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PRIMITIVE ACTIONS AND MAXIMAL SUBGROUPS OF LIE GROUPS

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0. Introduction

The classification of the primitive transitive and effective actions of Lie groups on manifolds is a problem dating back to Lie. The classification of the infinite dimensional infinitesimal actions was originally done by Cartan [3] and was made rigorous by some joint work of Guillemin, Quillen and Sternberg [8], whose proof was further simplified by Guillemin [7] recently by using some results of Veisfieler [19].

The classification of the primitive actions of a given finite dimensional Lie group is equivalent to that of the Lie subgroups of that group, which satisfy a certain maximality condition (see Prop. 1.5). This correspondence although more or less known seems never to have been stated in the literature (under the assumption that the leaves of a foliation are connected) so in § 1 we state it. The rest of § 1 is devoted to showing that the isotropy subalgebras of primitive actions are an intrinsically well-defined class, namely, they are the Lie algebras which correspond to maximal Lie subgroups and contain no proper ideals. We call these subalgebras primitive and hasten to add that this terminology does *not* agree with the use of "primitive" in [7], [8], [11], [12], [13] and [16]. In these articles a "primitive subalgebra" is a maximal Lie subalgebra which contains no proper ideals. In light of Theorem 1.10 we do feel that this is a more reasonable terminology. Also we show that every subalgebra which is "primitive" in the old sense is primitive in the new sense. The main result of this paper is that there exist primitive subalgebras which are not maximal subalgebras, i.e., there exist maximal Lie subgroups whose Lie algebras are not maximal subalgebras. In § 3 we classify the primitive, maximal rank, reductive subalgebras of the (complex) classical algebras giving many examples of primitive subalgebras which are, in fact, not maximal. § 2 and § 4 combined show that non-maximal primitive algebras exist only when the containing algebra is simple and the primitive subalgebra is reductive. The proofs of this involves essentially classifying the primitive subalgebras. In doing so we duplicate results of Morozov [15] on the classification of the maximal primitive subalgebras of non-simple algebras and results of Karpelevich [10] and Ochiai [16] on the

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classification of the maximal non-reductive subalgebras of the simple Lie algebras. For the latter result we use and generalize some work of Veisfieler [19].

The deepest work on the problem is still that of Dynkin [4], [5] who classified the maximal reductive subalgebras of the classical algebras. Clearly there is some overlap between § 3 and Dynkin's work. A comparison and statement of this overlap appears in [6].

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1. Preliminaries

Throughout this section G will denote a Lie group which acts on a manifold M. Let $L(G) = g$ denote the Lie algebra of G. For what follows G and M can be throught of as in either the real or complex category.

Definition 1.1. *A k-foliation* on M is a collection of k dimensional immersed submanifolds $\{F_m\}_{m \in M}$ such that $\forall m, m' \in M$,

(a) $m \in F_m$,

(b) F_m is connected and has a countable base for its topology,

(c) either $F_m = F_{m'}$ or $F_m \cap F_{m'} = \emptyset$.

The unique submanifold of the foliation containing the point m is called the *leaf through m.*

Definition 1.2. Let F be a foliation on M. Then F is *invariant under the action of G* iff $\forall a \in G, m \in M$,

$$aF_m = F_{am} ,$$

i.e., the action of G on M preserves the leaves of the foliation.

Note. $\exists$ two trivial foliations on any manifold; viz. foliation of the manifold (1) into points or (2) into connected components. These foliations are invariant under any Lie group action.

Definition 1.3. The action of G on M is *primitive* iff the only foliations on M invariant under the action of G are the trivial foliations.

The problem posed by Lie is to classify up to equivalence all of the primitive transitive and effective actions of Lie groups on manifolds. By standard results this is equivalent to determining (up to conjugacy) the set of closed subgroups P such that

(a) G acts primitively on G/P (using transitivity), and

(b) P contains no proper normal subgroups of G (using effectiveness).

Hence we have the following:

Definition 1.4. Let P be a closed subgroup of G. P is *primitive* iff

(a) P is proper,

(b) the standard action of G on G/P is primitive, and

(c) P contains no proper normal subgroups of G.

Primitivity of Lie subgroups of G can be translated into a "maximality condition" on those subgroups.

Proposition 1.5. *Let P be a closed subgroup of G. P is primitive iff*

(i) *P is proper,*

(ii) *P contains no proper normal subgroups of G, and*

(iii) *if H is a Lie subgroup of G containing P, then either* $\dim H = \dim P$ *or* $\dim H = \dim G$.

The proof of Proposition 1.5 follows directly from

Lemma 1.6. *Let I be a closed subgroup of G. Then $\exists$ a surjective correspondence from the set of all Lie subgroups of codimension k in G containing I to the set of all foliations of G/I of codimension k invariant under the action of G.*

Proof. Let H be a Lie subgroup of G, containing I, e be the coset of the identity in G/I, and F_e be the connected component of H/I containing e. Define $F_{ae} = aF_e$. We leave it to the reader to check that F is a well-defined foliation on G/I which is invariant under the action of G, and that $\operatorname{codim} F = \operatorname{codim} H$.

This correspondence is surjective. Let F be a foliation on G/I invariant under G. Define $H = \{a \in G \mid aF_e = F_e\}$. H is clearly a subgroup of G containing I. Let $r\colon G \to G/I$ be the canonical projection. Then $H = r^{-1}(F_e)$. Since r is a submersion and F_e is an immersed submanifold, H is an immersed submanifold. Hence H is a Lie subgroup and $\operatorname{codim} H = \operatorname{codim} F$.

We leave it to the reader to check that the foliation induced from H is just F.

For the rest of the paper we assume that G is connected.

Proposition 1.7. (i) *Let P be a closed maximal Lie subgroup of G which contains no proper normal subgroups of G. Then P is primitive.*

(ii) *Let P be a non-discrete primitive subgroup of G, and P^0 the connected component of the identity in P. Then*

$$\operatorname{norm}_G(P^0) \equiv \{x \in G \mid xP^0 = P^0x\}$$

is a closed maximal Lie subgroup of G. Moreover $\dim P^0 = \dim P = \dim \operatorname{norm} P^0$.

Proof. (i) follows directly from Proposition 1.5. (ii) Let $P' = \operatorname{norm}_G P^0$. Note that P being closed implies P^0 being closed which implies that P' is closed. By Proposition 1.5 either $\dim P = \dim P'$ or $\dim P' = \dim G$. In the latter case $P' = G$ since G is connected. But then P^0 is a proper normal subgroup of G. Since $P^0 \subset P$, we have a contradiction to the primitivity of P. Thus $\dim P = \dim P' = \dim P^0$.

Now we show that P' is a maximal Lie subgroup. Let K be a Lie subgroup of G containing P'. Then K contains P and by Proposition 1.5 either $\dim K = \dim P$ or $\dim K = \dim G$. In the latter case $K = G$ since G is connected. If

$\dim K = \dim P$, then $K^0 = P^0$. But then $K \subset \operatorname{norm}_G K^0 = \operatorname{norm}_G P^0$ and $K = P'$.

