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A FORMULA FOR THE RADIAL PART OF THE LAPLACE-BELTRAMI OPERATOR

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Let V be a manifold and H a Lie transformation group of V. Suppose $Du = 0$ is a differential equation on V, both the differential operator D and the function u assumed invariant under H. Then the differential equation will involve several inessential variables, a fact which may render general results about differential operators rather ineffective for the differential equation at hand. Thus although D may not be an elliptic operator it might become one after the inessential variables are eliminated (cf. [3, p. 99]).

This viewpoint leads to the general definition (cf. [7]) of the transversal part and radial part of a differential operator on V given in §§ 2 and 3. The radial part has been constructed for many special differential operators in the literature; see for example [1], [3], [4], [5], [8] for Lie groups, Lie algebras and symmetric spaces, [9], [6] for some Lorentzian manifolds. Our main result, formula (3.3) in Theorem 3.2, includes various known examples worked out by computations suited for each individual case. See Harish-Chandra [4, p. 99] for the Laplacian on a semisimple Lie algebra, Berezin [1] and Harish-Chandra [3, § 8] for the Laplacian on a semisimple Lie group, and Harish-Chandra [5, § 7] and Karpelevič [8, § 15] for the Laplacian on a symmetric space. The author is indebted to J. Lepowsky for useful critical remarks.

Notation. If V is a manifold and $v \in V$, then the tangent space to V at v will be denoted V_v; the differential of a differentiable mapping φ of one manifold into another is denoted $d\varphi$. We shall use Schwartz' notation $\mathscr{E}(V)$ (resp. $\mathscr{D}(V)$) for the space of complex-valued C^∞ functions (resp. C^∞ functions of compact support) on V. Composition of differential operators D_1, D_2 is denoted $D_1 \circ D_2$.

2. The transversal part of a differential operator

Let V be a manifold satisfying the second axiom of countability, and H a Lie transformation group of V. If $h \in H$, $v \in V$, let $h \cdot v$ denote the image of v under H and let H^v denote the isotropy subgroup of H at v. Let $\mathfrak{h}$ denote the Lie algebra of H. If $X \in \mathfrak{h}$, let X^+ denote the vector field on V *induced by* X, i.e.,

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$$(2.1) \qquad (X^+f)(v) = \left\{\frac{d}{dt} f(\exp tX \cdot v)\right\}_{t=0}, \quad f \in \mathcal{D}(V), \quad v \in V .$$

A C^∞ function f on an open subset of V is said to be *locally invariant* if $X^+f = 0$.

Lemma 2.1. *Suppose $W \subset V$ is a submanifold such that for each $w \in W$ the tangent spaces at w satisfy the condition*:

$$(2.2) \qquad V_w = W_w + (H \cdot w)_w \qquad (\textit{direct sum}).$$

Let $w_0 \in W$. Then there exists an open relatively compact neighborhood W_0 of w_0 in W and a relatively compact submanifold $B \subset H, e \in B$ such that the natural projection $\pi: H \to H/H^{w_0}$ is a diffeomorphism of B onto an open neighborhood U_0 of $\pi(e)$ in H/H^{w_0} and such that the mapping $\eta: (b, w) \to b \cdot w$ is a diffeomorphism of $B \times W_0$ onto an open neighborhood V_0 of w_0 in V.

Proof. Let $\mathfrak{h}^0$ denote the Lie algebra of H^{w_0}, and $\mathfrak{n} \subset \mathfrak{h}$ any subspace complementary to $\mathfrak{h}^0$. Then the mapping $\varphi: (X, w) \to \exp X \cdot w$ of $\mathfrak{n} \times W$ into V is regular at $(0, w_0)$. In fact, since $(d\varphi)_{(0,w_0)}$ fixes W_{w_0}, it suffices to prove

$$(2.3) \qquad (d\varphi)_{(0,w_0)}(\mathfrak{n} \times 0) = (H \cdot w_0)_{w_0} .$$

This however is clear from dimensionality considerations. Now the lemma follows from the standard fact that if $\mathfrak{n}_0$ is a sufficiently small neighborhood of 0 in $\mathfrak{n}$, then exp is a diffeomorphism of $\mathfrak{n}_0$ onto a submanifold $B \subset H$ diffeomorphic under π to an open neighborhood of w_0 in H/H^{w_0}.

