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PSEUDO-COMPACT SUBSETS OF INFINITE-DIMENSIONAL MANIFOLDS

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Introduction

In the last few years there has been considerable progress in the theory of Hilbert manifolds, and the answers to almost all questions turned out to be as simple as they possibly could be. In particular, as Eells and Elworthy have shown, every Hilbert manifold is diffeomorphic to an open subset of Hilbert space; a result of Kuiper and Burghelea then implies that two Hilbert manifolds are diffeomorphic if they are homotopically equivalent. One of the main reasons for this lack of complications is perhaps that there is nothing built into the definition of a Hilbert manifold which could play the role compact subsets play in finite dimensions. This let Palais to suggest that one should add structure to infinite-dimensional manifolds by specifying what subsets one considers a being "pseudo-compact". He also suggested a definition which is motivated by K. Uhlenbeck's notion of intrinsically bounded subsets of Sobolevmanifolds of sections.

The purpose of this paper is to investigate manifolds with pseudo-compact structure. In particular, we introduce the notion of ψ-boundaries (which correspond to compactifying boundaries of finite-dimensional manifolds), derive a strong invariant and use it to obtain the complete classification of a large class of manifolds with pseudo-compact structure as well as criteria for the existence of open embeddings. Finally, we compute our invariant for many concrete examples and end up with some deletion theorems for section manifolds which might have some interest in the theory of non-linear elliptic operators.

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1. Pseudo-compacta

Let M be a Banach manifold (i.e., a Hausdorf space locally homeomorphic to open subsets of suitable Banach spaces), and fix a (not necessarily maximal) atlas $\mathscr{A}$ for M.

Definition (*Palais*). A subset K of M is *pseudo-compact* (with respect to $\mathscr{A}$)

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if it is the finite union of sets of the form $\phi^{-1}(A)$ where the chart $\phi\colon U \cong \mathcal{O}$ (U open in M, and $\mathcal{O}$ open in the Banach space E) belongs to $\mathcal{A}$ and the subset A of $\mathcal{O}$ is bounded and closed in E.

We denote the set of pseudo-compacta by $\wp(M, \mathcal{A})$. M together with the structure given by $\wp(M, \mathcal{A})$ is called a *$\wp$-manifold*.

Compact subsets of M are pseudo-compact, and they are the only closed subsets which are pseudo-compact with respect to every atlas of M. Finite unions and closed subsets of pseudo-compacta are again pseudo-compact. The pseudo-compact neighborhoods of a pseudo-compactum K form a fundamental system of neighborhoods of K; in particular, M is "locally pseudo-compact". Also, if M is second countable, then it is the countable union of pseudo-compacta.

If now U is an open subset of M, we make U into a $\wp$-manifold by restricting $\mathcal{A}$ to U. Then $\wp(U, \mathcal{A} \mid U) = \{K \in \varphi(M, \mathcal{A}) \mid K \subset U\}$.

Definition. Let M_1, M_2 be $\wp$-manifolds and $f\colon M_1 \to M_2$ a continuous map. f is called a *$\wp$-map* if the image of each closed pseudo-compact subset of M_1 under f is contained in a closed pseudo-compact subset of M_2. f is called pseudo-proper if the inverse image of each pseudo-compactum in M_2 is pseudo-compact in M_1.

We now illustrate the concepts introduced so far by some examples.

a) Finite-dimensional manifolds.

Proposition 1.1. *If M is finite-dimensional, then for every atlas $\mathcal{A}$ of M*

$$\wp(M, \mathcal{A}) = \{\textit{compact subsets of } M\} .$$

b) Hilbert manifolds. A Hilbert manifold is a separable paracompact C^∞-manifold (without boundary unless explicitly mentioned) modelled on the separable ∞-dimensional Hilbert space H. The idea of the following proposition is due to D. Henderson who proved the corresponding result for (continuous) Fréchet manifolds.

Proposition 1.2. *For each Hilbert manifold M there exist two C^∞-charts ϕ_1, ϕ_2 whose domains cover M such that M itself is pseudo-compact with respect to $\mathcal{A} = \{\phi_1, \phi_2\}$.*

Proof. If x_1, x_2 are antipodal points in the Hilbert sphere S, then $M \simeq M \times S$ is covered by $M \times (S - \{x_1\})$ and $M \times (S - \{x_2\})$. Choose ϕ_i to be a suitable diffeomorphism between $M \times (S - \{x_i\})$ and a tubular neighborhood of a closed bounded infinite-codimensional imbedding of M into H.