We wish to classify those Lie subalgebras of $g = L(G)$ which are equal to $L(P)$ where P is some primitive subgroup.

The reader may verify the following:

Lemma 1.8. *Let P be a primitive subgroup of G, and $p = L(P)$. Then p contains no proper ideals of g.*

Definition 1.9. Let p be a proper subalgebra of g. Then p is *primitive* iff

(i) p contains no proper ideals of g,

(ii) $\operatorname{norm}_G P^0$ is a maximal Lie subgroup of G, where P^0 is the connected Lie subgroup of G corresponding to p and $\dim \operatorname{norm}_G P^0 = \dim P^0$.

Note that $L(\operatorname{norm}_G P^0) = \operatorname{norm}_g (p) \equiv \{a \in g \mid [a, p] \subset p\}$.

Theorem 1.10. *p is a primitive subalgebra of g iff p is the isotropy algebra corresponding to some primitive effective and transitive action.*

Proof. $\Leftarrow$ follows from Proposition 1.7 and Lemma 1.8.

$\Rightarrow$ Let $P = \operatorname{norm}_G P^0$. From the definition, $p = L(P)$. P is closed since $P = \{a \in G \mid \operatorname{Ad} a(p) = p\}$, and p is a closed subset of g. Let H be the normal subgroup of P whose elements act as the identity on G/P. By Lemma 1.8, H is discrete. Replacing G and P by G/H and P/H respectively, we still have $L(G) = g$ and $L(P) = p$, and what we did above holds. Now apply Proposition 1.7 (i) to P and get that P is primitive. Thus p is the isotropy algebra of some primitive transitive and effective action.

The definition of primitive subalgebra as given in Definition 1.9 seems to depend on which connected Lie group G one chooses to use. In fact, primitivity does not depend on such a choice, as shown by the following.

Lemma 1.11. *Let G be a connected Lie group with Lie algebra g, $\tilde{G}$ be the connected simply connected covering group of G, and $\varphi\colon \tilde{G} \to G$ be the covering projection. Let $D = \operatorname{Ker} \varphi$. Let h be a Lie subalgebra of g, H^0 and $\tilde{H}^0$ the associated connected Lie subgroups of G and $\tilde{G}$, respectively. Let $H = \operatorname{norm}_G H^0$ and $\tilde{H} = \operatorname{norm}_{\tilde{G}} \tilde{H}^0$. Then H is a maximal Lie subgroup of G iff $\tilde{H}$ is a maximal Lie subgroup of $\tilde{G}$. In particular, $\tilde{H} = \varphi^{-1}(H)$ and $H = \varphi(\tilde{H})$. Also* $\dim H = \dim \tilde{H}$.

The proof is straightforward and we leave it to the reader to verify.

2. A criterion for primitive subalgebras

Theorem 2.1. *Let p be a subalgebra of g containing no proper ideals of g. Let $P = \operatorname{norm}_G (P^0)$. Then p is primitive iff p is maximally invariant under* $\operatorname{Ad} P$, *i.e.,*

$$(*) \qquad \text{if } p \subset l \subset g \text{ and } \operatorname{Ad} P(l) \subset l, \text{ then either } l = p \text{ or } l = g\,.$$

Proof. Suppose p is primitive, and such an l be given. Let $L = \operatorname{norm}_G L^0$. *Claim*: L contains P. If $a \in P$ and $s \in l$, then

$$a \exp sa^{-1} = \exp \operatorname{Ad} a(s) \subset \exp l \subset L^0 .$$

So aL^0a^{-1} is contained in L^0 and a is in L, as claimed. Since P is a maximal Lie subgroup, $L = P$ or $L = G$. If $L = P$, then $\dim P^0 \leq \dim L^0 \leq \dim L = \dim P = \dim P^0$ by primitivity, so $l = p$. If $L = G$, then L^0 is normal in G and l is an ideal.

Now let l be an ideal containing p. We show $l = g$. L^0P is a subgroup since L^0 is normal, and it is also a Lie subgroup. By fiat define the connected component of the identity to be L^0. Clearly this is an analytic subgroup and is normal in L^0P. Also L^0P/L^0 is countable since P/P^0 is. Now L^0P contains P, and since P is a maximal Lie subgroup either $L^0P = P$ or $L^0P = G$. $L^0P = P$ implies $l = p$ which is a contradiction to primitivity so $L^0P = G$. Therefore $L^0 = G$, i.e., $l = g$.

Conversely, if $(*)$ holds we show that P is a maximal Lie subgroup. Assume L is a Lie subgroup containing P. Then $\operatorname{Ad} L(l) = l$ so that $\operatorname{Ad} P(l) = l$ and therefore $l = g$ or $l = p$. In the first case $L = G$, and in the latter L normalizes $L^0 = P^0$, and so L is contained in P, i.e., $L = P$.

Corollary 2.2. *Every maximal subalgebra p of g containing no proper ideals of g is primitive.*

Proposition 2.3. *Assume g is not simple. Let p be a primitive subalgebra of g. Then p is a maximal subalgebra. In fact, we can classify these algebras as Morosov originally did.*

(i) *If g is not semi-simple, then there exists an abelian ideal k such that $p \oplus k = g$, and p acts faithfully and irreducibly on k. (These are the so-called affine primitive examples).*

(ii) *If g is semi-simple, then there exists a simple algebra g_0 such that $g \cong g_0 \oplus g_0$ and $p = \{(x, x) \mid x \in g_0\}$, the diagonal of g under that isomorphism.*

Proof. (i) Let k be a minimal nonzero abelian ideal in g. Then $p + k$ is invariant under $\operatorname{Ad} P$, so $p + k = g$ by Theorem 2.1. Now $p \cap k$ is an ideal of g, since $[p + k, p \cap k] \subset [p, p \cap k] \subset p \cap k$. By primitivity $p \cap k = 0$ so $p \oplus k = g$. Let $l = \{x \in p \mid [x, k] = 0\}$. l is an ideal of g since $[l, p + k] = [l, p] \subset l$. Again by primitivity, $l = 0$ and p acts faithfully on k. To show p acts irreducibly on k, let k' be an invariant subspace of k. k' is an abelian subalgebra, and is also an ideal since $[p + k, k'] = [p, k'] \subset k'$. Hence by the minimality of k, either $k' = k$ or $k' = 0$.

(ii) Let k_1, k_2 be simple ideals of g. Then $p + k_1 = p + k_2 = g$ by Theorem 2.1 and the primitivity of p. Now $[p \cap k_1, p] \subset p \cap k_1$ and $[p \cap k_1, k_2] = 0$, so $p \cap k_1$ is an ideal of g contained in p. Thus $p \cap k_1 = 0 = p \cap k_2$ and $\dim g = \dim p + \dim k_1 = \dim p + \dim k_2$. This implies that all the simple ideals have the same dimension, say r. Suppose $g = k_1 \oplus \cdots \oplus k_s$. Then it is clear that $s = 2$ and $\dim p = \dim k_1 = \dim k_2$.