It was pointed out to me by R. Palais that the local integration of involutive distributions (Chevalley [3, p. 89]) shows that a submanifold W satisfying (2.2) always exists.

Now let us assume that V has a Riemannian structure g invariant under the action of H. Assuming furthermore that all the orbits of H have the same dimension, we shall with each differential operator D on V associate a new differential operator D_T on V which acts "transversally to the orbits".

Fix $s_0 \in V$ and let S denote the orbit $H \cdot s_0$. For each $s \in S$ consider the geodesics in V starting at s, perpendicular to S. If we take sufficiently short pieces of these geodesics, their union is a submanifold $S_s^\perp$ of V. Shrinking $S_{s_0}^\perp$ if necessary we may assume that it satisfies transversality condition (2.2) for W. Take w_0 as s_0, and let W_0, B and V_0 be as in the lemma. For $f \in \mathcal{E}(V)$ (or even for functions defined on V_0) we define a new function f_{s_0} on V_0 by

$$f_{s_0}(b \cdot w) = f(w) , \quad b \in B , \quad w \in W_0 .$$

We then define D_T by

$$(2.4) \qquad (D_T f)(s_0) = (D f_{s_0})(s_0) , \qquad s_0 \in V .$$

Since $B \cdot w$ is a neighborhood of w in the orbit $H \cdot w$, and since D decreases supports, the choice of B above is immaterial, and (2.4) is indeed a valid defini tion; the operator D_T decreases supports and is therefore a differential operator, which we call *the transversal part* of D.

Theorem 2.2. *Let V be a Riemannian manifold, H a Lie transformation group of isometries of V, all orbits assumed to have the same dimension. Let S be any H-orbit and let $\bar{f}$ denote restriction of a function f to S. Then the Laplace-Beltrami operators $L = L_V$ and L_S on V and S, respectively, satisfy*

$$(Lf)^- = L_S\bar{f} + (L_T f)^- \qquad f \in \mathscr{E}(V) . \tag{2.5}$$

Proof. Let $(y_1, \cdots, y_r)$ be any coordinate system on B such that $y_1(e) = \cdots = y_r(e) = 0$, and let $w \to (z_{r+1}(w), \cdots, z_n(w))$ be a coordinate system on W_0 such that the geodesics forming $S_{s_0}^{\perp}$ correspond to the straight lines through 0. Then we define a coordinate system $(x_1, \cdots, x_n)$ on V_0 by

$$\begin{aligned}(x_1(b \cdot w), \cdots, x_r(b \cdot w), x_{r+1}(b \cdot w), \cdots, x_n(b \cdot w))\\ = (y_1(b), \cdots, y_r(b), z_{r+1}(w), \cdots, z_n(w)) .\end{aligned}$$

The Laplace-Beltrami operator is given by

$$L = \sum_{p,q=1}^{n} g^{pq}(\partial_{pq} - \sum_t \Gamma^t_{pq}\partial_t) ,$$

where $\partial_p = \partial/\partial x_p$, $\partial_{pq} = \partial^2/\partial x_p \partial x_q$, g^{pq} is the inverse of the matrix $g_{pq} = g(\partial_p, \partial_q)$, and Γ^t_{pq} is the Christoffel symbol

$$\Gamma^r_{pq} = \tfrac{1}{2} \sum_s g^{rs}(\partial_q g_{ps} + \partial_p g_{qs} - \partial_s g_{pq}) .$$

