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REDUCTION OF THE CODIMENSION OF AN ISOMETRIC IMMERSION

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0. Introduction

Let $\psi: M^n \to \tilde{M}^{n+p}(\tilde{c})$ be an isometric immersion of a connected n-dimensional Riemannian manifold M^n into an $(n+p)$-dimensional Riemannian manifold $\tilde{M}^{n+p}(\tilde{c})$ of constant sectional curvature $\tilde{c}$. When can we reduce the codimension of the immersion, i.e., when does there exist a proper totally geodesic submanifold N of $\tilde{M}^{n+p}(\tilde{c})$ such that $\psi(M^n) \subset N$? We prove the following:

Theorem. *If the first normal space $N_1(x)$ is invariant under parallel translation with respect to the connection in the normal bundle and l is the constant dimension of N_1, then there exists a totally geodesic submanifold N^{n+l} of $\tilde{M}^{n+p}(\tilde{c})$ of dimension $n+l$ such that $\psi(M^n) \subset N^{n+l}$.*

This theorem extends some results of Allendoerfer [2].

1. Notation and some formulas of Riemannian geometry

Let $\psi: M^n \to \tilde{M}^{n+p}(\tilde{c})$ be as in the introduction. For all local formulas we may consider ψ as an imbedding and thus identify $x \in M^n$ with $\psi(x) \in \tilde{M}^{n+p}$. The tangent space $T_x(M^n)$ is identified with a subspace of the tangent space $T_x(\tilde{M}^{n+p})$. The normal space $T_x^\perp$ is the subspace of $T_x(\tilde{M}^{n+p})$ consisting of all $X \in T_x(\tilde{M}^{n+p})$ which are orthogonal to $T_x(M^n)$ with respect to the Riemannian metric g. Let ∇ (respectively $\tilde{\nabla}$) denote the covariant differentiation in M^n (respectively $\tilde{M}^{n+p}$), and D the covariant differentiation in the normal bundle. We will refer to ∇ as the tangential connection and D as the normal connection.

With each $\xi \in T_x^\perp$ is associated a linear transformation of $T_x(M^n)$ in the following way. Extend ξ to a normal vector field defined in a neighborhood of x and define $-A_\xi X$ to be the tangential component of $\tilde{\nabla}_X \xi$ for $X \in T_x(M^n)$. $A_\xi X$ depends only on ξ at x and X. Given an orthonormal basis $\xi_1, \cdots, \xi_p$ of $T_x^\perp$ we write $A_\alpha = A_{\xi_\alpha}$ and call the A_α's the second fundamental forms associated with $\xi_1, \cdots, \xi_p$. If $\xi_1, \cdots, \xi_p$ are now orthonormal normal vector fields in a neighborhood U of x, they determine normal connection forms $s_{\alpha\beta}$ in U by

$$D_X \xi_\alpha = \sum_\beta s_{\alpha\beta}(X) \xi_\beta$$

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for $X \in T_x(M^n)$. We let R^N denote the curvature tensor of the normal connection, i.e.,

$$R^N(X, Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}.$$

We then have the following relationships (in this paper Greek indices run from 1 to p):

$$\tilde{\nabla}_X Y = \nabla_X Y + \sum_\alpha g(A_\alpha X, Y)\xi_\alpha , \tag{1}$$

$$g(A_\alpha X, Y) = g(X, A_\alpha Y) , \tag{2}$$

$$\tilde{\nabla}_X \xi_\alpha = -A_\alpha X + D_X \xi_\alpha = -A_\alpha X + \sum_\beta s_{\alpha\beta}(X)\xi_\beta , \tag{3}$$

$$s_{\alpha\beta} + s_{\beta\alpha} = 0 , \tag{4}$$

$$(\nabla_X A_\alpha)Y - \sum_\beta s_{\alpha\beta}(X)A_\beta Y = (\nabla_Y A_\alpha)X - \sum_\beta s_{\alpha\beta}(Y)A_\beta X \tag{5}$$