Let $\pi_1 \colon g \to k_1$ be the canonical map with kernel k_2, and $\pi_2 \colon g \to k_2$ the map with kernel k_1. Then $\pi_i \mid p$ is a Lie algebra isomorphism so that $k_1 = k_2 = g_0$.

Hence the map $\pi_1 \oplus \pi_2 \colon g \to g_0 \oplus g_0$ is a Lie algebra isomorphism and the image of p is as promised.

3. The primitive, maximal rank, reductive subalgebras

For the rest of the paper all objects (i.e., groups and algebras) will be complex so G is assumed to be a complex simple connected Lie group with Lie algebra g. In this section we will compute the primitive, maximal rank, reductive subalgebras of the classical algebras. Recall that an algebra is reductive if it is the direct sum of its center with a semi-simple ideal.

Let Int_g be the group of all inner automorphism of g. If h is a subalgebra of g, denote by $\mathrm{Int}_g(h)$ the subgroup of Int_g whose elements also map h into itself.

Proposition 3.1. *Let p be a proper subalgebra of the simple Lie algebra g. Then p is primitive iff* $(*)$ *if m is a subalgebra of g containing p such that* $\mathrm{Int}_g(p)(m) = m$, *then $m = p$ or $m = g$, i.e., p is maximally invariant under the action of* $\mathrm{Int}_g(p)$.

Proof. Take G to be the adjoint group; then $\mathrm{Ad}\, P$ is just $\mathrm{Int}_g(p)$. Apply Theorem 2.1 noting that g is simple.

When p is of maximal rank we shall reduce this criterion for primitivity to a question about the finite Weyl group acting on a finite set. Recall that p is of maximal rank in g if there exists a Cartan subalgebra h of g contained in p.

Let W_h be the Weyl group relative to h, i.e., $W_h = \mathrm{Int}_g(h)/I(h)$ where

$$I(h) = \{\alpha \in \mathrm{Int}_g(h) \mid \alpha \text{ is the identity on } h\} .$$

Any Cartan subalgebra of g decomposes g into the direct sum of root spaces, i.e., $g = h \oplus \sum_\varphi e_\varphi$ where the φ's are the roots of g relative to h, and the e_φ's are the one-dimensional root spaces. Also

$$[e_\varphi, e_\psi] \subset e_{\varphi+\psi} \quad (\varphi \neq -\psi) , \quad [e_\varphi, h] \subset e_\varphi .$$

Also $[e_\varphi, e_\psi] = 0$ iff $\varphi + \psi$ is not a root. Define the one-dimensional subspaces x_φ of h by $x_\varphi = [e_{-\varphi}, e_\varphi]$. Let $f \in W_h$, and let α represent f in Int_g (h). Then

$$\text{(i)} \quad f(\varphi) = \varphi \circ \alpha^{-1} , \qquad \text{(ii)} \quad f(e_\varphi) = e_{f(\varphi)} , \qquad \text{(iii)} \quad f(x_\varphi) = x_{f(\varphi)}$$

are all well-defined and consistent ways of viewing the Weyl group action.

Now let m be any subalgebra of g containing h. Then h is a Cartan subalgebra of m, and m decomposes into $m = h \oplus \sum e_\varphi$ where the sum is taken over some subset of the roots. Let $K_m = \{\varphi \text{ a root on } h \mid e_\varphi \subset m\}$. Then K_m completely determines m. Note that if m is reductive, then $K_m = -K_m$. Let

$$W_m = \{f \in W_h \mid f(K_m) \subset K_m\} .$$

We now restate Proposition 3.1 for maximal rank subalgebras.

Proposition 3.2. *Let p be a maximal rank subalgebra of g containing h. Then p is primitive iff* (**) *if $p \subset m \subset g$, m is a subalgebra and $W_p \subset W_m$, then $m = p$ or $m = g$.*

Proof. Let m be a subalgebra of g containing p.

$\Leftarrow$ Assume (**) holds and $\mathrm{Int}_g(p)$ leaves m invariant. Let $f \in W_p$, and let $\alpha \in \mathrm{Int}_g(h)$ represent f. Then $\alpha(p) \subset p$ since $f \in W_p$. By assumption $\alpha(m) \subset m$ so $f \in W_m$. Hence $m = p$ or $m = g$. By Proposition 3.1, p is primitive.

$\Rightarrow$ Assume $W_p \subset W_m$. To show $p = m$ or $p = g$. By Proposition 3.1 it is sufficient to show that $\mathrm{Int}_g(p)(m) \subset m$. Let $\alpha \in \mathrm{Int}_g(p)$. Then $\alpha(h)$ is a Cartan subalgebra of g, and $\alpha(h) \subset p$. By the Cartan subalgebra conjugacy theorem there exists $\beta \in \mathrm{Int}_g(p)$ which also fixes m such that $\beta \circ \alpha(h) = h$. Thus $\beta \circ \alpha$ leaves h and p invariant, and defines an element in W_p. By assumption $W_p \subset W_m$ so $\beta \circ \alpha(m) \subset m$. Since $\beta(m) = m$, we must have that $\alpha(m) = m$. Thus $\mathrm{Int}_g(h)(m) = m$.

Corollary 3.3. *Let h be a Cartan subalgebra of a simple Lie algebra g. Then h is primitive iff the roots of g all have the same length. This happens in A_n, D_n (n arbitrary), E_6, E_7, and E_8.*

Note. The primitivity of the Cartan subalgebras of $\mathrm{sl}(2, C) = A_1$ was noted by Blattner.

Proof. W_h acts transitively on roots of a given length. If there is but one length, then the only algebra strictly containing h invariant under W_h is g. By Proposition 3.2, h is primitive.

The roots of a simple Lie algebra come in at most two different lengths. In all cases where the roots are of different lengths the longer root spaces plus the Cartan subalgebra form a proper subalgebra which is invariant under W_h. Again by Proposition 3.2, h is not primitive in this case.

We now describe the root systems of A_n, B_n, C_n and D_n. For a reference see [2]. Let $z_1, \cdots, z_n$ be an orthonormal basis for R^n, and let $Z_2^n = Z_2 \oplus \cdots \oplus Z_2$ (n-times) where Z_2 is the group $\{\pm 1\}$ under multiplication.

$A_{n-1}(n \geq 2)$: roots are $z_i - z_j$, $1 \leq i, j \leq n$, $i \neq j$ and $W_h = \mathrm{Per}(n)$, the permutation group on n letters acting in the obvious way on $z_1, \cdots, z_n$.

$B_n(n \geq 2)$: roots $\pm z_i, \pm z_i \pm z_j, 1 \leq i, j \leq n, i \neq j$ and $W_h = \mathrm{Per}(n) \oplus Z_2^n$. The action of the Z_2^n part of the Weyl group is given by: $f = (a_1, \cdots, a_n) \in Z_2^n$ acts on z_i by $f(z_i) = a_i z_i$.