Suppose $\psi \in \mathscr{E}(V_0)$ satisfies the condition

$$\psi(x_1, \cdots, x_n) \equiv \psi(0, \cdots, 0, x_{r+1}, \cdots, x_n) , \tag{2.6}$$

or equivalently

$$\psi(b \cdot w) = \psi(w) , \quad b \in B , \quad w \in W_0 .$$

Then

$$\psi = \psi_{s_0} , \qquad (L\psi)(s_0) = (L_T\psi)(s_0) . \tag{2.7}$$

On the other hand, suppose $\varphi \in \mathscr{E}(V_0)$ satisfies

$$\varphi(x_1, \cdots, x_n) \equiv \varphi(x_1, \cdots, x_r, 0, \cdots, 0) , \tag{2.8}$$

or equivalently

$$\varphi(b\cdot w) = \varphi(b\cdot s_0)\ , \quad b \in B\ , \quad w \in W_0\ .$$

For each set of real numbers $a_{r+1}, \cdots, a_n$, not all 0, the curve

$$t \longrightarrow (x_1(s_0), \cdots, x_r(s_0), a_{r+1}t, \cdots, a_n t)$$

is a geodesic in V. The differential equation for geodesics

$$\ddot{x}_i + \sum_{p,q} \Gamma^i_{pq}\dot{x}_p\dot{x}_q = 0$$

(dot denoting differentiation with respect to t) therefore shows that

$$\Gamma^i_{\alpha\beta}(s_0) = 0\ , \quad 1 \le i \le n\ , \quad r+1 \le \alpha, \beta \le n\ .$$

Since the geodesic is perpendicular to S at s_0,

(2.9) $$g_{i\alpha}(s_0) = g^{i\alpha}(s_0) = 0\ , \quad \text{for} \quad 1 \le i \le r\ , \quad r+1 \le \alpha \le n\ .$$

It follows that

$$(L\varphi)(s_0) = \sum_{1\le i,j\le r} g^{ij}(\partial_{ij}\varphi - \sum_{1\le k\le r} \Gamma^k_{ij}\partial_k\varphi)(s_0)\ .$$

But by (2.9), $\Gamma^k_{ij}(s_0)$ is the same for S and for V, so

(2.10) $$(L\varphi)(s_0) = (L_S\bar{\varphi})(s_0)\ .$$

But

$$L(\varphi\psi) = \varphi L\psi + 2g\,(\text{grad}\,\varphi, \text{grad}\,\psi) + \psi L\varphi\ ,$$

where for any $f \in \mathscr{E}(V_0)$,

$$\text{grad}\, f = \sum_{p,q} g^{pq}(\partial_p f)\partial_q\ .$$

Hence (2.9) implies

(2.11) $$L(\varphi\psi)(s_0) = \varphi(s_0)(L\psi)(s_0) + \psi(s_0)(L\varphi)(s_0)\ .$$

But φ_{s_0} is a constant function, so by (2.4) and (2.7)

$$\varphi_{s_0}(L\psi)(s_0) = L((\varphi\psi)_{s_0})(s_0) = (L_T(\varphi\psi))(s_0)\ .$$

Similarly, since $\bar{\psi}$ is a constant function, (2.10) implies

$$\psi(s_0)(L\varphi)(s_0) = L_S(\bar{\varphi}\bar{\psi})(s_0)\ .$$

This gives formula (2.5) for the function $f = \varphi\psi$, and since the linear combinations of such products form a dense subspace of $\mathscr{D}(V_0)$ the theorem follows by approximation.

Remark. The theorem remains true with the same proof if V is a manifold with a pseudo-Riemannian structure g provided g is nonsingular on S.

3. The radial part of a differential operator

Again let V be a manifold satisfying the second axiom of countability, and H a Lie transformation group of V. Suppose $W \subset V$ is a submanifold satisfying transversality condition (2.2) in Lemma 2.1.

Lemma 3.1. *Let D be a differential operator on V. Then there exists a unique differential operator $\Delta(D)$ on W such that*

$$(Df)^{-} = \Delta(D)\bar{f} \tag{3.1}$$

for each locally invariant function f on an open subset of V, the bar denoting restriction to W.