— Codazzi equation,

$$\begin{aligned}(\nabla_X s_{\alpha\beta})Y - (\nabla_Y s_{\alpha\beta})X &= 2(ds_{\alpha\beta})(X, Y)\\ &= X \cdot s_{\alpha\beta}(Y) - Y \cdot s_{\alpha\beta}(X) - s_{\alpha\beta}([X, Y])\\ &= g([A_\alpha, A_\beta]X, Y) + \sum_\gamma \{s_{\alpha\gamma}(X)s_{\gamma\beta}(Y) - s_{\alpha\gamma}(Y)s_{\gamma\beta}(X)\}\end{aligned} \tag{6}$$

— Ricci equation,

$$\begin{aligned}R^N(X, Y)\xi_\alpha &= \sum_\beta g([A_\alpha, A_\beta]X, Y)\xi_\beta\\ &= \sum_\beta \{2(ds_{\alpha\beta})(X, Y) + \sum_\gamma \{s_{\alpha\gamma}(Y)s_{\gamma\beta}(X) - s_{\alpha\gamma}(X)s_{\gamma\beta}(Y)\}\}\xi_\beta ,\end{aligned} \tag{7}$$

where X and Y are tangent to M^n.

The first normal space $N_1(x)$ is defined to be the orthogonal complement of $\{\xi \in T_x^\perp \mid A_\xi = 0\}$ in $T_x^\perp$. $\boldsymbol{R}^k$ will denote the k-dimensional Euclidean space, $S^k(1)$ the k-dimensional unit sphere in $\boldsymbol{R}^{k+1}$, and $H^k(-1)$ the k-dimensional simply connected space form of constant sectional curvature -1. All immersions, vector fields, etc., are assumed to be of C^∞.

2. Reducing the codimension of an isometric immersion

Let ψ: $M_n \to \tilde{M}^{n+p}(\tilde{c})$ be an isometric immersion of a connected n-dimensional Riemannian manifold M^n into an $(n+p)$-dimensional Riemannian manifold $\tilde{M}^{n+p}(\tilde{c})$ of constant sectional curvature $\tilde{c}$.

Lemma 1. *Suppose the first normal space $N_1(x)$ is invariant under parallel translation with respect to the normal connection and l is the constant dimension of N_1. Let $N_2(x) = N_1^\perp(x)$, where the orthogonal complement is taken in*

$T_x^\perp$, and for $x \in M^n$ let $\mathscr{S}(x) = T_x(M^n) + N_1(x)$. Then for any $x \in M^n$ there exists differentiable orthonormal normal vector fields $\xi_1, \cdots, \xi_p$ defined in a neighborhood U of x such that:

(a) *For any $y \in U$, $\xi_1(y), \cdots, \xi_l(y)$ span $N_1(y)$, and $\xi_{l+1}(y), \cdots, \xi_p(y)$ span $N_2(y)$,*

(b) *$\tilde{\nabla}_X \xi_\alpha = 0$ in U for $\alpha \geq l + 1$ and X tangent to M^n,*

(c) *The family $\mathscr{S}(y), y \in U$, is invariant under parallel translation with respect to the connection in $\tilde{M}^{n+p}$ along any curve in U.*

Proof. Since N_1 is invariant under parallel translation with respect to the normal connection, so is N_2. Let $x \in M^n$ and choose orthonormal normal vectors $\xi_1(x), \cdots, \xi_p(x)$ at x such that $\xi_1(x), \cdots, \xi_l(x)$ span $N_1(x)$ and $\xi_{l+1}(x), \cdots, \xi_p(x)$ span $N_2(x)$. Extend $\xi_1, \cdots, \xi_p$ to differentiable orthonormal normal vector fields defined in a normal neighborhood U of x by parallel translation with respect to the normal connection along geodesics in M^n. This proves (a).

