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THE RIEMANNIAN STRUCTURE OF CERTAIN FUNCTION SPACE MANIFOLDS

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Introduction

In this paper we shall examine the properties of a certain class of projection and Green's operators which are associated with the tangent bundle of a Sobolev space $H^k(X, Y)$ (defined below) of maps from a manifold X to a manifold Y. In § 2 we use these results to describe a class of functions on $H^k(X, Y)$ which satisfy Condition C (in the sense of Palais and Smale). In § 3 we derive an expression for the riemannian sectional curvature of $H^k(X, Y)$. One might hope that the property of having a sectional curvature of definite sign would be transferred from Y to $H^k(X, Y)$. However, this is not the case. We shall construct examples of spaces Y whose riemannian curvatures are non-negative (zero, non-positive) such that the riemannian curvatures of $H^k(S^1, Y)$ are indefinite. (§ 3 does not depend on the results of § 2, and may be read immediately after § 1.)

1. A. Notation and basic definitions

Hereafter X and Y denote smooth finite dimensional riemannian manifolds, X compact and without boundary. We shall suppose that Y is isometrically and smoothly embedded in a euclidean space $\boldsymbol{R}^q$ (which we may always do by a well-known theorem of Nash).

We recall some basic facts in global analysis: (For general references see [1], [3], [4] or [5].) Let $\langle , \rangle$ denote the standard inner product on $\boldsymbol{R}^q$, $d\mu$ a smooth measure on X, k a positive integer, and A a strictly positive strongly elliptic self-adjoint operator (with smooth coefficients) of order $2k$ on $C^\infty(X, \boldsymbol{R}^q)$, say $A = 1 + \Delta^k$. Let $(u, v)_k = \int_X \langle Au, v \rangle d\mu$, and let $\|\cdot\|_k$ denote the corresponding norm. Two such operators A give rise to equivalent norms, and $H^k(X, \boldsymbol{R}^q)$ is defined to be the completion of $C^\infty(X, \boldsymbol{R}^q)$ with respect to $\|\cdot\|_k$. For $k = 0$, set $A = I$. By a theorem of Rellich, for $k < l$, the natural injection $H^l(X, \boldsymbol{R}^q) \to H^k(X, \boldsymbol{R}^q)$ is dense and compact. A theorem of Sobolev asserts that the $\|\cdot\|_k$ topology is larger than the C^t topology when $k > \frac{1}{2}di(X) + t$. Hence when $2k > di(X)$ the elements of $H^k(X, \boldsymbol{R}^q)$ are continuous maps and one may define

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$H^k(X, Y) = \{f \in H^k(X, \boldsymbol{R}^q) \mid f(x) \in Y \text{ for all } x \in X\}$. $H^k(X, Y)$ with the induced topology is in fact a smooth infinite dimensional manifold modeled on a hilbert space, and inherits a riemannian structure from $H^k(X, \boldsymbol{R}^q)$. For $f \in H^k(X, Y)$, let $T_f H^k(X, Y) = \{\sigma \in H^k(X, \boldsymbol{R}^q) \mid \sigma(x) \in T_{f(x)}(Y) \text{ for all } x \in X\}$. Then $T_f H^k(X, Y)$ may be identified with the tangent space of $H^k(X, Y)$ at f.