Proof. Let $w_0 \in W$ and select W_0, B and V_0 as in Lemma 2.1. If $\varphi \in \mathscr{E}(W_0)$, we define f on V_0 by

$$f(b \cdot w) = \varphi(w) \ , \quad b \in B \ , \quad w \in W_0 \ .$$

The mapping $\varphi \to (Df)^{-}$ gives an operator $D_{w_0, W_0, B}$ of $\mathscr{E}(W_0)$ into itself. It is now an easy matter to verify that the linear transformation $\Delta(D)$ given by

$$(\Delta(D)\psi)(w_0) = (D_{w_0, W_0, B}\psi)(w_0)$$

is a well-defined differential operator on $\mathscr{E}(W)$, with the properties stated in the lemma.

The operator $\Delta(D)$ is called the *radial part* of D. We shall now give a formula for the radial part of the Laplace-Beltrami operator on V under a strengthening of transversality assumption (2.2); in fact we assume that each H-orbit intersects W just once and orthogonally.

Theorem 3.2. *Suppose V is a Riemannian manifold, H a closed unimodular subgroup of the Lie group of all isometries of V (with the compact open topology). Let $W \subset V$ be a submanifold satisfying the condition: For each $w \in W$,*

$$(H \cdot w) \cap W = \{w\}; \ V_w = (H \cdot w)_w \oplus W_w \ , \tag{3.2}$$

where $\oplus$ denotes orthogonal direct sum. Let L_V and L_W denote the Laplace-Beltrami operators on V and W, respectively. Then

$$\Delta(L_V) = \delta^{-\frac{1}{2}} L_W \circ \delta^{\frac{1}{2}} - \delta^{-\frac{1}{2}} L_W(\delta^{\frac{1}{2}}) \ , \tag{3.3}$$

where the function δ is the volume element ratio in (3.8) *below.*

Proof. Let V^* denote the subset $H \cdot W$ of V. Since the mapping $(h, w) \to h \cdot w$ of $H \times W$ into V has (by (3.2)) a surjective differential at each point, V^* is an open subset of V. Since H is closed, the isotropy subgroup H^w at each point $w \in W$ is compact and the orbit $H \cdot w$ is closed; if we fix a left invariant Haar measure on H and a Haar measure on H^w (with total measure 1), we obtain in a standard way an H-invariant measure $d\dot{h}$ on each orbit $H \cdot w = H/H^w$. Denoting by dv and dw the Riemannian measures on V and W, respectively, we shall prove that there exists a function $\delta \in \mathscr{E}(W)$ such that

$$\int_{V^*} F(v)dv = \int_W \delta(w) \left(\int_{H \cdot w} F(h \cdot w) d\dot{h} \right) dw , \qquad F \in \mathscr{D}(V^*) . \tag{3.4}$$

Let $w_0 \in W$. Because of the second part of (3.2) there exist a coordinate neighborhood W_0 of w_0 in W, a vector subspace $\mathfrak{m} \subset \mathfrak{h}$ of dimension $\dim V - \dim W$ and a neighborhood $\mathfrak{m}_0$ of 0 in $\mathfrak{m}$ such that the map

$$\eta \colon (X, w) \to \exp X \cdot w$$

is a diffeomorphism of $\mathfrak{m}_0 \times W_0$ onto an open neighborhood V_0 of w_0 in V. Let $(x_1, \cdots, x_r)$ be a Cartesian coordinate system on $\mathfrak{m}$, and $(x_{r+1}, \cdots, x_n)$ an arbitrary coordinate system on W_0. In the formulas below let $1 \leq i, j \leq r, r + 1 \leq \alpha, \beta \leq n$. Let the coordinate system $(x_1, \cdots, x_n)$ on V_0 be determined by

$$x_i (\exp X \cdot w) = x_i(X) , \qquad x_\alpha (\exp X \cdot w) = x_\alpha(w) .$$

Let g denote the Riemannian structure of V, and put $g_{pq} = g(\partial_p, \partial_q)$ as usual, so that

$$dv = \bar{g}^{\frac{1}{2}} dx_1 \cdots dx_n , \qquad dw = \bar{\gamma}^{\frac{1}{2}} dx_{r+1} \cdots dx_n ,$$

where

$$\bar{g} = |\det ((g_{pq})_{1 \leq p, q \leq n})| , \qquad \bar{\gamma} = |\det (g_{\alpha\beta})| . \tag{3.5}$$