Since N_1 and N_2 are invariant under parallel translation with respect to the normal connection, we have $D_X \xi \in N_1$ (respectively N_2) for $\xi \in N_1$ (respectively N_2). Let $\xi_1, \cdots, \xi_p$ be chosen as in (a). Then $s_{\alpha\beta} = 0$ in U for $1 \leq \alpha \leq l$, $l + 1 \leq \beta \leq p$ and $1 \leq \beta \leq l$, $l + 1 \leq \alpha \leq p$. Equations (6) and (7) imply that $R^N(X, Y)\xi = 0$ for $\xi \in N_2$, and since N_2 is also invariant under parallel translation with respect to the normal connection we conclude that for $\xi \in N_2(y)$, $y \in U$, the parallel translation of ξ with respect to the normal connection is independent of path in U. Thus $D\xi_\alpha = 0$ in U for $\alpha \geq l + 1$, and $s_{\alpha\beta} = 0$ in U for $l + 1 \leq \alpha \leq p$, $l + 1 \leq \beta \leq p$. Because of (3), we have $\tilde{\nabla}_X \xi_\alpha = 0$ for $\alpha \geq l + 1$ and X tangent to M^n, proving (b).

To prove (c) it suffices to show that $\tilde{\nabla}_X Z \in \mathscr{S}$ whenever $Z \in \mathscr{S}$ and X is tangent to M^n. This follows from (1) and (3) and (a) and (b) above.

We shall now prove our Theorem under the assumption that $\tilde{M}^{n+p}$ is simply connected and complete. We consider the cases $\tilde{c} = 0$, $\tilde{c} > 0$ and $\tilde{c} < 0$ separately.

Proposition 1. *The Theorem is true if $\tilde{M}^{n+p} = \boldsymbol{R}^{n+p}$.*

Proof. Let $x \in M^n$ and let $\xi_1, \cdots, \xi_p$, and U be as in Lemma 1. Define functions f_α on U by $f_\alpha = g(\vec{x}, \xi_\alpha)$ where $\vec{x}$ is the position vector. Then

$$X \cdot f_\alpha = \tilde{\nabla}_X f_\alpha = g(X, \xi_\alpha) + g(\vec{x}, \tilde{\nabla}_X \xi_\alpha) = 0$$

for $\alpha \geq l + 1$ and X tangent to U. Thus U lies in the intersection of $p - l$ hyperplanes, whose normal vectors are linearly independent, and the desired result is true locally; i.e., if $x \in M^n$ there exist a neighborhood U of x and a Euclidean subspace $\boldsymbol{R}^{n+l}$ such that $\phi(U) \subset \boldsymbol{R}^{n+l}$. To get the global result we use the connectedness of M^n. Let $x, y \in M^n$ with neighborhoods U and V respectively such that $U \cap V \neq \phi$ and $\phi(U) \subset \boldsymbol{R}_1^{n+l}$, $\phi(V) \subset \boldsymbol{R}_2^{n+l}$. Then

$$\phi(U \cap V) \subset \boldsymbol{R}_1^{n+l} \cap \boldsymbol{R}_2^{n+l} .$$

If $\boldsymbol{R}_1^{n+l} \neq \boldsymbol{R}_2^{n+l}$ then $\boldsymbol{R}_2^{n+l} \cap \boldsymbol{R}_2^{n+l} = \boldsymbol{R}^{n+k}$, $k < l$, and this implies that $\dim N_1(z) < l$ for $z \in U \cap V$. Since $\dim N_1 = \text{constant} = l$, we must have $\boldsymbol{R}_1^{n+l} = \boldsymbol{R}_2^{n+l}$. This proves the global result.

Proposition 2. *The Theorem is true if* $\tilde{M}^{n+p} = S^{n+p}(1)$.

Proof. Consider $S^{n+p}(1)$ as the unit sphere in $\boldsymbol{R}^{n+p+1}$ with center at the origin of $\boldsymbol{R}^{n+p+1}$. Let ξ be the inward pointing unit normal of S^{n+p}, $\bar{N}_1(x)$ be the first normal space for M^n considered as immersed in $\boldsymbol{R}^{n+p+1}$, $\bar{\nabla}$ be the Euclidean connection in $\boldsymbol{R}^{n+p+1}$, and $\xi_1, \cdots, \xi_p$ be chosen as in Lemma 1. Then $\bar{\nabla}_X \xi = -X$ and $\bar{\nabla}_X \xi_\alpha = \tilde{\nabla}_X \xi_\alpha$ for X tangent to M^n. It readily follows that $\bar{N}_1(x) = N_1(x) + \operatorname{span}\{\xi(x)\}$ and that $\bar{N}_1$ is invariant under parallel translation with respect to the normal connection for M^n considered as immersed in $\boldsymbol{R}^{n+p+1}$. Thus, by Proposition 1, there exists an $\boldsymbol{R}^{n+l+1}$ such that $\phi(M^n) \subset \boldsymbol{R}^{n+l+1}$, namely,

