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# REMARKS ON CURVATURE AND THE EULER INTEGRAND

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## 1. Review of the problem

We define an *n-dimensional curvature tensor* as an $n^4$-tuple $R = (R_{ijkl})$, $1 \leq i, j, k, l \leq n$, of real numbers satisfying the symmetry relations

$$R_{ijkl} = -R_{jikl} = R_{klij} \ , \tag{1}$$

and

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0 \ , \tag{2}$$

and we denote the vector space of all such curvature tensors by $K^n$.

The polynomial function $\chi^{2n}$: $K^{2n} \to \boldsymbol{R}$ defined by the formula

$$\chi^{2n}(R) = (-1)^n \sum_{i,j} \varepsilon_{i_1 \cdots i_{2n}} \varepsilon_{j_1 \cdots j_{2n}} R_{i_1 i_2 j_1 j_2} \cdots R_{i_{2n-1} i_{2n} j_{2n-1} j_{2n}}$$

will be called the Euler integrand in dimension $2n$ since, by the generalized Gauss-Bonnet theorem, the Euler characteristic of an oriented riemannian manifold $M$ of dimension $2n$ is obtained, up to a positive constant, by evaluating $\chi^{2n}$ on the components of the curvature tensor in orthonormal frames and integrating the resulting real valued function over $M$, using the volume element associated with the given riemannian metric.

It has been conjectured by H. Hopf that the Euler characteristic of an even dimensional riemannian manifold with positive sectional curvature is positive, and it may even be the case that the Euler integrand is positive in this situation. The present note is devoted to the presentation of some remarks on this question. We continue by fixing some more terminology.

The polynomial function $\sigma^n$: $K^n \times \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{R}$ defined by the formula

$$\sigma^n(R, x, y) = - \sum R_{ijkl} x_i y_j x_k y_l$$

may be called the sectional curvature function, since $\sigma^n(R, x, y)$ is, up to a positive constant, the sectional curvature of the plane spanned by $x$ and $y$, computed from the curvature tensor $R$. Of course, $x$ and $y$ really span a plane

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if and only if $\alpha^n(x, y) \neq 0$, where $\alpha^n: \boldsymbol{R}^n \times \boldsymbol{R}^n \to R$ is the polynomial function defined by

$$\alpha^n(x, y) = \sum_{i,j} (x_i y_j - x_j y_i)^2 .$$

$R$ is called *positive sectional* if $\sigma^n(R, x, y) > 0$ whenever $\alpha^n(x, y) \neq 0$.

The conjecture, "if $R \in K^{2n}$ is positive sectional, then $\chi^{2n}(R) > 0$" will be denoted by $\mathscr{C}^{2n}$. $\mathscr{C}^2$ is trivially true, and $\mathscr{C}^4$ has been proven by Milnor and Chern [1].

## 2. First remark on the conjectures $\mathscr{C}^{2n}$

**Theorem 1.** *For each $n$, there is a finite decision procedure for determining whether $\mathscr{C}^{2n}$ is true.*

The proof of Theorem 1 depends on a deep result of Seidenberg and Tarski concerning semi-algebraic sets, a subset of a vector space being called semi-algebraic if it is generated by unions and intersections from the solution sets of a finite number of polynomial equations and inequalities. The Seidenberg-Tarski theorem states that, if $V$ and $W$ are vector spaces, then the projection onto $V$ of a semi-algebraic subset of $V \times W$ is a semi-algebraic subset of $V$. The proof [2] of this theorem gives a finite procedure for constructing the equations and inequalities defining the projection from those defining the original set. Unfortunately, the procedure is too long to be used in practice even with the aid of a computer, so the conjectures $\mathscr{C}^{2n}$ should remain of interest to geometers and algebraists.

**Proposition 1.** *The set $P^n$ of positive sectional $n$-dimensional curvature tensors is semi-algebraic in $K^n$.*

*Proof.* Let $S^n \subseteq K^n \times \boldsymbol{R}^n \times \boldsymbol{R}^n$ be the semi-algebraic set

$$\{(R, x, y) \mid \alpha^n(x, y) \neq 0 \text{ and } \sigma^n(R, x, y) \leq 0\} ,$$

whose projection onto $K^n$ is the complement of $P^n$. By the Tarski-Seidenberg theorem and the obvious fact that the complement of a semialgebraic set is semialgebraic, $P^n$ is semialgebraic. q.e.d.