Corollary. *A subset K of a Hilbert manifold M is pseudo-compact with respect to the maximal C^∞-atlas of M if and only if K is the finite union of locally closed subsets of M (locally closed = intersection of an open subset with a closed one).*

Example. Choose an open halfspace H_+ in H, and take $\mathcal{A}$ to be the atlas for H consisting in the identity map of H and all C^∞-charts with domain in H_+.

Then H_+ is pseudo-compact in H, but its closure is not.

The corollary above indicates that we will have to consider non-maximal atlases if we want to obtain interesting ϕ-manifolds. In the following examples we will define standard ϕ-structures for some important Banach manifolds by using naturally arising atlases.

c) Banach spaces, spheres and Grassmannians. Let E be a (real or complex) infinite-dimensional Banach space. We give E (and similarly each open subset of E) its standard ϕ-structure by the atlas consisting in the identity map.

Also, if n is a natural number, we make the Grassmannian

$$\mathscr{G}_n(E) = \{n\text{-dimensional linear subspaces of } E\}$$

into a ϕ-manifold by the analytical atlas $\mathscr{A} = \{\phi_{F,G} \mid F$ and G are complementary subspaces in $E; F \in G_n(E)\}$, where the chart $\phi_{F,G}^{-1}$ has domain $L(F, G)$ and associates to each operator $u \in L(F, G)$ its graph; cf. [3], [7]. Now fix $\alpha \in (0, \pi/2)$, a pair (S, T) of complementary subspaces of E with S of arbitrary finite dimension, and denote the corresponding projections by P_S, P_T. Then $K_{(S,T,\alpha)} = \{F' \in \mathscr{G}_n(E) \mid \| p_T(x) \| \leq (tg\alpha) \| p_S(x) \|$ for all $x \in F'\}$ is a closed tubular neighborhood of the compact manifold $\mathscr{G}_n(S)$ in $\mathscr{G}_n(E)$.

Proposition 1.3. *A subset of $\mathscr{G}_n(E)$ is pseudo-compact if and only if it is closed and contained in $K_{(S,T,\alpha)}$ for some suitably chosen (S, T, α).*

Recall that $\mathscr{G}_n(E)$ is a classifying space for $GL(n, K)$ where K is the (real or complex) scalar field of E.

Corollary. *An n-plane bundle ξ over a paracompact basespace X is a subbundle of a finite-dimensional trivial vector bundle if and only if the classifying map of ξ can be chosen to map X into a pseudo-compact subset of $\mathscr{G}_n(E)$.*

The total space $\gamma_n(E)$ of the universal n-plane bundle over $\mathscr{G}_n(E)$ also can be given a standard ϕ-structure by a similarly naturally arising atlas of trivializations [7].

The sphere $S(E)$ of E can be considered to be the set of all (real) halflines in E starting at the origin. We make it into analytic ϕ-manifold by the atlas consisting in all central projections onto affine hyperplanes of E. Pseudo-compact subsets of $S(E)$ are closed and allow a characterization similar to Proposition 1.3. In particular, the projection of $S(E)$ onto the (real) projective space of E is a pseudo-proper ϕ-map.

d) Manifolds of sections. We adopt the terminology of [12]. Let E be a C^∞ fiber bundle over an n-dimensional compact manifold X, and let $\mathscr{M}$ be a Banach space valued section functor satisfying axioms (B § 2) and (B § 5), e.g., $\mathscr{M} = C^k$ for $k \geq 0$ or $\mathscr{M} = L_k^p$ for $pk > n$. Then the manifold $\mathscr{M}(E)$ of all $\mathscr{M}$-sections in E has a natural ϕ-structure given by the C^∞-atlas of all charts arising from open vector subbundles.

Definition. A subset of $\mathscr{M}(E)$ is *intrinsically bounded* if it lies in a finite union of sets each of which is contained and bounded in $\mathscr{M}(\xi_i)$ for some suitable open vector subbundle ξ_i of E.

