1}(V, X^{(r-1)}), Y^{(r)}) + Q_r(V, U, P_{r-1}(T, X^{(r-1)}), Y^{(r)}) \\ &\qquad + Q_r(T, V, P_{r-1}(U, X^{(r-1)}), Y^{(r)}) = 0\,. \end{aligned}$$

Consider the bundle morphism $Q_r\colon E^1 \otimes E^1 \otimes E^r \to E^r$ determined by the rule $\langle Q_r(U, T, X^{(r)}), Y^{(r)} \rangle = Q_r(U, T, X^{(r)}, Y^{(r)})$. Because of the anti-symmetry: $Q_r(U, T, Y^{(r)}, X^{(r)}) = -Q_r(U, T, X^{(r)}, Y^{(r)})$, we can transform (1) into

$$\begin{aligned} &\langle Q_r(U, T, Y^{(r)}), P_{r-1}(V, X^{(r-1)}) \rangle + \langle Q_r(V, U, Y^{(r)}), P_{r-1}(T, X^{(r-1)}) \rangle \\ &\qquad + \langle Q_r(T, V, Y^{(r)}), P_{r-1}(U, X^{(r-1)}) \rangle = 0\,, \end{aligned}$$

and using the duality formula (2) of Chapter I, we get

$$\langle L_r(V, Q_r(U, T, Y^{(r)})) + L_r(T, Q_r(V, U, Y^{(r)})) + L_r(U, Q_r(T, V, Y^{(r)})), X^{(r-1)}\rangle = 0$$

for any $X^{(r-1)}$. Thus the sum on the left-hand side of the inner product is zero. Nor for any fixed section $Y^{(r)}$ we can use algebraic Lemma 2, so that $Q_r(U, T, Y^{(r)}) = 0$, $Q_r(U, T, X^{(r)}, Y^{(r)}) = 0$. (Cf. [2, Theorem III].)

Theorem 2. *Let $G_{r,C} = \{E^k, P_k\}^r$ be a Riemannian geometry having a prolongation $G_{r+1,C} = \{E^k, P_k\}^{r+1}$ of type $t \geq 2$.*

(i) *If $G'_{r+1,C} = \{E'^k, P'_k\}^{r+1}$ is another prolongation of $G_{r,C}$, then* $\dim E'^{r+1} \geq \dim E^{r+1}$.

(ii) *$G_{r,C}$ cannot be integrable.*

Proof. (i) Consider the r-th Gaussian equation:

$$P_r(U, T, X^{(r)}, Y^{(r)}) = Q_r(U, T, X^{(r)}, Y^{(r)}) \qquad \text{in } G_{r+1,C} ,$$
$$P'_r(U, T, X^{(r)}, Y^{(r)}) = Q_r(U, T, X^{(r)}, Y^{(r)}) \qquad \text{in } G'_{r+1,C} .$$

The exterior product $E^r \wedge E^r \to M$ can be made a *Riemannian vector bundle* if we introduce an inner product $\mathfrak{S}$ on it as follows: first we define $\mathfrak{S}$ on 2-vectors, namely,

$$\mathfrak{S}(U^{(r)} \wedge T^{(r)}, X^{(r)} \wedge Y^{(r)}) = \langle U^{(r)}, X^{(r)}\rangle\langle T^{(r)}, Y^{(r)}\rangle - \langle U^{(r)}, Y^{(r)}\rangle\langle T^{(r)}, X^{(r)}\rangle \tag{2}$$

for any vectors $U^{(r)}, T^{(r)}, X^{(r)}, Y^{(r)}$ of the same fibre $E^r_{|x}$, and then we can extend $\mathfrak{S}$ by the linearity (cf. [6, p. 43]). Choose $x \in M$ and a basis $\{\xi^1, \cdots, \xi^d\}$ in $E^{r+1}_{|x}$ ($d = \dim E^{r+1}$). Further put $A^iX = -L_{r+1}(X, \xi^i)$ for any $X \in E^1_{|x}$ and $i = 1, \cdots, d$. Then we can write

$$\begin{aligned} P_r(U, X^{(r)}) &= \sum_{i=1}^{d} \langle A^iU, X^{(r)}\rangle\xi^i , \\ P_r(U, T, X^{(r)}, Y^{(r)}) &= \mathfrak{S}\left(\sum_{i=1}^{d} A^iT \wedge A^iU, X^{(r)} \wedge Y^{(r)}\right), \end{aligned} \tag{3}$$