Because of the orthogonality in (3.2) we have

$$g_{i\alpha}(w) = 0 , \qquad w \in W_0 . \tag{3.6}$$

But if $h = \exp X$ ($X \in \mathfrak{m}_0$) then our choice of coordinates implies for the differential dh,

$$dh \left(\frac{\partial}{\partial x_\alpha} \right)_w = \left(\frac{\partial}{\partial x_\alpha} \right)_{h \cdot w} , \qquad dh \left(\frac{\partial}{\partial x_i} \right)_w = \sum_{j=1}^{r} a_{ij} \left(\frac{\partial}{\partial x_j} \right)_{h \cdot w} ,$$

where $a_{ij} \in \mathbf{R}$. Hence $g_{\alpha\beta}(h \cdot w) = g_{\alpha\beta}(w)$ and using (6), $g_{i\alpha}(h \cdot w) = 0$; consequently

$$\bar{g}(h \cdot w) = \det (g_{ij})(h \cdot w) \bar{\gamma}(w) . \tag{3.7}$$

However

$$|\{\det(g_{ij})\}^{\frac{1}{2}}(h\cdot w)|\,dx_1\cdots dx_r(h\cdot w)$$

is just the Riemannian volume element $d\sigma_w$ on the orbit $H\cdot w$. Thus, if $F\in\mathscr{D}(V_0)$ we obtain from the Fubini theorem and (3.7) that

$$\int_V F(v)dv = \int_W \bar{\gamma}^{\frac{1}{2}}(w)\left(\int_{H\cdot w} F(p)d\sigma_w(p)\right)dx_{r+1}\cdots dx_n(w)\ .$$

But $d\sigma_w$ is invariant under H, so it must be a scalar multiple of $d\dot{h}$,

$$d\sigma_w = \delta(w)d\dot{h}\ . \tag{3.8}$$

This proves (3.4) for all $F\in\mathscr{D}(V_0)$; then it holds also if F has support inside $h\cdot V_0$ for some $h\in H$. But as w_0 runs through W, the sets $h\cdot V_0$ form a covering of V^*. Passing to a locally finite refinement and a corresponding partition of unity, (3.4) follows for all $F\in\mathscr{D}(V^*)$.

Let $\dot{F}(w)$ denote the inner integral in (3.4), so that

$$\dot{F}(w) = \int_{H\cdot w} F(h\cdot w)d\dot{h}\ . \tag{3.9}$$

It is a routine matter to verify that the mapping $F\rightarrow\dot{F}$ is surjective, i.e.,

$$\mathscr{D}(V^*)^{\cdot} = \mathscr{D}(W)\ . \tag{3.10}$$

For the determination of $\Delta(L_V)$ we first observe that

$$\Delta(L_V) = L_W + \text{lower order terms.} \tag{3.11}$$

This is clear from the coordinate expression for L_V together with (3.6) if we also note that the vector fields $\partial/\partial x_i$ are tangential to the H-orbits. Next we recall that L_V is symmetric with respect to dv, i.e.,

$$\int_V (L_Vf_1)(v)f_2(v)dv = \int_V f_1(v)(L_Vf_2)(v)dv \tag{3.12}$$

for all $f_1, f_2\in\mathscr{D}(V^*)$. But then this relation holds for all $f_2\in\mathscr{E}(V^*)$. In particular we can use it on f_2 invariant under H. Applying (3.4) to the left hand side of (3.12) we obtain

$$\int_W \delta(w)f_2(w)\left(\int_{H\cdot w}(L_Vf_1)(h\cdot w)d\dot{h}\right)dw\ . \tag{3.13}$$

But for each $v\in V$ the isotropy subgroup H^v is compact, so by invariance of L_V

$$(L_V)_v\left(\int_H f_1(h\cdot v)\,dh\right) = \int_H (L_Vf_1)(h\cdot v)dh\ .$$