Proposition 1.4. *A subset of $\mathscr{M}(E)$ is pseudo-compact if and only if it is closed and intrinsically bounded.*

In her work on the calculus of variations [15] K. Uhlenbeck introduced the concept of intrinsically bounded sets for $\mathscr{M} = L_k^p$, gave various equivalent characterizations, and used the related notion of pseudo-properness (which coincides with ours) as a criterion for certain functions (e.g., energy functionals) to satisfy condition (C). One can extend one of her results to prove the following

Proposition 1.5. *Let $D: C^\infty(E) \to C^\infty(E')$ be a (non-linear) differentiable operator of order r between the fiber bundles E and E' over X. If $pk > n$, then D extends to a smooth ϕ-map from $L_{k+r}^p(E)$ to $L_k^p(E')$.*

2. Boundaries and ends of ϕ-manifolds

Let M be a Hilbert manifold. We call two continuous maps $j_1: N_1 \to M$, $j_2: N_2 \to M$ (N_1, N_2 are other Hilbert manifolds) equivalent if there is a homotopy equivalence $h: N_1 \to N_2$ such that $j_2 \cdot h \sim j_1$. The collection of such equivalence classes of maps into M is denoted by $B(M)$. For later use we notice that each element in $B(M)$ can be represented by the inclusion map of an open subset of M (cf. [8]).

Also let ϕ_1, ϕ_2 be two ϕ-structures on M, i.e., $\phi_i = \phi(M, \mathscr{A}_i)$ for some atlas $\mathscr{A}_i$. We call ϕ_1 and ϕ_2 equivalent if there is a ϕ-diffeomorphism $f: (M, \phi_1) \simeq (M, \phi_2)$(i.e., both f and f^{-1} are smooth ϕ-maps) homotopic to the identity.

Definition. Let M be a Hilbert manifold with a ϕ-structure ϕ. We say M has a *ϕ-boundary* (or ϕ admits a ϕ-boundary) if there exists a Hilbert manifold $\overline{M}$ with interior M and boundary $\partial\overline{M}$ such that the complements of the open neighborhoods of $\partial\overline{M}$ in $\overline{M}$ are just the closed pseudo-compact subsets of M.

If ϕ admits a ϕ-boundary as above we can, using an (one-sided) collar of $\partial\overline{M}$, homotop the inclusion $j': \partial\overline{M} \subset \overline{M}$ in an obvious way into a map $j: \partial\overline{M} \to M$. The corresponding equivalence class $[j] \in B(M)$ depends only on the equivalence class of the ϕ-structure ϕ on M and is denoted by $b(\phi)$.

Classification theorem. *For each Hilbert manifold M the invariant $b(\phi)$ determines a bijective correspondence between equivalence classes of ϕ-structures on M which admit ϕ-boundaries, and $B(M)$.*

Proof. We construct an inverse for b. Each element of $B(M)$ can be represented by a closed infinite-codimensional imbedding $j: N \to M$, [8]. By fiberwise deletion within a closed tubular neighborhood $E(1)$ of N we obtain a diffeomorphism between M and the complement of a smaller tubular neighborhood $E(1/2)$. Use this diffeomorphism to add the boundary of $E(1/2)$ as boundary $\partial\overline{M}$ to M. The atlas consisting in all charts whose domain is the complement of a closed neighborhood of $\partial\overline{M}$ in $\overline{M} = M \cup \partial\overline{M}$ gives M a ϕ-structure ϕ_j. By various ambient isotopy theorems, [6], [8], the equivalence class of ϕ_j

depends only on $[j]$. Also clearly $b(\phi_j) = [j]$. On the other hand, let ϕ be a ϕ-structure on M which admits a ϕ-boundary. Fix $\overline{M}$, $\partial\overline{M}$ as in the definition above, and choose a collar $C \simeq \partial\overline{M} \times [0, 1)$ around $\partial\overline{M}$ in $\overline{M}$. Furthermore observe that by [2] $\overline{M}$ is Palais stable, i.e., $\overline{M} \simeq \overline{M} \times H$. Now we represent $b(\phi)$ by $j:\partial\overline{M} \simeq \partial\overline{M} \times \{1/2\} \times \{0\} \subset M \times H$, and, in the construction of ϕ_j, we choose the tubular neighborhood $E(1)$ of $j(\partial\overline{M})$ to be a suitable subfibration of $C \times H$ over $j(\partial\overline{M})$. Using the techniques of [8, Theorem 2.5] fiberwise it is then possible to find a diffeomorphism from $M \times H$ onto itself, homotopic to the identity map and taking ϕ over into ϕ_j (since it extends to the respective ϕ-boundaries). Hence the original ϕ-structure ϕ is equivalent to the ϕ-structure ϕ_j induced by $b(\phi)$. q.e.d.