(cf. [5, Chapter I]). Similarly, we introduce maps $A'^jX = L'_{r+1}(X, \eta^j)$, $j = 1, \cdots, d'$, on $E^1_{|x}$, where $\{\eta^1, \cdots, \eta^{d'}\}$ is an orthonormal basis of $E'^{r+1}_{|x}$. Then

$$P'_r(U, T, X^{(r)}, Y^{(r)}) = \mathfrak{S}\left(\sum_{j=1}^{d'} A'^jT \wedge A'^jU, X^{(r)} \wedge Y^{(r)}\right).$$

Finally, from the Gaussian equations we get in $E^r_{|x} \wedge E^r_{|x}$ the following identity:

$$\sum_{i=1}^{d} A^iU \wedge A^iT = \sum_{j=1}^{d'} A'^jU \wedge A'^jT .$$

Let U, T be linearly independent vectors of a distinguished subspace $E_0 \subset E^1_{|x}$

$= T(M)_{|x}$ with respect to the homomorphism $L_{r+1,x}\colon E^1_{|x} \otimes E^{r+1}_{|x} \to E^r_{|x}$ ($\dim E_0 \geq 2$). Then $A^1U, \cdots, A^dU, A^1T, \cdots, A^dT$ are linearly independent vectors, and the element $\sum_{i=1}^d A^iU \wedge A^iT$ of $E^r_{|x} \wedge E^r_{|x}$ is of rank d (see [10]). Thus $\sum_{j=1}^{d'} A'^jU \wedge A'^jT$ is also of rank d, and we cannot have $d' < d$. Hence $d' \geq d$ as stated. (Cf. [2, Theorem V].)

(ii) For the vectors U, T given above we have $\sum_{i=1}^d A^iU \wedge A^iT \neq 0$. Hence by (3), $P_r(U, T, X^{(r)}, Y^{(r)})$ is a nonzero function, and so is $Q_r(U, T, X^{(r)}, Y^{(r)})$.

Theorem 3. *Let $G_{r,C}$ be a Riemannian geometry, and $G_{r+1,C}$, $G'_{r+1,C}$ its prolongations of types t, t' respectively.*

(i) *If $t \geq 3$, $t' \geq 3$, then $G'_{r+1,C} \cong G_{r+1,C}$,*

(ii) *If $t \geq 3$ and $\dim E'^{r+1} = \dim E^{r+1}$, then $G'_{r+1,C} \cong G_{r+1\,C}$.*

Proof. (i) According to Theorem 2, (i), we have $\dim E'^{r+1} = \dim E^{r+1}$ and our assertion is reduced to (ii).