Proposition 1 implies that there exist finitely many polynomial inequalities in in the $R_{ijkl}$'s such that, given any curvature tensor, one could determine whether it is positive sectional by evaluating the polynomials and checking whether the results satisfy the inequalities. (There are no equations, because, as is easily verified, $P^n$ is an open subset of $K^n$.) It would be useful to know these inequalities explicitly. They could be used, for example, in a computer procedure to generate a random sample of the elements of $P^{2n}$, on which $\chi^{2n}$ could be evaluated for an empirical test of $\mathscr{C}^{2n}$.

*Proof of Theorem* 1. Let $T^{2n} \subseteq \boldsymbol{R}^0 \times K^{2n}$ be the set

$$\{(0, R) \mid R \in P^{2n} \text{ and } \chi(R) \leq 0\} .$$

$T^{2n}$ being semialgebraic, so is its projection $U^{2n}$ onto $\boldsymbol{R}^0$, which is empty if and only if $\mathscr{C}^{2n}$ is true. But one may decide by checking a finite number of polynomial inequalities, derivable in a finite number of steps from the data of the problem, whether the unique element 0 of $\boldsymbol{R}^0$ lies in $U^{2n}$.

### 3. The importance of the Bianchi identity

An $n^4$-tuple $R$ satisfying the relations (1) of § 1 will be called a generalized curvature tensor, whether or not it satisfies the Bianchi identities (2). We denote the space of all generalized curvature tensors by $\tilde{K}^n$. All the functions and definitions in the previous paragraphs extend in the obvious way to $\tilde{K}^n$, and there is a corresponding sequence of conjectures $\tilde{\mathscr{C}}^{2n}$.

First, we observe that the Chern-Milnor proof of $\mathscr{C}^4$ is even a proof of $\tilde{\mathscr{C}}^4$ (i.e., the Bianchi identity is never used), and that $\tilde{\mathscr{C}}^2$ is trivially true. The following result suggests the source of some of the difficulty in proving $\mathscr{C}^{2n}$ for $n > 2$.

**Theorem 2.** *$\tilde{\mathscr{C}}^{2n}$ is false for $n > 2$.*

*Proof.* Let $R \in \tilde{K}^{2n}$ be defined by the formulas

$$(3) \qquad R_{1234} = -R_{2134} = -R_{1243} = R_{3412} = -1 ,$$

$$(4) \qquad R_{5612} = -R_{6512} = -R_{5621} = R_{1256} = 1 ,$$

$$(5) \qquad R_{3456} = -R_{4356} = -R_{3465} = R_{5634} = -1 ,$$

$$(6) \qquad R_{2k-1,2k,2k-1,2k} = -R_{2k,2k-1,2k-1,2k} = -R_{2k-1,2k,2k,2k-1} = -1 , \quad \text{for } 4 \leq K \leq n ,$$

$$(7) \qquad R_{ijkl} = 0 , \qquad \text{for all other values of } i, j, k, l .$$

It is not hard to see that the part of $R$ coming from (3), (4), and (5) contributes nothing to sectional curvature. What is left is the curvature tensor of the product of a 6-dimensional flat space and $(n - 3)$ 2-dimensional spaces of positive curvature, so $R$ is non-negative sectional. (We will make it positive in a moment.)

Now all the non-vanishing terms in $\chi^{2n}(R)$ may be shown to be equal, by even permutations of the indices, to

$$(-1)^n \cdot R_{1234} R_{3456} R_{5612} R_{7878} \cdots R_{2n-1,2n,2n-1,2n} ,$$

which equals $(-1)^n \cdot (-1) \cdot 1 \cdot (-1) \cdot (-1)^{n-3} = (-1)^{2n-1} = -1$, so that $\chi^{2n}(R)$ is negative.

Letting $S$ be any positive sectional curvature tensor (for instance, the one for constant sectional curvature), we write $R_\delta$ for $R + \delta \cdot S$. Since the function $\sigma$ is linear in its first argument, $R_\delta$ is positive sectional for $\delta > 0$. For $\delta$ sufficiently small, the continuity of $\chi$ implies that $\chi(R_\delta) < 0$, and $R_\delta$ is a counterexample to $\tilde{\mathscr{C}}^{2n}$.

## References

[1] S. S. Chern, *On curvature and characteristic classes of a Riemannian manifold*, Abh. Math. Sem. Univ. Hamburg **20** (1956) 117–126.
[2] N. Jacobson, *Lectures in abstract algebra*, Vol. 3, Van Nostrand, Princeton, 1964.

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