It follows from this proof that in the above definition $\overline{M}$ is unique up to a diffeomorphism which, when restricted to M, is homotopic to the identity.

Corollary. *Let M_1, M_2 be two Hilbert manifolds with ϕ-structures ϕ_1, ϕ_2 which admit ϕ-boundaries. Then the following conditions are equivalent:*

(i) *M_1 and M_2 are ϕ-diffeomorphic.*

(ii) *If $b(\phi_i)$ is represented by j_i: $N_i \to M_i$ $(i = 1, 2)$, then there are homotopy equivalences $N_1 \sim N_2$ and $M_1 \sim M_2$ which make the diagram*

$$\begin{array}{ccc} N_1 & \xrightarrow{j_1} & M_1 \\ \wr & & \wr \\ N_2 & \xrightarrow{j_2} & M_2 \end{array}$$

homotopy commutative.

(iii) *$\overline{M}_1$ and $\overline{M}_2$ are diffeomorphic.*

Definition. An *open ϕ-imbedding* of a Hilbert ϕ-manifold M into H is a ϕ-diffeomorphism from M onto an open subset of H (with its standard ϕ-structure).

Suppose that M has a ϕ-boundary and $\partial\overline{M} \neq \phi$. Then such an open ϕ-imbedding exists if and only if there is a diffeomorphism from $\overline{M}$ onto a closed submanifold (with boundary) of H which maps M onto an open subset of H. The techniques in [1] then imply

Proposition 2.1. *Let M be a Hilbert manifold with a ϕ-structure which admits a ϕ-boundary such that $\partial\overline{M} \neq \phi$. Then there is an open ϕ-imbedding of M into H if and only if there exist a pair of countable CW-complexes (A, B) and a homotopy equivalence h: $\partial\overline{M} \sim B$ such that $\partial\overline{M} \cup_h A$ is contractible.*

In particular, one can use a construction of Burghelea to show that a Hilbert manifold admits an open ϕ-imbedding into H with respect to infinitely many, but not all of its ϕ-structures.

Proposition 2.2. *Let M be a Hilbert ϕ-manifold which is not pseudo-compact itself but has a countable family of closed pseudo-compacta $K_1, K_2, \cdots$ such that each closed pseudo-compact subset of M is contained in some K_i.*

Then M has no ϕ-boundary.

Corollary. *Hilbert space H, its unit sphere, its Grassmannians $G_n(H)$ and the universal vector bundles $\gamma_n(H)(n = 1, 2, \cdots)$ do not have ϕ-boundaries. In particular, H and its open unit ball are not ϕ-diffeomorphic.*

Definition. Let M be an arbitrary ϕ-manifold. An open subset U of M is called an *ideal ϕ-boundary* for M if each closed pseudo-compactum in M is contained in another closed pseudo-compactum K such that $M - K \subset U$ and the inclusion is a homotopy equivalence. The connected components of U represent the *ends* of M.

We call two open subsets U_1 and U_2 of a Banach manifold M equivalent if there is a homotopy equivalence $h: U_1 \xrightarrow{\simeq} U_2$ which makes the diagram

$$\begin{array}{ccc} & M & \\ \hookrightarrow\!\!\!\!\nearrow & & \nwarrow\!\!\!\!\hookleftarrow \\ U_1 & \xrightarrow[h]{\simeq} & U_2 \end{array}$$

homotopy commutative. The set of such equivalence classes of open subsets of M is denoted by $B(M)$. If now M has an ideal ϕ-boundary U with respect to some ϕ-structure ϕ, then the equivalence class of U is a well defined invariant $b(\phi) \in B(M)$. In case M is a Hilbert ϕ-manifold which has a ϕ-boundary then it also has an ideal ϕ-boundary, e.g., take the interior of a collar of $\partial\overline{M}$. The above definition of $B(M)$ and $\phi(b)$ coincides with the previous one after an obvious identification.