(ii) Denote $d = \dim E^{r+1} = \dim E'^{r+1}$. For any point $x \in M$ there is a neighborhood $\mathscr{U} \ni x$ such that the vector bundles E^{r+1}, E'^{r+1} are completely parallelizable over $\mathscr{U}$. Moreover, we can construct sections ξ^i, η^i of E^{r+1}, E'^{r+1} respectively, $i = 1, \cdots, d$, such that $\langle \xi^i, \xi^j \rangle = \delta_{ij}$, $\langle \eta^i, \eta^j \rangle = \delta_{ij}$ on $\mathscr{U}$. Put $A^iX = L_{r+1}(X, \xi^i)$, $A^{*j}X = L_{r+1}(X, \eta^j)$ for any vector field X on $\mathscr{U}$ and for $i, j = 1, \cdots, d$. Similarly as in the proof of Theorem 2 we get the identity $\sum_{i=1}^d A^iU \wedge A^iT = \sum_{j=1}^d A^{*j}U \wedge A^{*j}T$ for any two vector fields U, T on $\mathscr{U}$. According to Lemma 3 there is a differentiable matrix function $s\colon \mathscr{U} \to 0(d)$ such that $A^i = \sum_{j=1}^d s_{ij}A^{*j}$. Putting $\xi'^i = \sum_{j=1}^d s_{ij}\eta^j$, $A'^i = L_{r+1}(*, \xi'^i)$, for $i = 1, \cdots, d$ on $\mathscr{U}$, we have $\langle \xi'^i, \xi'^j \rangle = \delta_{ij}$, and $A'^i = A^i$ ($i = 1, \cdots, d$). Define an isometric bundle map $\varphi_{\mathscr{U}}\colon G'_{r+1,C\,|\mathscr{U}} \to G_{r+1,C\,|\mathscr{U}}$ as follows: on the common part $G_{r,C\,|\mathscr{U}}$ put $\varphi_{\mathscr{U}} =$ identity, and $\varphi_{\mathscr{U}}(\xi') = \sum_{i=1}^d \langle \xi', \xi'^i \rangle \xi^i$ for any $\xi' \in E'^{r+1}_{|y}$, $y \in \mathscr{U}$. Then for any $U \in T(M)_{|y}$, $X^{(r)} \in E'^r_{|y}$ we have $\varphi_{\mathscr{U}}(P'_r(U, X^{(r)}) = \varphi_{\mathscr{U}}(\sum_{i=1}^d \langle A'^iU, X^{(r)} \rangle \xi'^i) = \sum_{i=1}^d \langle A^iU, X^{(r)} \rangle \xi^i = P_r(U, X^{(r)})$, (see (3)). Hence $\varphi_{\mathscr{U}}$ is a local equivalence.

Choose a covering $\{\mathscr{U}_i, i \in I\}$ of M by open sets such that for any $i \in I$ we have a local equivalence φ_i of $G'_{r+1,C}$ onto $G_{r+1,C}$ over $\mathscr{U}_i$. In any intersection $\mathscr{U}_i \cap \mathscr{U}_j \neq \emptyset$ we have $\varphi_i = \varphi_j$, and hence we get a global equivalence $\varphi\colon G'_{r+1,C} \to G_{r+1,C}$. q.e.d.

Theorems 1, 3 together imply the following "Rigidity Theorem" which generalizes Allendoerfer's Theorem I (see Introduction).

Theorem 4. *Let $G_{r,C}$ be a Riemannian geometry of genus r. If a prolongation $G_{r+1,C}$ of type $t \geq 3$ exists, then it is integrable and unique exactly up to an equivalence.*

The following theorem generalizes a classical result of T. Y. Thomas [11].

Theorem 5. *Let $G_{r,C} = \{E^k, P_k\}^r$ be a Riemannian geometry, and $G_{r+1,C} = \{E^k, P_k\}^{r+1}$ its prolongation such that $\dim E^{r+1} = 1$.*

(i) *$G_{r+1,C}$ is of type $t < 2$ if and only if $G_{r,C}$ is integrable.*

(ii) *If $G_{r+1,C}$ is of type $t \geq 2$ at a point $x \in M$, then $t(x) = \dim M - z(x)$, where*

$$z(x) = \dim \{U \in T(M)_{|x} : Q_r(U, T, X^{(r)}, Y^{(r)}) = 0 ,\\ T \in T(M)_{|x}, X^{(r)}, Y^{(r)} \in E^r_{|x}\} .$$

Hence the type of $G_{r+1,C}$ is uniquely determined by its subgeometry $G_{r,C}$.

Proof. Let ξ be a unit vector of $E^{r+1}_{|x}$, and define a homomorphism $A: T(M)_{|x} \to E^r_{|x}$ by $AX = L_{r+1}(X, \xi)$. Then the r-th Gaussian equation can be written in the form

$$Q_r(U, T, X^{(r)}) = \langle AT, X^{(r)} \rangle AU - \langle AU, X^{(r)} \rangle AT .$$

(Cf. the proof of Theorem 1.) From now on we can proceed exactly as in [5, p. 42–43].

Now we derive a prolongation theorem for canonical connections.