$$\boldsymbol{R}^{n+l+1} = T_x(M^n) + N_1(x) + \operatorname{span}\{\xi(x)\} ,$$

for any $x \in M^n$. Hence $\boldsymbol{R}^{n+l+1}$ contains ξ and therefore passes through the origin of $\boldsymbol{R}^{n+p+1}$. Thus

$$\phi(M^n) \subset \boldsymbol{R}^{n+l+1} \cap S^{n+p}(1) = S^{n+l}(1) .$$

Proposition 3. *Our theorem is true if* $\tilde{M}^{n+p} = H^{n+p}(-1)$.

Proof. It is convenient to consider H^{n+p} as being in a Minskowski space E^{n+p+1}. Let E^{n+p+1} be a Minskowski space with global coordinates $x^0, \cdots, x^{n+p}$ and pseudo-Riemannian metric g determined by the quadratic form

$$g(x, y) = -x_0 y_0 + x_1 y_1 + \cdots + x_{n+p} y_{n+p} .$$

Consider the submanifold H^{n+p} defined by

$$-x_0^2 + x_1^2 + \cdots + x_{n+p}^2 = -1, \; x_0 > 0.$$

The pseudo-Riemannian metric $g(\ ,\ )$ on E^{n+p+1} induces a Riemannian metric on H^{n+p} such that H^{n+p} becomes a simply connected Riemannian manifold of constant sectional curvature -1 (cf. [4, p. 66]). Let $\xi = \vec{x}$, the position vector. Then for $x \in H^{n+p}$, $\xi(x)$ is normal to H^{n+p} and $g(\xi(x), \xi(x)) = -1$. Let $\bar{\nabla}$ be the Euclidean connection on E^{n+p+1}, i.e., the connection arising from g; and define A by $\bar{\nabla}_X \xi = -AX$ for X tangent to H^{n+p}. Then $A = -I$ and

$$\bar{\nabla}_X Y = \tilde{\nabla}_X Y - g(AX, Y)\xi$$

for X, Y tangent to H^{n+p}. The minus sign, rather than a plus sign as in (1), occurs in the last equation because g is indefinite. Let $\xi_1, \cdots, \xi_p$ be as in Lemma 1 and consider M^n as isometrically immersed in E^{n+p+1}. Then $\tilde{\nabla}_X \xi_\alpha$

$\bar{\nabla}_X \xi_\alpha$ for X tangent to M^n. In a way similar to the argument in Proposition 2 we can show that

$$W(x) = \mathscr{S}(x) + \text{span}\,\{\xi(x)\} = T_x(M^n) + N_1(x) + \text{span}\,\{\xi(x)\}$$

is invariant under parallel translation with respect to the Euclidean connection in E^{n+p+1}. Thus, in a way similar to the argument in Proposition 1, there exists an $(n+l+1)$-dimensional plane E^{n+l+1} $(=W(x)$ for any $x \in M^n)$ such that $\psi(M^n) \subset E^{n+l+1}$. We may assume that the point $x_0 = 1$, $x_k = 0$ for $k \geq 1$ is in $\psi(M^n)$. Then, since E^{n+l+1} contains ξ and passes through the point $x_0 = 1$, $x_k = 0$ for $k \geq 1$, we conclude that E^{n+l+1} is perpendicular to the $x_0 = 0$ plane and passes through the origin of E^{n+p+1}. Thus $H^{n+p} \cap E^{n+l+1}$ is totally geodesic in H^{n+p}, and

$$\psi(M^n) \subset H^{n+l}(-1) = H^{n+p}(-1) \cap E^{n+l+1}.$$

Clearly completeness is not essential in Propositions 1, 2, and 3 in the sense that if $\tilde{M}^{n+p}$ is a connected open set of $\boldsymbol{R}^{n+p}$, S^{n+p}, or H^{n+p} then Propositions 1, 2, and 3 remain true. Thus when $\tilde{M}^{n+p}(\tilde{c})$ is neither simply connected nor complete we obtain the local result: if $x \in M^n$, then there exists a neighborhood U of x such that $\psi(U)$ is contained in a totally geodesic submanifold N_U^{n+l} of $\tilde{M}^{n+p}$. We obtain the global result (the Theorem) by a connectedness argument similar to the connectedness argument in Proposition 1.