Now putting here $v = w$ we get the inner integral in (3.13) equal to $(\Delta(L_V)\dot{f}_1)(w)$; thus the left hand side of (3.12) is

$$\int_W (\Delta(L_V)\dot{f}_1)(w)\bar{f}_2(w)\delta(w)dw ,$$

the bar denoting restriction to W. But using the H-invariance of $L_V f_2$, formula (3.4) and the definition of radial part, the right hand side of (3.12) reduces to

$$\int_W \dot{f}_1(w)(\Delta(L_V)\bar{f}_2)(w)\delta(w)dw .$$

But in view of (3.10) the functions $\dot{f}_1$ (and of course the $\bar{f}_2$) fill up $\mathscr{D}(W)$, so the equality of the two last expressions implies that $\Delta(L_V)$ is symmetric with respect to $\delta(w)dw$. Now since L_W is symmetric with respect to dw, a simple computation shows that the composition $\delta^{-\frac{1}{2}}L_W \circ \delta^{\frac{1}{2}}$ is symmetric with respect to $\delta(w)dw$ and it clearly agrees with L_W up to lower order terms. Thus by (3.11) the symmetric operators $\Delta(L_V)$ and $\delta^{-\frac{1}{2}}L_W \circ \delta^{\frac{1}{2}}$ agree up to an operator of order ≤ 1. But this operator, being symmetric, must be a function, and now (3.3) follows by applying the operators to the function 1.

It is of interest to generalize Theorem 3.2 to pseudo-Riemannian manifolds V. If V has a pseudo-Riemannian structure g, which for each $w \in W$ is non-degenerate on the closed orbit $H\cdot w$, and if each H^w $(w \in W)$ is compact, then Theorem 3.2 remains valid. In fact, the isotropy group H^v is then compact for each $v \in V^*$, so no change is necessary in the proof.

When a semisimple Lie group H acts on its Lie algebra by the adjoint representation, the regular elements of a Cartan subalgebra constitute a transversal submanifold W where the isotropy subgroup H^w is the same for all $w \in W$. This then provides an example for the following variation of Theorem 3.2.

Theorem 3.3. *Let the assumptions be as in Theorem* 3.2 *except that V has only a pseudo-Riemannian structure g. Then formula* (3.3) *remains valid if we further assume that*

(i) *for each $w \in W$ the orbit $H\cdot w$ is closed and g is non-degenerate on it,*

(ii) *H^w is the same for all $w \in W$, and its Lie algebra is its own normalizer in the Lie algebra of H.*

Proof. Put $H^0 = H^w$ $(w \in W)$ and $\dot{h} = hH^0$, and fix an H-invariant measure $d\dot{h}$ on the coset space H/H^0. Such a measure exists since each orbit $H\cdot w$ has an H-invariant measure $d\sigma_w$ defined as above. If γ is a geodesic in V tangential to W at w then γ is left fixed by each $h \in H_0$. Thus (ii) implies $\gamma \subset W$ so W is a totally geodesic submanifold of V. Defining δ by (3.8) the only part of the proof above which requires change is the justification of the formula

$$\int_{H\cdot w} (L_V f_1)(h\cdot w)d\dot{h} = (\Delta(L_V)\dot{f}_1)(w) . \tag{3.14}$$

For this we use Theorem 2.2 and the subsequent remark to split L_V into its "orbital part" and transversal part. The orbital part gives integral 0 over $H\cdot w$, so in the integral (3.14) we can replace L_V by its transversal part $L_{V,T}$. Putting $f_1^h(w) = f_1(h\cdot w)$ for $h \in H$, we have, by the H-invariance of $L_{V,T}$,

$$(L_{V,T}f_1)(h\cdot w) = (L_{V,T}(f_1^h))(w) ,$$

which, by the definition of transversal part and radial part, equals $\Delta(L_V)(\overline{f_1^h})(w)$, W being totally geodesic. But then the left hand side of (3.14) equals

$$\int_{H/H^0} \Delta(L_V)_w(f_1(\dot{h}\cdot w))d\dot{h} ,$$

which equals $(\Delta(L_V)\dot{f}_1)(w)$ because now $\dot{h}$ and w are independent variables. This proves (3.14) and therefore also Theorem 3.3.

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