Theorem 6. *Let $E = \{E^k, P_k\}^r$ be a graded Riemannian bundle, and suppose that*

a) *E is of type $t \geq 4$,*

b) *a sequence $\nabla^{(1)}, \cdots, \nabla^{(r-1)}$ of canonical connections exists in $\{E^k, P_k\}^{r-1}$. Then the sequence $\nabla^{(1)}, \cdots, \nabla^{(r-1)}$ can be prolonged to a canonical sequence $\nabla^{(1)}, \cdots \nabla^{(r)}$ in E if and only if the function $P_{r-1}(U, T, X^{(r-1)}, Y^{(r-1)})$ satisfies the Bianchi identity.*

Proof. It is sufficient to prove the part "if". Let $x \in M$ be given. Then in a neighborhood $\mathscr{U}$ of x we can choose sections $\xi_1, \cdots, \xi_d$ of E^r ($d = \dim E^r$) such that $\langle \xi_i, \xi_j \rangle = \delta_{ij}$ on $\mathscr{U}$. We have bundle morphisms $A_\alpha: E^1_{|\mathscr{U}} \to E^{r-1}_{|\mathscr{U}}$, $\alpha = 1, \cdots, d$, defined by $A_\alpha X = -L_r(X, \xi_\alpha)$.

Denote by $\mathfrak{S}$ the Riemannian product on the vector bundle $E^{r-1} \wedge E^{r-1}$ defined similarly as in the proof of Theorem 2. Then we can see easily that, on $\mathscr{U}$, the Bianchi identity for $P_{r-1}(U, T, X^{(r-1)}, Y^{(r-1)})$ can be written in the fom:

$$\mathfrak{S}\Big(\sum_{\alpha=1}^{d} \{\Delta_\alpha(U, T) \wedge A_\alpha V + \Delta_\alpha(T, V) \wedge A_\alpha U + \Delta_\alpha(V, U) \wedge A_\alpha T\} ,\\ X^{(r-1)} \wedge Y^{(r-1)}\Big) = 0 , \tag{4}$$

where we put

$$\Delta_\alpha(U, T) = (\nabla_U A_\alpha)(T) - (\nabla_T A_\alpha)(U) ,\\ (\nabla_U A_\alpha)(T) = \nabla_U^{(r-1)} A_\alpha(T) - A_\alpha(\nabla_U^{(1)} T) , \qquad \alpha = 1, \cdots, d .$$

According to Lemma 4, there is a unique linear differential matrix form (ω_α^β) on $\mathscr{U}$, with values in the Lie algebra $\mathfrak{o}(d)$ of the orthogonal matrix group $O(d)$, such that

$$\Delta_\alpha(U, T) = \sum_{\beta=1}^{d} \omega_\alpha^\beta(U) A_\beta T - \sum_{\beta=1}^{d} \omega_\alpha^\beta(T) A_\beta U \tag{5}$$

for any sections $U, T\colon \mathscr{U} \to E^1$. (In the notation of Lemma 4 we have $\omega_\alpha^\beta = w_{\beta\alpha}.$).

Denote by $\mathscr{F}_{\mathscr{U}}(E^r)$ the principal bundle of all orthogonal frames of the Riemannian vector bundle $E^r_{|\mathscr{U}}$, and consider its cross-section $\theta = \{\xi_1, \cdots, \xi_d\}$. Then there is a unique linear connection $\tilde{\nabla}^{(r)}$ in $\mathscr{F}_{\mathscr{U}}(E^r)$ such that for its connection form $\omega^{(r)}$ we have $\theta_* \omega^{(r)} = (\omega_\alpha^\beta)$. (See [6] and [10]. In the classical notation, $D\xi_\alpha = \sum_\beta \omega_\alpha^\beta \xi_\beta$, $D =$ the absolute differential.)

