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THE SCHWARZIAN AS A CURVATURE

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1. Introduction

If $z = z(s)$ is a C''' function of a real variable or a regular function of a complex variable, and if $z'(s) \neq 0$, the *Schwarzian derivative* of z is

$$\frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2 = \left(\frac{z''}{z'}\right)' - \frac{1}{2}\left(\frac{z''}{z'}\right)^2 .$$

It is known to be invariant under linear fractional transformations $w = (az + b)/(cz + d)$. In this note we shall interpret the Schwarzian as the natural invariant (curvature) of an equivalence problem for curves in the projective line. This interpretation will make its invariance transparent, whereas usually it seems to be an accidental by-product of calculation. The moving frame method of *E.* Cartan is the natural tool for this study.

2. Moving frames

Let $\boldsymbol{P}^1$ denote the complex projective line or the oriented real projective line. As usual we represent the points of $\boldsymbol{P}^1$ by non-zero vectors $\boldsymbol{x}$ in affine space A^2, where $\lambda \boldsymbol{x}$ and $\boldsymbol{x}$ represent the same point of $\boldsymbol{P}^1$ if $\lambda \neq 0$.

We shall use the area function

$$[\boldsymbol{x}, \boldsymbol{y}] = x_1 y_2 - x_2 y_1$$

on A^2, an alternating bilinear functional.

A *frame* for $\boldsymbol{P}^1$ consists of a pair $\boldsymbol{x}, \boldsymbol{y}$ of points in A^2 such that $[\boldsymbol{x}, \boldsymbol{y}] = 1$.

We handle the real and complex cases simultaneously so that *function* means either a C''' (real or complex valued) function on a real open interval or a regular function on a simply connected complex domain. (Later square roots are needed.)

Let $s \to \{\boldsymbol{x}(s), \boldsymbol{y}(s)\}$ be a function into frames (i.e., a moving frame). Then

$$\boldsymbol{x}' = a\boldsymbol{x} + b\boldsymbol{y} , \qquad \boldsymbol{y}' = c\boldsymbol{x} + d\boldsymbol{y} ,$$

where $a = a(s)$, etc. Differentiate $[\boldsymbol{x}, \boldsymbol{y}] = 1$ to obtain $[\boldsymbol{x}', \boldsymbol{y}] + [\boldsymbol{x}, \boldsymbol{y}'] = 0$, $a + d = 0$. Hence

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$$x' = ax + by\ , \qquad y' = cx - ay\ .$$

These are the structure equations for a moving frame.

3. Curves

We study mappings $\phi\colon \boldsymbol{D} \to \boldsymbol{P}^1$ where $\boldsymbol{D}$ is a domain. Two mappings ϕ and ψ are *equivalent* if there is a projective transformation π on $\boldsymbol{P}^1$ such that $\psi = \pi \circ \phi$.

Such a mapping may be thought of as a curve in $\boldsymbol{P}^1$. Note that a change of parameter is *not* allowed, unlike the usual situation in curve theory, so that the corresponding problem of finding conditions under which mappings are equivalent is not trivial, in spite of the one-dimensionality of the ambient space.

Given a mapping $\phi = \phi(s)$, we choose a moving frame $x(s), y(s)$ so that for each $s, x(s)$ is a representative of $\phi(s)$. The problem is to choose the frame so that its structure equations are as simply as possible. Let

$$x' = ax + by\ , \qquad y' = cx - ay$$

be the structure equations for one particular choice of frame. As usual in these situations, one must distinguish cases.

4. Extreme cases

Suppose $b = 0$ for all s. Then for $\lambda = \lambda(s) \neq 0$,

$$(\lambda x)' = (\lambda' + a\lambda)x\ .$$

Choose $\lambda \neq 0$ so that $\lambda' + a\lambda = 0$. Then $\lambda x = x_0$ is a constant representative of the mapping ϕ, hence ϕ is constant. This is an extreme; we pass to the opposite extreme—intermediate cases are hopeless—which may be thought of as the generic case.

Suppose b is never 0. Change frame to x_1, y_1 by

$$x = hx_1\ , \qquad y = h^{-1}y_1\ ,$$

where h will be determined. The new frame x_1, y_1 has structure equations $x_1' = a_1x_1 + b_1y_1$, etc., hence

$$x' = h'x_1 + ha_1x_1 + hb_1y_1\ .$$

But $x' = ax + by = ahx_1 + bh^{-1}y_1$, therefore

$$b = h^2b_1\ .$$

Since b is never 0, we can choose h so that $b_1 = 1$ (complex case, or real case with $b > 0$) or $b_1 = -1$ (real case, $b < 0$).

Remark. The conditions $b \equiv 0$ or $b > 0$ are invariant. From the structure equations $[x, x'] = [x, ax + by] = b$. But any other representative of $\phi(s)$ is λx where $\lambda \neq 0$, and

$$[\lambda x, (\lambda x)'] = [\lambda x, \lambda' x + \lambda x'] = \lambda^2 [x, x'] = \lambda^2 b \ .$$

Clearly $b \equiv 0$ if and only if $\lambda^2 b = 0$. In the real case, $b > 0$ if and only if $\lambda^2 b > 0$.

We shall consider in detail only the case $b_1 = 1$ and merely state the analogous results in the second case.