$C_n(n \geq 3)$: roots are $\pm 2z_i, \pm z_i \pm z_j$, $1 \leq i, j \leq n$, $i \neq j$ and $W_h = \mathrm{Per}(n) \oplus Z_2^n$.

$D_n(n \geq 4)$: roots are $z_i \pm z_j$, $1 \leq i, j \leq n$, $i \neq j$ and $W_h = \mathrm{Per}(n) \oplus Z_2^n$ subject to the constraint that if $f = (a_1, \cdots, a_n) \in Z_2^n$ then $\prod_{i=1}^{n} a_i = 1$.

The following ideas will be used repeatedly:

(i) If p is reductive, maximal rank, then $K_p = -K_p$.

(ii) If φ and ψ are roots in K_p, and $\varphi + \psi$ is a root in g, then $\varphi + \psi \in K_p$. Just note that $[e_\varphi, e_\psi] \neq 0$ iff $\varphi + \psi$ is a root and $[e_\varphi, e_\psi] \subset e_{\varphi+\psi}$. So if p is an algebra and $\varphi + \psi$ is a root, then $[e_\varphi, e_\psi] = e_{\varphi+\psi} \subset p$, i.e., $\varphi + \psi \in K_p$.

(iii) If m is a vector space which is the direct sum of the Cartan subalgebra h and certain root spaces, then K_m makes perfectly good sense. To check that m is a subalgebra, it is sufficient (by (ii)) to show that K_m is closed under addition, i.e., to show that if $\varphi, \psi \in K_m$ and $\varphi + \psi$ is a root, then $\varphi + \psi \in K_m$.

Let p be a maximal rank, reductive subalgebra of A_{n-1}, B_n, C_n, or D_n. Then p defines an equivalence relation $\sim$ on the numbers $1, \cdots, n$.

Define $i \sim j$ if $i = j$ or $z_i + z_j \in K_p$ or $z_i - z_j \in K_p$. $\sim$ is clearly reflexive and symmetric. We show that $\sim$ is transitive: Let $i \sim j$ and $j \sim k$. We can assume that i, j, and k are all distinct. Now $i \sim j$ implies that $z_i - z_j$ or $z_i + z_j$ is in K_p. Thus either $z_i - z_j$ and $z_j - z_i$ are in K_p or $z_i + z_j$ and $-z_j - z_i$ are in K_p. Similarly for j and k.

Assume $\pm(z_i - z_j) \in K_p$. If $\pm(z_j - z_k) \in K_p$, then $(z_i - z_j) + (z_j - z_k)$ is a root in K_p since p is a subalgebra. So $i \sim k$. If $\pm(z_j + z_k) \in K_p$, then $(z_i - z_j) + (z_j + z_j)$ is a root in K_p and again $i \sim k$. Similarly if $\pm(z_i + z_j) \in K_p$. In any case the reductivity of p gives that $\sim$ is an equivalence relation.

By a suitable relabelling of $1, \cdots, n$ we can assume that the equivalence classes come in blocks. The length of an equivalence block is the number of equivalent numbers in the block.

Lemma 3.4. *Let $f \in \operatorname{Per}(n) \cap W_p$. Then f preserves the equivalence relation, i.e., $i \sim j$ implies $f(i) \sim f(j)$.*

Proof. By definition since $f(z_i) = z_{f(i)}$.

Lemma 3.5. *Let p be a primitive, maximal rank, reductive subalgebra of g. If $i \sim j$, then either $\pm z_i \pm z_j$ and $\pm 2z_i$ are in K_p (subject to the constraint that the roots are also in g), or else there exists exactly one equivalence class.*

Proof. Let m be the vector space defined by

$$K_m = K_p \cup \bigcup_{i,j} \{\pm z_i \pm z_j\} \cup \bigcup_{i=1}^{n} \{\pm 2z_i\},$$

when the added roots are in g. We show that m is a subalgebra by a case by case examination. For $A_{n-1}, K_m = K_p$ since p is assumed to be reductive. For B_n, the roots of K_p are of the form $z_i \pm z_j$ or $\pm z_i$. First note that $i \sim j$ and $\pm z_i \in K_p$ implies that $\pm z_j \in K_p$, since p is a reductive subalgebra. (E.g., if $z_i - z_j$ and $-z_i$ are in K_p, then $-z_j \in K_p$ since p is a subalgebra. Using $K_p = -K_p$ we have the result for any of the possibilities.) A root in $K_m - K_p$ has the form $z_i \pm z_j$. To show m is a subalgebra when g is B_n we need only to show that if φ is in K_m and $\varphi + (\pm z_i \pm z_j)$ is a root ($\pm z_i \pm z_j$ a root in $K_m - K_p$), then it is in K_m. φ must have the form $\pm z_i$ or $\pm z_i \pm z_j$. In the first case the only possibility for $\varphi + (\pm z_i \pm z_j)$ to be a root is $\pm z_j$, which is a root in K_p as noted above since $\pm z_i \pm z_j \in K_m$ implies $i \sim j$. If φ has the form $\pm z_k \pm z_l$, then the sum with $\pm z_i \pm z_j$ is a root only if k or l is equal to either i or j. But then $i \sim j$ and $k \sim l$, and that equality gives $i \sim j \sim k \sim l$ so the sum of the roots is in K_m by definition of K_m. D_n is the same as B_n

except that we do not have the z_i's as roots and we may use the same arguments as in B_n. In C_n just note that $2z_i + (-z_i \pm z_j) = z_i \pm z_j$ so that as in B_n the relation $i \sim j$ is not disturbed. So we have shown that in every case m is a subalgebra.

By Lemma 3.4 the permutations in W_p leave K_m invariant. Clearly the action of $Z_2^n \cap W_p$ does not disturb K_m. Thus W_m contains W_p, and by Proposition 3.2, $m = p$ or $m = g$. If $m = g$, then there is exactly one equivalence class; if $m = p$ we have our result.

Theorem 3.6. *Let p be a primitive, maximal rank, reductive subalgebra of one of the classical Lie algebras. Then all of the equivalence classes have the same length or there exist at most two equivalence classes. Moreover, having fixed a Cartan subalgebra in p, the root structure, and the equivalence relation as above, we obtain the following restrictions*:

(i) *In A_{n-1}, all blocks are of the same length.*

(ii) *In B_n, there exist at most two blocks.*

If there are exactly two blocks, then the roots $\pm z_i$ are in K_p for all i in one of the blocks, say the second block. There exists one primitive subalgebra with one block, namely, the algebra of longer roots

$$q_n = h \oplus \sum_{i,j} e_{\pm z_i + z_j} .$$

(iii) *In C_n, the algebras with more than one block contain $e_{\pm 2z_i}$ for all i, i.e., all of the longer roots. There exists one primitive subalgebra with one block, namely,*

$$h \oplus \sum_{i,j} e_{z_i - z_j} .$$

(iv) *In D_n, there exist two primitive subalgebras of one block when n is even, namely,*

$$h \oplus \sum_{i,j=1}^{n} e_{z_i - z_j} , \qquad h \oplus \sum_{i,j=1}^{n-1} e_{z_i - z_j} \oplus \sum_{i=1}^{n-1} e_{\pm (z_i + z_n)} .$$

Finally all of the above algebras are primitive and give a complete classification of the primitive, maximal rank, reductive subalgebras of the classical algebras.