For $u \in C^\infty(X, \boldsymbol{R}^q)$, define $\|u\|_{-k} = \sup\{(u, v)_0/\|v\|_k \mid v \in H^k(X, \boldsymbol{R}^q)\}$, and let $H^{-k}(X, \boldsymbol{R}^q)$ denote the completion of $C^\infty(X, \boldsymbol{R}^q)$ with respect to $\|\cdot\|_{-k}$. It can be shown that $H^{-k}(X, \boldsymbol{R}^q)$ is a hilbert space, which is dual to $H^k(X, \boldsymbol{R}^q)$, the bilinear pairing being given by $(\ ,\ )_0$; i.e., for every continuous linear functional l on $H^k(X, \boldsymbol{R}^q)$ there exists a unique $u \in H^{-k}(X, \boldsymbol{R}^q)$ such that $l(v) = (u, v)_0$. The proof of this and of certain other basic theorems involves the construction of a Green's operator G satisfying the relation $(u, v)_0 = (Gu, v)_k$ for all u, $v \in H^k(X, \boldsymbol{R}^q)$. One shows that G extends to an isometry $H^{-k}(X, \boldsymbol{R}^q) \to H^k(X, \boldsymbol{R}^q)$ and defines $(u, v)_{-k} = (Gu, v)_0$. G and A are inverse isomorphisms $H^{-k}(X, \boldsymbol{R}^q) \leftrightarrow H^k(X, \boldsymbol{R}^q)$. In paragraph 1$C$, we shall construct analogous operators $\tilde{G}_f$ and $\tilde{A}_f$ on the spaces $T_f H^k(X, Y)$.

By means of the spectral representation of A (or G), spaces $H^\alpha(X, \boldsymbol{R}^q)$ are defined for each $\alpha \in \boldsymbol{R}$, and the collection of spaces thus obtained are shown to satisfy the theorems of Rellich and Sobolev.

Finally, we remark that this theory is usually discussed in a more general setting: Collections of spaces $\{H^k(\xi)\}$ and $\{H^k(\xi^1)\}$ are constructed where ξ^1 is a fibre sub-bundle of a riemannian vector bundle ξ over X. The case we are considering is $\xi = X \times \boldsymbol{R}^q$, $\xi^1 = X \times Y$, but the results of this paper can be easily extended to the more general case.

1. B. The projection operators P_f^0, P_f^k

Hereafter we write $\hat{H}^k = H^k(X, \boldsymbol{R}^q), H^k = H^k(X, Y)$. To avoid the appearance of inessential constants, we choose the operators A so that $\|\cdot\|_k \leq \|\cdot\|_l$ for $k < l$. k will denote a fixed positive integer with $2k > di(X)$.

For $f \in H^k, T_f H^k$ can be identified with a linear subspace of $\hat{H}^k$, and for $i = 0, k$ we let P_f^i represent the projection $\hat{H}^k \to T_f H^k$ which is orthogonal with respect to $(\ ,\ )_i$. Let $N_f^i = I - P_f^i$; then the following relations are easy consequences of the definitions and properties of orthogonal projections:

$$\|N_f^i u\|_i = \inf\{\|u - \xi\|_i \mid \xi \in T_f H^k\}\ . \tag{1}$$

$$(P_f^i)^2 = P_f^i\,;\ (P_f^i u, v)_i = (u, P_f^i v)_i\,;\ P_f^0 P_f^k = P_f^k\,;\ P_f^k P_f^0 = P_f^0\ . \tag{2}$$

$$\|N_f^0 u\|_0 \leq \|N_f^k u\|_0 \leq \|N_f^k u\|_k \leq \|N_f^0 u\|_k\ . \tag{3}$$

$$P_f^0 A N_f^k = P_f^k G N_f^0 = 0\ , \tag{4}$$

where here, as always, A denotes the operator which defines the inner product $(\ ,\ )_k$, and G denotes the corresponding Green's operator. Note that (2) defines P_f^k as the projection whose range is the range of P_f^0 and which is orthogonal with respect to $(\ , \ )_k$. The relations (3) are a direct consequence of (1). Also, from (1) it follows that $(P^0_f u)(x) = P_{f(x)}u(x)$ where for $y \in Y, P_y$ is the orthonormal projection $R^q \to T_y(Y)$.

To prove (4), we have $(P^0_f A N^k_f u, v)_0 = (A N^k_f u, P^0_f v)_0 = (N^k_f u, P^0_f v)_k = (N^k_f u, P^k_f P^0_f v)_k = 0$. The other part of (4) is proved in the same way.

It is known that the map $f \to P^0_f$ is continuous in the norm topology of H^k; i.e., $f \to f_1$ in H^k inplies $\|P^0_f - P^0_{f_1}\|_k \to 0$, [4, p. 112].

