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AUTOMORPHISMS AND INTEGRABILITY OF PLANE FIELDS

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A p-plane field on an n-dimensional manifold is a section in the bundle associated to the tangent bundle with fiber the Grassmann manifold of p-planes in affine space $\boldsymbol{R}^n$. It is integrable if each point has a neighborhood U homeomorphic to affine space in such a way that the restriction of the plane field to U is carried by the induced tangent map onto a field of parallel planes. Since a field of parallel planes in $\boldsymbol{R}^n$ is preserved by any translation, the restriction to U of the field admits a transitive abelian group of automorphisms, that is, homeomorphisms such that their tangent maps take the field onto itself. In this paper, we shall prove the converse.

Theorem. *A p-plane field is integrable if and only if each point has a neighborhood homeomorphic to affine space on which the restriction of the field admits a transitive abelian group of automorphisms. The homeomorphisms occurring in the definition of integrability and in the automorphism groups are of the same class C^k for some $k = 0, 1, \cdots, \infty$.*

This theorem follows immediately from the preceding remarks and the following lemma:

Lemma 1. *Let G be a transitive abelian subgroup of the group of homeomorphisms of class C^k of $\boldsymbol{R}^n$, where $k = 0, 1, \cdots, \infty$. Then G is conjugate to the group of translations, and the conjugating element is unique up to an affine map.*

Indeed, suppose the lemma holds. Let $f\colon U \to \boldsymbol{R}^n$ be a homeomorphism, G_1 be a transitive abelian group of automorphisms of the restriction of the field to U, and T be the group of translations in $\boldsymbol{R}^n$. Then there exists a homeomorphism g of $\boldsymbol{R}^n$ such that

$$gfG_1f^{-1}g^{-1} = T \ .$$

Hence the tangent map induced by gf takes the given p-plane field into one preserved by the translation group of $\boldsymbol{R}^n$, that is, a parallel field. Hence the p-plane field is integrable as required.

It remains to prove Lemma 1. The idea of the proof is to topologize the given

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group G so that it becomes an abelian topological group homeomorphic to $\boldsymbol{R}^n$, hence isomorphic in the category of topological groups to the additive group of $\boldsymbol{R}^n$. This isomorphism will be used to construct the required conjugating homeomorphism. We first state and prove some additional lemmas required for the proof of Lemma 1.

Lemma 2. *A transitive abelian subgroup of the homeomorphism group of* $\boldsymbol{R}^n$ *is simply transitive.*

Proof. Since the group is transitive and abelian, all the isotropy subgroups are conjugate and identical; the latter means that any element, which leaves one point fixed, leaves all points fixed and therefore is the identity. Hence there cannot be two distinct elements which carry a given point to another given point.

Lemma 3. *The normalizer of the translation group in the homeomorphism group of* $\boldsymbol{R}^n$ *is the affine group.*

Proof. It is well-known that the translation group is normal in the affine group. On the other hand, suppose f is a homeomorphism such that

$$fTf^{-1} = T ,$$

where T is the translation group. Let $f(0) = x_0$ and set

$$g(x) = f(x) - x_0 , \qquad x \in \boldsymbol{R}^n .$$

Then given any y, there is a z such that for all x

$$g(x + y) = f(x + y) - x_0 = f(x) + z - x_0 .$$

Setting $x = 0$, we get $g(y) = z$, so

$$g(x + y) = f(x) + g(y) - x_0 = g(x) + g(y) .$$

Since g is continuous, this equation implies that it is linear, and hence that f is affine as required.

We can now proceed with the proof of Lemma 1. Let G be given the point-open topology, that is, the topology generated by all sets of the form

$$M(x, W) = \{f \mid f(x) \in W\} ,$$

where $x \in \boldsymbol{R}^n$ and W is an open set of $\boldsymbol{R}^n$. Since G is abelian, it is easily proved that

$$M(x, W) = M(0, h(W)) ,$$

where $h(x) = 0$. Let g_x denote the unique element of G such that $g_x(0) = x$, and let ϕ: $\boldsymbol{R}^n \to G$ be defined by $\phi(x) = g_x$. Clearly, ϕ is a homeomorphism, and the group operation is continuous as a function of each factor separately.

G is a topological group by a theorem of Ellis [1], is isomorphic in the category of topological groups to a Lie group by a theorem to which many authors have contributed [2, p. 184], and must be the additive group of $\boldsymbol{R}^n$ by the classification theorem for abelian Lie groups [2, p. 187]. Hence there exists a continuous open isomorphism $\eta\colon G \to \boldsymbol{R}^n$. Let $\mathscr{D}$ be the homeomorphism group of $\boldsymbol{R}^n$, and $\psi\colon \mathscr{D} \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ be its natural action. Define $\rho\colon \boldsymbol{R}^n \to T$ by taking $\rho(a)$ to be the translation which takes 0 to a. Then we have the commutative diagram:

$$
\begin{array}{ccc}
G \times \boldsymbol{R}^n & \xrightarrow{\psi} & \boldsymbol{R}^n \\
\Big\downarrow{\scriptstyle \mathrm{id}\, x\phi} & & \Big\downarrow{\scriptstyle \phi} \\
G \times G & \xrightarrow{\circ} & G \\
\Big\downarrow{\scriptstyle \eta \times \eta} & & \Big\downarrow{\scriptstyle \eta} \\
\boldsymbol{R}^n \times \boldsymbol{R}^n & \xrightarrow{+} & \boldsymbol{R}^n \\
\Big\downarrow{\scriptstyle \rho x\, \mathrm{id}} & & \Big\downarrow{\scriptstyle \mathrm{id}} \\
T \times \boldsymbol{R}^n & \xrightarrow{\psi} & \boldsymbol{R}^n
\end{array}
$$

By following around the full diagram in both directions, we obtain

$$\eta\phi G\phi^{-1}\eta^{-1} = T$$

as required. If ζ is any other conjugating element, then $\eta\phi\zeta^{-1}$ lies in the normalizer of T, so by Lemma 2 it is affine. Let $\psi^*\colon \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ be defined by

$$\psi^*(a, x) = \psi(\eta^{-1}(a), x) \ .$$

By a theorem of Bochner and Montgomery [2, p. 212], if each element of G is differentiable of class C^k, then ψ^* is also of class C^k in all its variables simultaneously. If $\eta^{-1}(a) = \phi(y)$, then

$$\psi^*(a, 0) = \phi^{-1}(\eta^{-1}(a)) \ ,$$

so $(\eta\phi)^{-1}$ is also of class C^k. Its Jacobian is nowhere zero since the action ψ^* is generated by n independent commuting vector fields, none of which can have any zero points because of simple transitivity. This completes the proof of Lemma 1, and with it, the theorem.

References

[1] R. Ellis, *Locally compact transformation groups*, Duke Math. J. **24** (1957) 119–125.

[2] D. Montgomery & L. Zippen, *Topological transformation groups*, Interscience, New York, 1955; and references contained therein.

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