Note. The results for blocks of length greater than two in A_{n-1} were suggested by A. Borel.

Proof. Let g be a classical algebra not equal to B_n. Suppose there exist two blocks of the same length. Let m be the subalgebra defined by "equivalencing" all blocks of the same length, i.e., $m = p \oplus \sum e_\varphi$ where $\varphi = \pm z_i \pm z_j$, and i and j are in blocks of the same length. To check that m is actually a subalgebra, we use the same case by case techniques used in the proof of the last lemma. By Lemma 3.4, W_p is contained in W_m. By primitivity $m = p$ or

$m = g$. By assumption $m \neq p$, so $m = g$. Hence all blocks must have the same length.

Suppose all of the blocks are of different lengths, then let m be formed from p by "equivalencing" two of the blocks, i.e., $m = p \oplus \sum e_{\pm z_i \pm z_j}$ where i is in one of the blocks and j is in the other. By Lemma 3.4 an element of W_p must preserve each block so W_m contains W_p and $m = g$ by primitivity. Hence there must have been only two blocks.

(ii) In B_n, the roots are $\pm z_i$, $\pm z_i \pm z_j$. Suppose $z_i \in K_p$, Then $\pm z_j \in K_p$ for all $j \sim i$ since p is a reductive subalgebra. Also if $z_i, z_j \in K_p$, then $\pm z_i \pm z_j \in K_p$ since p is a subalgebra. Thus equivalence blocks either have or do not have their $\pm z_i$'s, and there exists only one block with its z_i's in K_p, say the last block. If p contains some z_i's, then form m from p by "equivalencing" all blocks without z_i's. By Lemma 3.4, W_p is contained in W_m and $m \neq g$, so by primitivity $m = p$, i.e., there were but two blocks to begin with. If no z_i's are in p, then p is contained in the algebra of longer roots. By primitivity p is the algebra of longer roots.

(i) In A_{n-1}, suppose p consists of two blocks of different lengths. Let $m = p \oplus \sum e_\varphi$ where $\varphi = z_i - z_j$ with i in the first block and j in the second block. Clearly $m \neq g$ and $m \neq p$. But by Lemma 3.4, W_m contains W_p, hence p is not primitive.

(iii) Let p be an algebra with more than one block. By Lemma 3.5. K_p contains $2z_i$ for all i, which are the longer roots of C_n. If p has one equivalence block, we then claim that $\pm z_i \pm z_j \in K_p$ for some i and j implies that $p = g$. First $\pm z_i \pm z_j \in K_p$ for all i and j. Choose $k \neq i, j$. By assumption $z_i - z_k$ or $z_i + z_k$ is in K_p, say $z_i - z_k$. Since p is a subalgebra, $z_j - z_k$ and $-z_i - z_k$ are in K_p. Since p is reductive, $\pm z_i \pm z_k$ are in K_p. It is then easy to see that the long roots must be in K_p so that $p = g$. If p is a proper algebra, it is determined by the roots $z_1 + \text{sign}\,(i) z_i$ $(i = 2, \cdots, n)$ since

$$\begin{aligned} z_i + \text{sign}\,(i, j) z_j &= z_i + \text{sign}\,(i) z_1 - \text{sign}\,(i) z_1 - \text{sign}\,(i)\, \text{sign}\,(j) z_j \\ &= z_i - \text{sign}\,(i)\, \text{sign}\,(j) z_j\ , \end{aligned}$$

i.e.,

$$\text{sign}\,(i, j) = -\text{sign}\,(i)\, \text{sign}\,(j)\ .$$

By choosing the element $(1, -\text{sign}\,(2), \cdots, -\text{sign}\,(n))$ from the Z_2^n factor of the Weyl group we see that p is conjugate via an inner automorphism to the algebra where $\text{sign}\,(i) = -1$ for $i = 2, \cdots, n$, i.e.,

$$p = h \oplus \sum_{i,j} e_{z_i - z_j}\ .$$

We show the primitivity of p here. Let m be a subalgebra of g properly containing p. Then K_m contains a root of the form $\pm 2z_i$, $\pm(z_i + z_j)$. Say the sign is $+$. Then K_m contains all such roots $2z_i$ and $z_i + z_j$ since m is a subalgebra containing p. Now the element $(-1, \cdots, -1) \in Z_2^n$ is in W_p since it takes $z_i - z_j$ to $z_j - z_i$. Applied to $z_i + z_j$ we get that $-z_i - z_j$ is in K_m if m is invariant under W_p. So $m = g$, and p is primitive.

(iv) Suppose that p has one equivalence block. Then as above p is determined by sign (2), $\cdots$, sign (n). In D_n, we cannot take arbitrary elements from Z_2^n, since we need the elements with an even number of -1's. Again up to inner automorphism, we can assume that sign $(2) = \cdots =$ sign $(n-1) = -1$. So p is determined by sign (n). Now we divide our discussion into two cases: n even and n odd.

(a) *n even.* The element $(-1, \cdots, -1) \in Z_2^n$ is in W_p. Using the same arguments as in C_n we see that p is primitive.

(b) *n odd.* For any element $(a_1, \cdots, a_n)$ of the Weyl group to be in W_p, we need that a_i have the same sign for all i. Since $\prod_{i=1}^{n} a_i = 1$, $a_i = 1$ for all i. So W_p consists entirely of permutations.

Embed p in

$$m = p \oplus \sum_{i,j=1}^{n} e_{z_i+z_j} \qquad (\text{sign}\,(n) = -1)$$

to get p not primitive. Similarly for sign $(n) = 1$. Also, note that the two primitive algebras given for even n are not conjugate via an inner automorphism.

We must now show that all of these algebras are primitive. We have already settled the question when there is exactly one block. When there are two blocks of different lengths, these algebras are all maximal and hence primitive. Therefore we may assume that we are in the case where all of the blocks have the same length k. Consider the following cyclic permutation:

$$f_s^t = (sk + 1, (t + 1)k, sk + 2, (t + 1)k - 1, \cdots, (s + 1)k, tk + 1)\,.$$

f_s^t interchanges the s-th and the t-th block. So $f_s^t \in W_p$ since f_s^t does not disturb any of the roots in the other blocks.

We now show that the block decompositions give primitive algebras in A_{n-1}, and note that almost exactly the same argument yields the results for C_n and D_n.

Let p_k (k divides n) be the subalgebra of A_{n-1} given by the blocks of length k. Let m be an algebra with W_{p_k} contained in W_m, and suppose m contains but is not equal to p_k. We show that $m = g$, and thus p_k is primitive. By assumption there exists $\varphi \in K_m - K_p$. Let $\varphi = z_i - z_j$. Number the blocks from 0 to $(n/k) - 1$. Then the s-th block includes $sk + 1, \cdots, (s + 1)k$. Assume i is in the s-th block and j is in the t-th block. Since arbitrary permutations within blocks are in W_p, we have that $z_{i'} - z_{j'}$ with i' in the s-th block and j' in the t-th block are in K_m. Now since f_s^t shows that we can permute the blocks arbitrarily, we get that $z_{i'} - z_{j'}$ is in K_m where i' is in any block and j' is in any other block, i.e., $K_m = K_g$ and p is primitive.