Proposition. *Let j: $M \to H$ be a C^{k+2} isometric embedding of a manifold M into a hilbert space H, and let P_x: $H \to H$ denote the orthogonal projection of H onto $M_x = T_x(M)$ (identified with a closed subspace of H). Then $x \to P_x$ is a C^k map $M \to L(H, H)$.*

To prove the proposition let $u \in H, v \in M_x$. Then $P_x u = dj_x u^1$ for some $u^1 \in M_x$, and $(u^1, v)_{M_x} = (dj_x u^1, dj_x v)_H = (P_x u, dj_x v)_H = (u, P_x dj_x v)_H = (u, dj_x v)_H = (dj^*_x u, v)_{M_x}$. Hence $u^1 = dj^*_x u$, and therefore

$$P_x u = dj_x dj^*_x u \ . \tag{5}$$

More precisely, if we write ϕ for the composition $M \times H \xrightarrow{dj^*} T(M) \xrightarrow{dj} M \times H$, then $P_x = \phi(x, \cdot)$, and the differentiability of P is a consequence of the differentiability of ϕ. (In writing out the details, one would use the fact that ϕ is linear in the second variable, and that the maps $x \to \|\phi(x, \cdot)\|$ $x \to \|d\phi(x, \cdot)\|$ are continuous.)

1. C. The spaces $T_f H^{-k}$

Let $\|u\|_{-\bar{k}} = \sup \{(u, v)_0 \mid v \in T_f H^k, \|v\|_k = 1\}$, and let $T_f H^{-k}$ be the completion of, say, $T_f H^k$ with respect to $\|\cdot\|_{-\bar{k}}$.

Theorem. *Suppose the symbol of A is a multiple of the identity matrix. Then $T_f H^{-k}$ is a hilbert space which is dual to $T_f H^k$, the bilinear pairing being given by $(\ , \ )_0$.*

Proof. We shall first prove the theorem for the case when f is smooth, the more general statement being obtained by a limit process. Let $A_f = P^0_f A \mid \text{image}\,(P^0_f)$. Then if f is smooth we may consider A_f to be an operator on the smooth sections of the lifted bundle $f^*T(Y)$. $\tilde{A}_f$ is strongly elliptic since, decomposing every $\sigma \in C^\infty(X, \boldsymbol{R}^q)$ into a tangential and normal component, we see that the symbol of $\tilde{A}_f$ is the symbol of A "cut down" to the dimension of Y. From the relation $(\tilde{A}_f u, v)_0 = (Au, v)_0$; $u, v \in T_f H^k$, it can be seen that $\tilde{A}_f$ is self-adjoint and strictly positive. Hence we can apply the standard theory to obtain a Green's operator $\tilde{G}_f$ satisfying the relation $(u, v)_0 = (\tilde{G}_f u, v)_k$ for all $u, v \in T_f H^k$, and the proof proceeds exactly as indicated in Paragraph A, $T_f H^k$ and $T_f H^{-k}$ now playing the roles of $\hat{H}^k$ and $\hat{H}^{-k}$, respectively. Before proceeding we note the following identity

$$P^k_f = \tilde{G}_f P^0_f A \ , \tag{6}$$

whose proof consists in verifying that this expression for P^k_f satisfies the relation (2) which define P^k_f as the projection which is orthogonal with respect to $(\, , \,)_k$, and whose range is the range of P^0_f.

Now let f be any element of H^k. (We cannot now use the standard theory since $f^*T(Y)$ may only be of class C^0.) To complete the proof we have to construct a Green's operator $\tilde{G}_f$. Let $\{f_n\}$ be a sequence of smooth maps in H^k which converge to f in H^k-norm. Multiplying (6) on the right by G, we obtain $P^k_f G = \tilde{G}_f P^0_f$, ($f$ smooth). This motivates defining $\tilde{G}_f = \lim P^k_{f_n} G \mid \text{image}\,(P^0_f)$. A simple calculation shows that $(u, v)_0 = (\tilde{G}_f u, v)_k$ for all $u, v \in T_f H^k$, and the proof proceeds as before. Also, it is easy to see that (6) now holds for any $f \in H^k$.