The associated linear connection $\nabla^{(r)}$ in the vector bundle $E^r_{|\mathscr{U}}$ is semi-Riemannian, and we have

$$\nabla_U^{(r)} \xi_\alpha = \sum_{\beta=1}^{\alpha} \omega_\alpha^\beta(U) \xi_\beta \qquad \text{for any } U \in E^1_{|\mathscr{U}} \text{ and } \alpha = 1, \cdots, d\ .$$

Then formula (5) becomes

$$\begin{aligned} &\nabla_T^{(r-1)} L_r(U, \xi_\alpha) - \nabla_U^{(r-1)} L_r(T, \xi_\alpha) + L_r([U, T], \xi_\alpha) \\ &\quad = L_r(U, \nabla_T^{(r)} \xi_\alpha) - L_r(T, \nabla_U^{(r)} \xi_\alpha) = 0\ , \qquad \alpha = 1, \cdots, d\ , \end{aligned}$$

and hence for any section of $E^r_{|\mathscr{U}}$, $X^{(r)} = \sum_{\beta=1}^d f^\beta \xi_\beta$, we get

$$\begin{aligned} &\nabla_U^{(r-1)} L_r(T, X^{(r)}) - \nabla_T^{(r-1)} L_r(U, X^{(r)}) - L_r([U, T], X^{(r)}) \\ &\qquad + L_r(U, \nabla_T^{(r)} X^{(r)}) - L_r(T, \nabla_U^{(r)} X^{(r)}) = 0\ . \end{aligned}$$

The last identity is just the *dual form* of the corresponding Codazzi equation (see the proof of [7, Proposition 11]). Thus $\nabla^{(r)}$ is a canonical connection in $E^r_{|\mathscr{U}}$ prolonging the sequence $\nabla^{(1)}, \cdots, \nabla^{(r-1)}$. Since a canonical connection exists in any small coordinate neighborhood of M, from the uniqueness property we see that $\nabla^{(r)}$ can be extended to the whole E^r.

Corollary. *Under the assumptions of Theorem* 6, *if the* $(r-1)$*-th Gaussian equation*

$$P_{r-1}(U, T, X^{(r-1)}, Y^{(r-1)}) = Q_{r-1}(U, T, X^{(r-1)}, Y^{(r-1)})$$

holds in E, then the sequence $\nabla^{(1)}, \cdots, \nabla^{(r-1)}$ *can be prolonged uniquely to a canonical sequence* $\nabla^{(1)}, \cdots, \nabla^{(r)}$. (Cf. Theorem II in the Introduction.)

Remark. Despite of the formal similarity of Theorem B and Theorem 6 it seems to the author that there is no real connection between those two results. Let us remark that any maximal Riemannian graded bundle is of type $t = 0$.

Theorem 1 and the above corollary together imply

Theorem 7. *Let* $\{E^k, P_k\}^r$ *be a graded Riemannian bundle of type* $t \geq 4$. *If*

(a) *the subbundle* $\{E^k, P_k\}^{r-1}$ *is a Riemannian geometry* $G_{r-1,C}$,

(b) *the* $(r-1)$*-th Gaussian equation* $P_{r-1}(U, T, X^{(r-1)}, Y^{(r-1)}) = Q_{r-1}(U, T, X^{(r-1)}, Y^{(r-1)})$ *holds involving C, then* $\{E^k, P_k\}^r$ *is an integrable Riemannian geometry* $G_{r,C}$.

Let $\{S^kT(M), H_k\}^{r-1}$ be a maximal graded Riemannian bundle in the normal form, and consider a bundle map $H_r^*: S^rT(M) \otimes S^rT(M) \to M \times \boldsymbol{R}$ such that, on each fibre the induced bilinear form is symmetric and positively semi-definite of rank $q = \text{const.}$ on M. Then the nullspaces Z_x, $x \in M$, of our bilinear forms determine a vector subbundle Z of $S^rT(M)$, and we can form a *Riemannian vector bundle* $E^r = S^rT(M)/Z$ of dimension q over M, so that we have a canonical bundle map $\rho: S^rT(M) \to E^r$ over M.