In C_n and D_n we note that by an appropriately chosen element of Z_2^n we can show that if $z_{sk} - z_{sk+1} \in K_m$ then $z_{sk} + z_{sk+1} \in K_m$. In C_n, take the element with -1's in the s-th block (from $sk + 1, \cdots, (s + 1)k$). In D_n, when the number of blocks is greater than two, take the element with -1's in the s-th block and in any other block aside from the s-th block so that the number of -1's is even. If the number of blocks is two, the algebra is maximal. q.e.d.

The following table contains a list of all of the maximal rank, primitive, reductive subalgebras of the classical algebras. By T^k we denote the k-dimensional center of the subalgebra.

$A_{n-1}(n \geq 2)$:	$\underbrace{A_{s-1} \oplus \cdots \oplus A_{s-1}}_{r} \oplus T^{r-1}$	$s \cdot r = n$ and $s > 1$
	T^{n-1}	the Cartan subalgebra
$B_n(n \geq 2)$:	$D_{n-1} \oplus T^1$	the algebra of longer roots
	$D_m \oplus B_{n-m}$	$1 \leq m \leq n - 1$
$C_n(n \geq 3)$:	$C_m \oplus C_{n-m}$	$1 \leq m \leq n - 1$ (two blocks)
	$A_{n-1} \oplus T^1$	the one algebra with one block
	$\underbrace{C_s \oplus \cdots \oplus C_s}_{r}$	$s \cdot r = n$ and $s \geq 1$
	$\underbrace{A_1 \oplus \cdots \oplus A_1}_{n}$	the algebra of longer roots
$D_n(n \geq 4)$:	$D_{n-1} \oplus T^1$	
	$D_m \oplus D_{n-m}$	$2 \leq m \leq n/2$
	$\underbrace{D_s \oplus \cdots \oplus D_s}_{s}$	$s \cdot r = n$ and $s > 1$
	T^n	the Cartan subalgebra
	$A_{n-1} \oplus T^1$	when n is even embedded in two non-inner automorphic ways.

In all of these one must take into account the isomorphisms

$$A_1 = B_1 = C_1\ , \quad D_2 = A_1 \oplus A_1\ , \quad B_2 = C_2\ , \quad A_3 = D_3\ .$$

4. The nonreductive primitive subalgebras

We now classify the non-reductive primitive subalgebras of the complex simple Lie algebras. This classification shows that these subalgebras are all maximal (the main theorem of the section) and is obtained by extending some results of Veisfieler. The non-reductive maximal subalgebras are then classified along with the classification of the graded Lie algebras of the kind which Veisfieler considered. The result duplicates the classification of the non-reductive maximal subalgebras which was obtained by Karpelevich [10] and is included only because it follows simply from the results needed for the main theorem. The results on the Veisfieler gradings although not stated before are implicit in some work of Ochiai [15].

We assume that the reader is familiar with Guillemin's description of the Veisfieler gradings [7, §§ 7 and 8], and we will not include any proofs for the statements on graded and filtered algebras, which we make here and are either already proved or essentially proved in [7].

Let g be a complex simple Lie algebra, and G an associated connected Lie group. Let $p \subset g$ be a primitive non-reductive subalgebra, and P^0 the connected Lie subgroup of G associated with p and $P = \mathrm{norm}_G P^0$. The Veisfieler gradings were obtained for the case where p is actually a maximal subalgebra—we extend the methods to the case where p is an arbitrary primitive subalgebra.

Let $l^0 = p$, and choose l^{-1} to be a minimally invariant subspace of g containing p properly which is invariant under $\mathrm{Ad}\, P$.

Define

$$l^{-i-1} = l^{-i} + [l^{-i}, l^{-i}] , \quad l^i = \{x \in l^{i-1} | [x, l^{-1}] \subset l^{i-1}\} \quad \forall i \geq 1 .$$

Lemma 4.1. (a) $[l^i, l^j] \subset l^{i+j}$, $\forall i, j$.

(b) $\mathrm{Ad}\, P$ *leaves* l^i *invariant* $\forall i$.

(c) $\exists k, m$ *(nonnegative integers) such that*

$$g = l^{-k} \supset \cdots \supset l^0 = p \supset \cdots \supset l^{m+1} = \{0\} ,$$

i.e., $\exists$ *a well-defined filtration on* g.

The filtration on g yields a corresponding graded Lie algebra

$$\tilde{g} = \tilde{g}^{-k} \oplus \cdots \oplus \tilde{g}^m$$

where $\tilde{g}^i = l^i / l^{i+1}$. Let $\tilde{g}^- = \tilde{g}^{-1} \oplus \cdots \oplus \tilde{g}^{-k}$ and $\tilde{g}^+ = \tilde{g}^1 \oplus \cdots \oplus \tilde{g}^m$.

Lemma 4.2. (i) $[\tilde{g}^i, \tilde{g}^j] \subset \tilde{g}^{i+j}$, $\forall i, j$.

(ii) $\tilde{g}^{-1}$ *generates* $\tilde{g}^-$ *as a Lie algebra, in fact* $\tilde{g}^{-i} = [\tilde{g}^{-1}, \tilde{g}^{-i+1}]$.

(iii) *If* $\tilde{x} \in \tilde{g}^0 \oplus \tilde{g}^+$ *and* $[\tilde{x}, \tilde{g}^{-1}] = 0$, *then* $\tilde{x} = 0$.

(iv) $\mathrm{Ad}\, P$ *acts irreducibly on* $\tilde{g}^{-1}$.

(v) $\tilde{g}^0$ *acts completely reducibly on* $\tilde{g}^{-1}$ *under the adjoint action.*

(vi) $\tilde{g}^+ \neq 0$. (The nonreductivity of p is needed here.)

Proof. (i)-(iv) are standard. (v) Let V be any nonzero subspace of $\tilde{g}^{-1}$ on which $\tilde{g}^0$ acts irreducibly. Let $\beta \in \mathrm{Ad}\, P$. Then

$$[\beta(V), \tilde{g}^0] = [\beta(V), \beta(\tilde{g}^0)] = \beta[V, \tilde{g}^0] \subseteq \beta(V) ,$$

so $\tilde{g}^0$ leaves invariant and clearly acts irreducibly on $\beta(V)$. Let $V' = \sum_{\beta \in \mathrm{Ad}\, P} \beta(V)$,

V' is nonzero and invariant under $\operatorname{Ad} P$. By (iv), $V' = \tilde{g}^{-1}$. So $\tilde{g}^{-1}$ is written as the sum of irreducible subspaces. Hence $\tilde{g}^0$ acts completely reducibly on $\tilde{g}^{-1}$. (vi) Suppose $\tilde{g}^+ = 0$. Then $\tilde{g}^0 = l^0 = P$.