1. D. The gradients $\nabla^k E, \nabla^0 E$

Let E be a C^1 function on H^k. The gradient $\nabla^k E(f)$ of E at f is defined by the relation $dE_f(v) = (\nabla^k E(f), v)_k$ for all $v \in T_f H^k$. Now the map $v \longrightarrow dE_f(v)$ is a continuous linear functional on $T_f H^k$, hence there exists an element of $T_f H^{-k}$, denoted by $\nabla^0 E$ and called the formal H^0 (or L^2) gradient of E, which satisfies the relation $dE_f(v) = (\nabla^0 E(f), v)_0$. Hence

$$\nabla^k E(f) = \tilde{G}_f \nabla^0 E(f) = P^k G \nabla^0 E(f) \; ; \tag{7}$$

and for C^1 functions E, F,

$$(\nabla^k E(f), \nabla^k F(f))_k = (\nabla^0 E(f), \nabla^0 F(f))_{-\bar{k}} \, , \tag{8}$$

and therefore

$$\|\nabla^k E(f)\|_k = \|\nabla^0 E(f)\|_{-\bar{k}} \, . \tag{9}$$

For later application it is important to note that although $T_f H^{-k} \supset \hat{H}^{-k}$ (since $T_f H^k \subset \hat{H}^k$), we can write $\nabla^0 E(f) \in \hat{H}^{-k}$; i.e., we can extend the map $v \longrightarrow (\nabla^0 E(f), v)_0$ to a continuous linear functional on $\hat{H}^k$: for $v \in \hat{H}^k$, define $(\nabla^0 E(f), v)_0 = (\nabla^k E(f), v)_k$.

2. Condition C

A. Following Palais and Smale we say that a C^1 function F on H^k satisfies Condition C iff every sequence of points $\{f_n\}$ in H^k for which $\{F(f_n)\}$ is bounded and $\|\nabla^k F(f_n)\|_k$ is not bounded away from zero contains a convergent subsequence (converging to a critical point of F). We say that F satisfies Condition H iff every component of H^k contains a critical point of F. This is the same as saying that every $f \in H^k$ is homotopic to a critical point of F. Suppose F is bounded below on each component of H^k, say, $F \geq 0$, and that F satisfies Condition C. Then Palais [3] has shown that every component of H^k contains a point at which F assumes an absolute minimum. Hence, if $F \geq 0$, Condition C implies Condition H.

F will be said to satisfy Condition Γ iff $f \to \nabla^0 F(f)$ is a weak-strong continuous map $H^k \to \hat{H}^{-k}$ in the sense that $f_n \xrightarrow{\text{weak}, H^k} f$ implies $\nabla^0 E(f_n) \xrightarrow{H^{-k}} \nabla^0 E(f)$. (Cf. the remarks at the end of Paragraph 1D.) For examples, see Paragraph 2D below.

Remark 1. Note that as a consequence of the theorems of Rellich and Sobolev, H^k is a weak closed subspace of $\hat{H}^k$. For weak convergence in $\hat{H}^k$ implies strong convergence in $\hat{H}^{k-\alpha}$ for any $\alpha > 0$, and if α satisfies $2(k - \alpha) > di(X)$, strong convergence in $\hat{H}^{k-\alpha}$ implies C^0 convergence.

Remark 2. Using (7) it is easy to show that F satisfies Condition Γ iff the map $f \to \nabla^k F(f)$ is a weak-strong continuous map from H^k to $\hat{H}^k$. Suppose that F is a positive C^2 function on H^k, and consider the heat equation $df(t)/dt = -\nabla^k F(f(t))$, with initial condition $f(0) = f$. It is known that this equation has infinite positive escape time, so that $\|\nabla^k F(f_t)\|_k$ is not bounded away from zero along the trajectory (Palais [3]). Therefore we get the following proposition: If F is a positive C^2 function on H^k which satisfies Condition Γ, and if the solution to the heat equation $df/(t)dt = -\nabla^k F(f_t)$ with initial condition $f(0) = f$ is bounded in $\hat{H}^k$ norm, then f is homotopic to a critical point of F. (Cf. Eells [1]. The condition he imposes on F is that the map: $f \to \nabla^k F(f)$ be compact.)