We may assume then that a frame x, y has been chosen so that

$$x' = ax + y \ , \qquad y' = cx - ay \ .$$

Now make the change of frame

$$x = x_1 \ , \qquad y = -ax_1 + y_1 \ .$$

Then $x_1' = x' = ax + y = y_1$, so $a_1 = 0$. Without the subscript notation, we have found a moving frame for ϕ such that

$$x' = y \ , \qquad y' = -kx \ ,$$

where $k = k(s)$. We shall call x, y a *natural moving* frame for ϕ.

5. Invariance of k

Now we show that x, y is determined up to sign and that k is an invariant. For suppose x_1, y_1 is a second natural frame so that

$$x_1' = y_1 \ , \qquad y_1' = -k_1 x_1 \ .$$

Then $x = \lambda x_1$ with $\lambda \neq 0$, hence

$$\begin{aligned} y &= x' = \lambda' x_1 + \lambda y_1 \ , \\ 1 &= [x, y] = [\lambda x_1, \lambda' x_1 + \lambda y_1] = \lambda^2 [x_1, y_1] = \lambda^2 \ . \end{aligned}$$

It follows that $\lambda = \pm 1, x_1 = \pm x, y_1 = x_1' = \pm x' = \pm y, -k_1 x_1 = y_1' = \pm y = \mp kx = -kx_1$. Consequently $k_1 = k$.

6. Formula for k

We next develop a practical formula for k. Suppose that ϕ is given by an (affine) representative $s \to z(s)$. Let $x(s), y(s)$ be a natural frame. Then

$$z = \lambda x \ ,$$

where $\lambda(s)$ is never 0. Differentiate:

$$\begin{aligned} z' &= \lambda' x + \lambda y \ , \\ z'' &= (\lambda'' - \lambda k)x + 2\lambda' y \ , \\ z''' &= (\cdots)x + (3\lambda'' - \lambda k)y \ . \end{aligned}$$

Now form the various areas:

$$\begin{aligned} [z, z'] &= \lambda^2 \ , \\ [z, z''] &= 2\lambda\lambda' \ , \\ [z, z'''] &= 3\lambda\lambda'' - \lambda^2 k \ , \\ [z', z''] &= 2(\lambda')^2 - \lambda\lambda'' + \lambda^2 k \ . \end{aligned}$$

Hence

$$\begin{aligned} &[z, z'''] + 3[z', z''] = 6(\lambda')^2 + 2\lambda^2 k \ , \\ &\frac{[z, z'']^2}{[z, z']} = 4(\lambda')^2 \ , \\ &[z, z'''] + 3[z', z''] - \frac{3}{2}\frac{[z, z'']^2}{[z, z']} = 2\lambda^2 k \ . \end{aligned}$$

It follows that

$$2k = \frac{[z, z'''] + 3[z', z'']}{[z, z']} - \frac{3}{2}\left(\frac{[z, z'']}{[z, z']}\right)^2 .$$

Note. According to a remark in § 4, the condition b never 0 is equivalent to $[z, z']$ never 0; the condition $b > 0$ is equivalent to $[z, z'] > 0$ in the real case.

In the other real case, $b < 0$, the *natural frame* x, y satisfies

$$x' = -y \ , \qquad y' = kx \ ,$$

and the equation for k is the same as that above.

7. The Schwarzian

Now suppose a real or complex valued function $z(s)$ is given. It may be considered as the non-homogeneous coordinate of a point in P^1. Thus define ϕ by $s \to (1, z(s)) = z(s)$. Then

$$z' = (0, z'), \quad z'' = (0, z''), \quad z''' = (0, z''') \ ,$$

$$[z, z'] = z', \quad [z, z''] = z'', \quad [z, z'''] = z''', \quad [z', z''] = 0 .$$

The condition for the existence of a natural frame is that z' never be 0, and the curvature formula specializes to

$$2k = \frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2 .$$

8. Constant *k*

We shall determine the mappings ϕ with constant k. We consider several cases.

(1) $k = 0$. The structure equations are

$$\boldsymbol{x}' = \pm \boldsymbol{y} , \qquad \boldsymbol{y}' = 0 .$$

Thus $\boldsymbol{y} = \boldsymbol{b}, \boldsymbol{x} = \boldsymbol{a} \pm \boldsymbol{b}s, [\boldsymbol{a}, \boldsymbol{b}] = 1$. If the mapping is given by a function z, then $(1, z) = \lambda(\boldsymbol{a} \pm \boldsymbol{b}s)$. By eliminating λ, we see that $z = (a_2 \pm b_2 s)/(a_1 \pm b_1 s)$ is linear fractional.

(2) $k = c^2 \neq 0$. The structure equations are

$$\boldsymbol{x}' = \pm \boldsymbol{y} , \qquad \boldsymbol{y}' = \mp c^2 \boldsymbol{x} ,$$

hence

$$\boldsymbol{x}'' + c^2 \boldsymbol{x} = 0 .$$

Integrate:

$$\boldsymbol{x} = \boldsymbol{a} \cos cs + \boldsymbol{b} \sin cs ,$$

where $c[\boldsymbol{a}, \boldsymbol{b}] = \pm 1$ since $[\boldsymbol{x}, \boldsymbol{x}'] = \pm 1$.

If the mapping is given by a function z, then $(1, z) = \lambda \boldsymbol{x}$, hence

$$z = (a_2 \cos cs + b_2 \sin cs)/(a_1 \cos cs + b_1 \sin cs) .$$

(3) $k = -c^2 \neq 0$. The result is similar to Case (2) with cos and sin replaced by cosh and sinh.

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