By (iii) and (iv) $\tilde{g}^0$ acts faithfully and completely reducibly on $\tilde{g}^{-1}$. Hence by a standard theorem (see [8, p. 81]) $\tilde{g}^0 = p$ is reductive, a contradiction.

Theorem 4.3 (*Extended Veisfieler Lemma*). *Let $\tilde{g} = \tilde{g}^{-k} \oplus \cdots \oplus g^m$ be a finite dimensional Lie algebra satisfying the conclusions of* Lemma 4.2, *i.e.*,

(a) $\tilde{g}^{-1}$ *generates $\tilde{g}^-$ as a Lie algebra.*

(b) $\operatorname{Int}_g(p)$ *acts irreducibly on $\tilde{g}^{-1}$.*

(c) *If $\tilde{x} \in \tilde{g}^0 \oplus \tilde{g}^+$ and $[\tilde{x}, \tilde{g}^{-1}] = 0$, then $\tilde{x} = 0$.*

(d) $\tilde{g}^+ \neq 0$.

Then (i) *$\tilde{g}$ is semi-simple,* (ii) $k = m$, (iii) $\tilde{g}^{-i} \cong (\tilde{g}^i)^*$.

The proof is essentially the same as that of Lemma 8.1 in Guillemin [7]. We leave the details of the changes to the reader.

Note that since $\tilde{g}$ is semi-simple the degree derivation is inner, i.e., there exists $D \in \tilde{g}^0$ such that $\operatorname{ad} D \mid \tilde{g}^i = i \times$ identity.

Proposition 4.4. *g is isomorphic to $\tilde{g}$ as Lie algebras. Under this isomorphism $p \cong \tilde{g}^0 \oplus \tilde{g}^+$.*

The proof of this result is the same as the case where p was assumed maximal. We only need the existence of the element $D \in \tilde{g}^0$. Hence we can write g as a graded Lie algebra $g = g^{-k} \oplus \cdots \oplus g^k$ where $p = g^0 \oplus g^+$.

Proposition 4.5. *g^0 is of maximal rank.*

Proof. Let $h \subset g^0 \subset p$ be a Cartan subalgebra of g^0. We show h is a Cartan subalgebra of g. Now h is a Cartan subalgebra of g iff

(i) h is nilpotent, and

(ii) $\operatorname{norm}_{g_0}(h) = h$.

Clearly (i) is unaffected by what algebra h is contained in, hence we need only to show that $\operatorname{norm}_g(h) = \operatorname{norm}_{g_0}(h)$. Every subalgebra q of g containing D is graded since $\operatorname{ad} D\colon q \to q$, and q is then graded by the eigenspace decomposition. We note first that $D \in h$ because $D \in$ center g_0.

Let $\operatorname{norm}_g(h) = q^{-k} \oplus \cdots \oplus q^k$, and $x \in q^i$, $i \neq 0$. Then $[D, x] = ix$, but $[D, x] \in [h, x] \subset h \subset g^0$ by choice of x. Thus $x \in g^0 \cap g^i$ and $x = 0$, so that we have that $\operatorname{norm}_g(h) \subset g_0$ and hence $\operatorname{norm}_g(h) = \operatorname{norm}_{g_0}(h) = h$.

Definition 4.6. A subalgebra of a simple Lie algebra is *parabolic* if it contains a maximal solvable subalgebra. Give a Cartan subalgebra h of g and a system Π of simple roots on h. Then the algebra $h \oplus \sum_{\varphi \geq 0} e_\varphi$ is a maximal solvable subalgebra. Let $Q = \sum_{\varphi \geq 0} e_\varphi$.

Proposition 4.7. *In the notation above, there exists a system of simple roots $\alpha_1, \cdots, \alpha_n$ where rank $g = n$ such that $g^+ \subset Q_\alpha$. Moreover $p = g^0 \oplus g^+$ is parabolic.*

Proof. $h \oplus g^+$ is a solvable subalgebra and is hence contained in a maximal solvable subalgebra in $m \oplus Q_\Pi$ where m is some Cartan subalgebra of g and Π

is a system of simple roots on m. Now m and h are both Cartan subalgebras of the algebra $m \oplus Q_\Pi$. By the Cartan subalgebra conjugacy theorem there exists an inner automorphism β of $m \oplus Q_\Pi$ such that $\beta(h) = m$. β extends to be an inner automorphism of g. Define a system of simple roots on h by $\alpha'_i = \alpha_i \circ \beta$ on $h \cdot \Pi' = \{\alpha'_1, \cdots, \alpha'_n\}$. Then $Q_{\Pi'} = \beta^{-1}(Q_\Pi)$.

Since β preserves $m \oplus Q_\Pi$ we have that $h \oplus Q_{\Pi'} \subset m \oplus Q_\Pi$. By dimension they are equal. Therefore $h \oplus g^+ \subset h \oplus Q_{\Pi'}$.

Now $g^+ \subset Q_{\Pi'}$. In fact, let $x \in g^i$, $i > 0$. Then $ix = [D, x] \in [h, h \oplus Q_{\Pi'}] \subset Q_{\Pi'}$. Since $i \neq 0$, we have that $x \in Q_{\Pi'}$. Next note that $e_\varphi \subset g^i$ iff $\varphi(D) = i$, where φ is a nonzero root. In fact, let $x \in e_\varphi$. Since $D \in h$, $[D, x] = \varphi(D)x$. Thus x is always an eigenvector for $\operatorname{ad} D$ and clearly the above holds. Let φ be a positive root. Suppose $\varphi(D) < 0$. Then $e_{-\varphi} \subset g^+ \subset Q_{\Pi'}$ since $-\varphi(D) > 0$. But then $e_{-\varphi}$ is in the space generated by positive root spaces, a contradiction since $-\varphi$ is a negative root. Thus if $\varphi > 0$, then $\varphi(D) \geq 0$, i.e., $e_\varphi \subset g^0 \oplus g^+$. Hence $h \oplus Q_{\Pi'} \subset g^0 \oplus g^+$, and $g^0 \oplus g^+$ is parabolic.

Theorem 4.8. *If p is a primitive and parabolic subalgebra of a simple Lie algebra, then p is maximal. In particular, any primitive, non-reductive subalgebra is maximal.*

Proof. If p is parabolic, $P = P^0$ (see [2, Theorem 4, p. 24]). Hence P can be a maximal Lie subgroup only if p is a maximal Lie subalgebra.

We can now assume that g^0 acts irreducibly on g^{-1} (see Guillemin [7]).