Remarks. It is an easy consequence of Codazzi's equation that if the type number of ψ (see [3, vol. II, p. 349]) is greater than or equal to two and N_1 has constant dimension, then N_1 is invariant under parallel translation with respect to the normal connection. To prove this last remark, let l be the dimension of N_1 and choose orthonormal normal vectors $\xi_1, \cdots, \xi_p$ in a neighborhood U of x such that $\xi_1, \cdots, \xi_l$ span $N_1(y)$ for $y \in U$ (cf. § 3). Since the type number of the immersion is greater than or equal to two, there exist X and Y tangent to M^n such that $A_j X$ and $A_j Y$, $1 \leq j \leq l$, are linearly independent. Codazzi's equation then implies that

$$\sum_{\beta=1}^{l} s_{\alpha\beta}(X) A_\beta Y = \sum_{\beta=1}^{l} s_{\alpha\beta}(Y) A_\beta X \ ,$$

for $\alpha \geq l+1$, since $A_\beta = 0$ for $\beta > l$. Since $A_\beta Y$ and $A_\beta X$, $1 \leq \beta \leq l$, are linearly independent we conclude that $s_{\alpha\beta}(X) = s_{\alpha\beta}(Y) = 0$ for $\alpha > l \geq \beta$. But, for any Z tangent to M^n, we have

$$\sum_{\beta=1}^{l} s_{\alpha\beta}(X) A_\beta Z = \sum_{\beta=1}^{l} s_{\alpha\beta}(Z) A_\beta X \ .$$

Thus $s_{\alpha\beta}(Z) = 0$ for $\alpha > l \geq \beta$. We conclude that $D_Z \xi \in N_1$ if Z is tangent to M^n and $\xi \in N_1$. Thus N_1 is invariant under parallel translation with respect to the normal connection.

3. The higher normal spaces

Let $\psi: M^n \to \tilde{M}^{n+p}(\tilde{c})$ be as in § 1, and h the second fundamental form of the immersion, i.e., for X, Y tangent to M^n, $h(X, Y)$ is the normal component of $\tilde{\nabla}_X Y$. Equation (1) of § 1 may be written as

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) .$$

Following Allendoerfer [1] we define the normal spaces as follows. The first normal space $N_1(x)$ is defined to be the

$$\operatorname{span} \{h(X, Y) \mid X, Y \in T_x(M^n)\} .$$

Choosing orthonormal normal vectors $\xi_1, \cdots, \xi_p$ at x such that $\xi_1, \cdots, \xi_l$ span $N_1(x)$, where l is the dimension of $N_1(x)$, and using (1) one easily sees that this agrees with our previous definition for $N_1(x)$ given in § 1. Suppose $N_1, \cdots, N_k$ have been defined such that $N_i \perp N_j$ for $i \neq j$. If

$$N_1(x) + \cdots + N_k(x) \neq T_x^\perp$$

define $N_{k+1}(x)$ as follows: Let

$$L(x) = \operatorname{span} \{(D_{Z_1}(D_{Z_2}(\cdots (D_{Z_k}(h(Z_{k+1}, Z_{k+2}))) \cdots)))_x\} ,$$

where $Z_1, \cdots, Z_{k+2}$ are vector fields tangent to M^n. If

$$L(x) \cap (N_1(x) + \cdots + N_k(x))^\perp$$

is not equal to $\{0\}$, where the orthogonal complement is in $T_x^\perp$, define $N_{k+1}(x)$ to be

$$L(x) \cap (N_1(x) + \cdots + N_k(x))^\perp .$$

Otherwise define $N_{k+1}(x)$ to be

$$(N_1(x) + \cdots + N_k(x))^\perp .$$

It is clear that we may speak of the last normal space.