Now we can define canonically a graded Riemannian bundle $\{E^k, P_k\}^r$ prolonging $\{S^kT(M), H_k\}^{r-1}$: the Riemannian inner product on E^r is determined by H_r^*, and the bundle map $P_{r-1}: T(M) \otimes S^{r-1}T(M) \to E^r$ is defined as follows: for any base element $X_{i_1} \bigcirc \cdots \bigcirc X_{i_{r-1}}$ of $S^{r-1}T(M)_{|x}$, $x \in M$, and for any $X \in T(M)_{|x}$ we put

$$P_{r-1}(X, X_{i_1} \bigcirc \cdots \bigcirc X_{i_{r-1}}) = \rho(X \bigcirc X_{i_1} \bigcirc \cdots \bigcirc X_{i_{r-1}}) .$$

Now let $\{S^kT(M), H_k\}^{r-1}$ be a Riemannian geometry $G_{r-1,C}$, and $B_1, \cdots, B_{r-1}$ its Bompiani forms. Similarly as in Chapter I, § 12, one can define the meaning for a symmetric $2r$-form $B_r(X_1, \cdots, X_{2r})$ to be *positively semi-definite of rank q with respect to $B_1, \cdots, B_{r-1}$ involving C*. If that is the case, we get a unique prolongation of $G_{r-1,C}$ to a graded Riemannian bundle $E = \{E^k, P_k\}^r$, $E^r = S^rT(M)/Z$, such that

(i) B_r is the r-th Bompiani form of E,

(ii) the $(r-1)$-th Gaussian equation holds in E.

This fact follows easily from the considerations of [8] (see Theorem 2 and the proof of Theorem 1).

It is also clear what we mean by saying that *B_r is of relative type t with respect to $B_1, \cdots, B_{r-1}$, involving C*.

In the following immersion theorem we shall limit ourselves to the euclidean case $C = 0$.

Theorem 8. *Let $B_1(X_1, X_2), \cdots, B_r(X_1, \cdots, X_{2r})$ be symmetric differential forms of degrees $2, \cdots, 2r$ respectively on a simply connected manifold M. Suppose that $B_k(X_1, \cdots, X_{2k})$ is relatively positive with respect to $B_1, \cdots, B_{k-1}$ for $k = 1, \cdots, r-1$, and that $B_r(X_1, \cdots, X_{2r})$ is positively semi-definite of rank q and of relative type $t \geq 4$ with respect to $B_1, \cdots, B_{r-1}$. Then there is, exactly up to a rigid motion, a unique immersion $\varphi: M \to E^n$ ($n = \sum_{k=1}^{r-1} C_{m+k-1}^k + q$) such that $B_1, \cdots, B_r$ are the Bompiani forms of φ. Moreover, the form B_r is determined uniquely by $B_1, \cdots B_{r-1}$.*

Remark. With respect to a general local coordinate system $\{u^1, \cdots, u^m\}$ on M, the proper "integrability conditions" of the theorem are given by the equations meaning that the relative rank of B_r is $\leq q$; this requires that all the minors of degree $q + 1$ of a Gramm determinant are zero. Besides this, we have two well-determined systems of inequalities: one expressing the positive definiteness (or positive semi-definiteness of rank $\geq q$) of B_k with respect to

$B_1, \cdots, B_{k-1}$ $(k = 1, \cdots, r)$, and the other expressing the assumption $t \geq 4$ for a *fixed* q. One can see that all the equations and inequalities considered are relations between the local components of the forms $B_1, \cdots B_r$ and their derivatives up to order $2r - 2$.

Theorem 8 has the following concequence for $r = 2$.

Theorem 9. *Let M be a simply connected Riemannian manifold of dimension $m \geq 4$. If there is a symmetric 4-form $B_2(U, T, X, Y)$ on M, which is positively semi-definite of rank q and of the relative type $t \geq 4$ with respect to the Riemannian metric on M, then the following are true*:

(i) *$B_2(U, T, X, Y)$ is determined uniquely by the Riemannian metric of M,*

(ii) *M is of class q,*

(iii) *any two isometric immersions $\varphi, \varphi' : M \to E^{m+q}$ differ by a rigid motion,*

(iv) *$B_2(U, T, X, Y)$ is the common second Bompiani form of all isometric immersions $\varphi : M \to E^{m+q}$.*

(See [2, Theorem VI.)

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