Theorem 4.9. *Let $h \subset g^0$ be a Cartan subalgebra, and $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ the set of simple roots given by Proposition 4.7. Then $\alpha_i(D)$ is either 0 or 1 for all i, and there exists exactly one i for which $\alpha_i(D) = 1$.*

Proof. (a) $\alpha_i(D)$ is either 0 or 1 ($1 \leq i \leq n$), and $e_{\alpha_i} \subset g^0 \oplus g^+$ since α_i is positive. Therefore $[e_{\alpha_i}, g^{-1}] \neq 0$ by property (iii) of Lemma 4.2. g^{-1} is the direct sum of root spaces e_φ where $\varphi(D) = -1$. Hence there exists a root space e_φ such that $[e_\varphi, e_{\alpha_i}] \neq 0$, i.e., $\varphi + \alpha_i$ is a root. Let $\varphi = \sum n_j \alpha_j$ be the decomposition of φ into simple roots. φ is a negative root, so $n_j \leq 0$ for all j. Now $\varphi + \alpha_i$ is a root. If $\alpha_i(D) > 1$, then $(\varphi + \alpha_i)(D) > 0$, and $\varphi + \alpha_i$ is a positive root, implying that all $n_j \geq 0, j \neq i$, i.e., $\varphi = -\alpha_i$. But then $\varphi(D) = -1$ implies $\alpha_i(D) = 1$, a contradiction, so $0 \leq \alpha_i(D) \leq 1$.

(b) Since $D \neq 0$, there exists α_i such that $\alpha_i(D) \neq 0$, i.e., $\alpha_i(D) = 1$. Define $V_i = \sum e_\varphi$ where $\varphi(D) = -1$. If $\varphi = \sum n_j \alpha_j$, then $n_i \neq 0$. First we show V_i to be invariant under g_0 where $g_0 = h \oplus \sum_{\psi(D)=0} e_\psi$. Clearly $[h, V_i] \subset V_i$. Now either $[e_\psi, e_\varphi] = 0$, or $\varphi + \psi$ is a root and $[e_\psi, e_\varphi] \subset e_{\varphi+\psi}$. If $\varphi + \psi$ is a root, then $(\varphi + \psi)(D) = -1$. Let $\psi = \sum m_j \alpha_j$ be the expansion of ψ in terms of simple roots. Then $m_i = 0$; otherwise, assuming ψ is positive we obtain

$$\psi(D) \geq m_i \alpha_i(D) \geq \alpha_i(D) = 1 > 0 ,$$

a contradiction. Similarly if ψ is a negative root. So in terms of its expansion in simple roots $\varphi + \psi$ has a nonzero coefficient on α_i since φ does. Hence $e_{\varphi+\psi} \subset V_i$. Now $e_{-\alpha_i} \subset V_i$, hence $V_i \neq 0$. Since g_0 acts irreducibly on g^{-1}, we have $V_i = g^{-1}$. Suppose that $\alpha_j(D) = 1$ also with $j \neq i$. Then $e_{-\alpha_j} \subset V_j = g^{-1} = V_i$. But cleary $e_{-\alpha_j}$ is not contained in V_i. So this cannot happen.

Corollary 4.10. *In the Veisfieler graded algebras (those graded algebras satisfying* (ii), (iii), (vi) *of Lemma* 4.2 *and having g^0 act irreducibly on g^{-1}), there exist a Cartan subalgebra h of g^0 and a system of simple roots on h with a distinguished simple root α such that*

(i) $g^0 = h \oplus \sum e_\varphi$ *where the coefficient on α in the expansion of φ in terms of simple roots is zero,*

(ii) $g^i = \sum e_\varphi$ *where the coefficient of α in the expansion of φ in terms of simple roots is i.*

Corollary 4.11. *Let p be a non-reductive maximal subalgebra of a simple Lie algebra g. Then p is of maximal rank, and there exist a Cartan subalgebra $h \subset p$ and a system of simple roots with a distinguished simple root α such that $p = h \oplus \sum e_\varphi$ where the expansion of φ in terms of simple roots has a nonnegative coefficient on α.*

Added in proof. We should note that the primitive, maximal rank, reductive subalgebras of the (complex) exceptional Lie algebras have been classified in the paper: M. Golubitsky & B. Rothschild, *Primitive subalgebras of exceptional Lie algebras,* Pacific J. Math. **39** (1971) 371–393.

Bibliography

[1] A. Borel & J. de Siebenthal, *Les sous-groupes fermés de rang maximum des groupes de Lie clos,* Comment. Math. Helv. **23** (1949) 200–221.

[2] N. Bourbaki, *Groupes et algebres de Lie,* Hermann, Paris, 1968, Chapitres 4, 5 and 6.

[3] E. Cartan, *Les groupes de transformations continus, infinis, simples,* Ann. Sci. École Norm. Sup. **26** (1909) 93–161.

[4] E. B. Dynkin, *Semisimple subalgebras of semisimple Lie algebras,* Mat. Sb. (N.S.) **30** (72) (1952) 349–462; English transl. Amer. Math. Soc. Transl. (2) **6** (1957) 111–244.

[5] ——, *The maximal subgroups of the classical groups,* Trudy Moskov Mat. Obšč. **1** (1952) 39–166; English transl., Amer. Math. Soc. Transl. (2) **6** (1957) 245–378.

[6] M. Golubitsky, *Primitive actions and maximal subgroups of Lie groups.* Thesis, Massachusetts Institute of Technology, 1970.

[7] V. Guillemin, *Infinite dimensional primitive Lie algebras,* J. Differential Geometry **4** (1970) 257–282.

[8] V. Guillemin, D. Quillen & S. Sternberg, *The classification of the irreducible complex algebras of infinite type,* J. Analyse Math. **18** (1967) 107–112.

[9] N. Jacobson, *Lie algebras,* Interscience, New York, 1962.

[10] F. I. Karpelevich, *On nonsemisimple maximal subalgebras of semisimple Lie algebras,* Dokl. Acad. Nauk SSSR (N.S.) **76** (1951) 775–778, (Russian).

[11] S. Kobayashi & T. Nagano, *On filtered Lie algebras and geometric structures.* I, J. Math. Mech. **13** (1964) 875–907.

[12] ——, *On filtered Lie algebras and geometric structures.* II, J. Math. Mech. **14** (1965) 513–521.

[13] ——, *On filtered Lie algebras and geometric structures*. III, J. Math. Mech. **14** (1965) 679–706.

[14] V. V. Morosoff, *Sur les groups primitifs,* Mat. Sb. (N.S.) **5** (47) (1939) 355–390, (Russian).

[15] V. V. Morozov, *On nonsemisimple maximal subgroups of simple groups,* Dissertation, Kazan Gos. Univ. 1943, (Russian).

[16] T. Ochiai, *Classification of the finite nonlinear primitive Lie algebras,* Trans. Amer. Math. Soc. **124** (1966) 313–322.

[17] I. M. Singer & S. Sternberg, *The infinite groups of Lie and Cartan. Part I* (*The transitive groups*), J. Analyse Math. **15** (1965) 1–114.

[18] S. Sternberg & M. Hausner, *The structure of semisimple Lie algebras,* RIAS Publications, TR 58–9, 1958.

[19] B. Yu. Veisfieler, *Infinite dimensional filtered Lie algebras with graded Lie algebras,* Funkcional Anal. i. Priložen. **2** (1) (1968) 94–95.

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