Note the following lemma.

Lemma. *If each $N_k(x)$ has constant dimension n_k, then there exist orthonormal normal vector fields $\xi_1, \cdots, \xi_p$ in a neighborhood U of x such that $\xi_{n_1+\cdots+n_{k-1}+1}, \cdots, \xi_{n_k}$ span $N_k(y)$ for $y \in U$.*

Proof. Choose vector fields X_i and Y_i, $1 \leq i \leq n_1$, in a neighborhood of x such that $(h(X_k, Y_i))_x$ are linearly independent and span $N_1(x)$. Since $h(X_i, Y_i)$, $1 \leq i \leq n_1$, are differentiable normal vector fields in a neighborhood of x and linearly independent at x, they are linearly independent in a neighborhood of x. But N_1 has constant dimension and $h(X_i, Y_i) \in N_1$; using the Gram-

Schmidt orthogonalization process we obtain orthonormal normal vector fields $\xi_1, \cdots, \xi_{n_1}$ in a neighborhood U of x such that $\xi_1, \cdots, \xi_{n_1}$ span $N_1(y)$ for $y \in U$. Now suppose $\xi_1, \cdots, \xi_{n_1+\cdots+n_k}$ have been found with the desired property. If N_{k+1} is the last normal space, then

$$N_{k+1} = (N_1 + \cdots + N_k)^{\perp} .$$

By using an orthonormal basis of the normal space in a neighborhood of x and $\xi_1, \cdots, \xi_{n_1+\cdots+n_k}$ above, it is clear that we may find an orthonomal basis of N_{k+1} in a neighborhood of x. If N_{k+1} is not the last normal space, then we may obtain $\bar{\xi}_i$, $n_1 + \cdots + n_k + 1 \leq i \leq n_1 + \cdots + n_{k+1}$, in a neighborhood V of x, by various choices of the vector fields $Z_1, \cdots, Z_{k+2}$ so that

(a) each $\bar{\xi}_i$ is of the form

$$D_{Z_1}(D_{Z_2}(\cdots (D_{Z_k}(h(Z_{k+1}, Z_{k+2}))) \cdots)) ,$$

(b) $\bar{\xi}_i(y) \in N_{k+1}(y)$ for $y \in V$,

(c) $\bar{\xi}_i(x)$ are linearly independent and span $N_{k+1}(x)$.

By the differentiability of $\bar{\xi}_i$, they are linearly independent in a neighborhood of x. By (b) and the constant dimension of N_{k+1}, they span N_{k+1} in a neighborhood of x. Use the Gram-Schmidt orthogonalization process to obtain the desired result.

Thus, when each N_k has constant dimension, each N_k is a differentiable vector bundle. We also note that when each N_k has constant dimension we may replace $L(x)$ in the definition of $N_{k+1}(x)$ by

$$\text{span}\,\{(D_X\xi)_x \,|\, X \in T_x(M^n),\ \xi \text{ a local cross section for } N_k \text{ near } x\} .$$

If N_1 is invariant under parallel translation with respect to the normal connection, then there are only two normal spaces N_1 and $N_2 = N_1^{\perp}$.

Let $N(x)$ be a subspace of $T_x^{\perp}$ such that $N(x) \supset N_1(x)$. If N is invariant under parallel translation with respect to the normal connection, then by replacing $\mathscr{S}(x) = T_x(M^n) + N_1(x)$ by $T_x(M^n) + N(x)$ in Lemma 1 we may prove the following:

Thorem. *Let ϕ: $M^n \to \tilde{M}^{n+p}(\tilde{c})$ be as in § 1. If $N \supset N_1$ and N is invariant under parallel translation with respect to the normal connection and l is the dimension of N, then there exists a totally geodesic submanifold N^{n+l} of $\tilde{M}^{n+p}(\tilde{c})$ such that $\phi(M^n) \subset N^{n+l}$.*

For example, though N_1 may not be invariant under parallel translation with respect to the normal connection, we may have $N_1 + N_2$ invariant under parallel translation with respect to the normal connection.

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