Ideal ϕ-boundaries generalize the notion of ϕ-boundaries considerably. We will see in the next section that in most concrete cases naturally arising standard ϕ-structures admit ideal ϕ-boundaries (in contrast to the corollary of Proposition 2.2). On the other hand, it also follows from the corollary that the generalized invariant $b(\phi)$ does no longer completely determine ϕ.

3. Deleting pseudo-compact subsets

Definition. Let M be a ϕ-manifold. M has the *pseudo-compacta deletion property* (*PDP*) if each closed pseudo-compactum in M is contained in another one K such that the inclusion $M - K \subset M$ is a homotopy equivalence. In other words, M has *PDP* if and only if it is an ideal ϕ-boundary of itself.

Proposition 3.1. *No finite-dimensional non-empty manifold satisfies PDP.*

This follows from a standard argument using homology theory. In contrast, *PDP* seems to hold for many infinite-dimensional ϕ-manifolds, especially for those with natural standard structures.

The next two propositions follow from known deletion theorems for infinite codimensional submanifolds and from the considerations in 1*b*) (proof of Proposition 1.2.) and 1*c*).

Proposition 3.2. *Each Hilbert manifold is diffeomorphic to an open subset* *of H, which with respect to its standard ϕ-structure has PDP (and even has a ϕ-boundary).*

Proposition 3.3. *If E is an infinite-dimensional Banach space, then E, $S(E)$, $G_n(E)$ and $\gamma_n(E)$, $n = 1, 2, \cdots$, (see 1c)) have PDP.*

Theorem. *Let E be a C^∞ fiber bundle over a compact n-dimensional manifold X. Then the section manifolds $C^k(E)$, $k = 0, 1, \cdots$, and $L_k^p(E)$, $pk > n$, have PDP.*

Idea of proof. For simplicity let us consider only $C^0(E)$. Each pseudo-compactum of $C^0(E)$ is contained in one of the form $K = \bigcup_{i=1}^{r} C^0(B_i)$, where B_i is a closed ball subbundle of E. It suffices to show that the inclusion of $C^0(E) - K$ into $C^0(E)$ induces an isomorphism of homotopy groups [13, Theorem 15]. Choose $s_0 \in C^0(E), x_1, \cdots, x_r \in X$, disjoint neighborhoods U_i of x_i and trivializations T_i of $E \mid U_i$, such that $s_0 \mid U_i$ is constant with respect to T_i and $s_0(U_i) \subset E - B_i \, (i = 1, \cdots r)$. Now let an arbitrary element of $\pi_q(C^0(E), s_0)$ be represented by a map $\alpha\colon S^q \to C^0(E)$ which maps a neighborhood of the base-point of S^q into s_0. For $y \in S^q$ the section $\alpha(y)$ can be deformed first into a section which is constant with respect to T_i in a neighborhood of x_i, and then into a section which maps x_i into $s_0(x_i)$ or at least some point of U_i into $E - B_i$, and this in a uniform way. (For the latter deformation we use the map α, evaluated at x_i.) Hence the homotopy homomorphism $\pi_q(C^0(E) - K, s_0) \to \pi_q(C^0(E), s_0)$ is onto, and a similar deformation argument also gives injectivity.

Corollary. *Let E, E' be C^∞-fiber bundles over the compact n-dimensional manifold without boundary X, and let $D_{k+r}\colon L^2_{k+r}(E) \to L^2_k(E')$ extend an elliptic nonlinear differential operator D of order r from E to E'. Then for each pseudo-compactum K in $L^2_{k+r}(E)$ the analytic index of D (see* [12, p. 94]) *is independent of $D_{k+r} \mid K$.*

While Propositions 3.2 and 3.3 are consequences of known deletion theorems, it seems to be hard to reduce the above theorem to standard deletion methods for compacta. One step in that direction however may be the following observation. The coordinate changes in the standard atlas for a Banach space, its sphere, Grassmannians etc. are weakly continuous, and therefore these manifolds have in a natural way a weak topology. Recently J. Dowling [4] showed that also the coordinate transformations between charts of $L_k^p(E)$, which arise from open vector subbundles of a fiber bundle E, are sequentially weakly continuous. Thus there is a natural (sequentially) weak topology on $L_k^p(E)$. In a forthcoming paper we will study the relationship between pseudo-compacta and weak topologies and also give a full proof of the last theorem.

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