B. A strongly elliptic self-adjoint of order $2k$ on $C\ (X, \boldsymbol{R}^q)$ will be said to be admissible, if $2k > di(X)$, and either A is strictly positive or Y is compact. The following theorem was proved by Saber [4], [6].

Theorem. *Let A be admissible, and for $f \in \hat{H}^k$ let $\hat{F}(f) = \frac{1}{2}(Af, f)_0$. Let $F = \hat{F} | H^k$. Then F satisfies Condition C.*

An easy proof of this theorem is provided by a result of K. Uhlenbeck [4. p. 113], which asserts that a bounded sequence $\{f_n\}$ in H^k contains a subsequence $\{f_n^1\}$ for which $\|N^0_{f_m^1}(f_m^1 - f_n^1)\|_k \to 0$ as $m, n \to \infty$. From (3), we see that N^0 can be replaced by N^k in this statement. Now if A is strictly positive, we may write $F(f) = \frac{1}{2}\|f\|_k^2$. Hence $\nabla^k F(f) = P_f^k f$. Suppose $\{f_n\}$ satisfies the hypothesis of Condition C; i.e., $\|f_n\|_k \leq$ constant and $P^k_{f_n} f_n \to 0$. Then $(f_m, f_m - f_n)_k = (f_m, P^k_{f_m}(f_m - f_n))_k + (f_m, N^k_{f_m}(f_m - f_n))_k = (P^k_{f_m} f_m, f_m - f_n) + (f_m, N^k_{f_m}(f_m - f_n))$, and it is easy to see that $\{f_n\}$ contains a Cauchy subsequence. (The other case will be treated below. Also, note that the symbol of A is not required to be a multiple of the identity matrix.)

C. The following theorem is the principal result of this section.

Theorem. *Let A be an admissible operator whose symbol is a multiple of the identity matrix, and J be a C^1 function on H^k which is bounded below and satisfies Condition Γ. Let $F(f) = \frac{1}{2}(Af, f)_0 + J(f)$. Then F satisfies Condition C.*

Proof. First suppose that A is strictly positive, so that, as in B, we can write $F(f) = \frac{1}{2}\|f\|_k^2 + J(f)$ and $\nabla^k F(f) = P_f^k f + \nabla^k J(f)$. Then, using (7), we can write $\nabla^k F(f) = P^k_f(f + G\nabla^0 J(f))$. Suppose $\{f_n\}$ is a sequence in H^k such that $|F(f_n)| \leq$ constant and $\nabla^k F(f_n) \to 0$. Then, since J is bounded below, $\|f_n\|_k \leq$ constant. Hence, by extracting a subsequence, we may suppose that $f_n \xrightarrow{\text{weak}, H^k} f$ for some f, and using Condition Γ, $G\nabla^0 J(f_n) \xrightarrow{H^k} \xi$ for some ξ. Therefore, since $\|P^k_{f_n}\| \equiv 1$, we have $P^k_{f_n}(G\nabla^0 J(f_n) - \xi) \xrightarrow{H^k} 0$. Hence $\nabla^k F(f_n) = P^k_{f_n}(f_n + \xi) + o(1)$. Now write $f^*_n = f_n + \xi$, and let Y^* be the translated manifold $Y + \xi$. Since $P^0_{f^*} = P^0_f$, it follows that $P^k_{f^*} = P^k_f$ for all f. Therefore, we have $\|f^*_n\|_k \leq$ constant and $P^k_{f^*_n} f^*_n \to 0$, and we can apply Saber's theorem to obtain a convergent subsequence of $\{f^*_n\}$.

Now suppose that A is not necessarily strictly positive, but that Y is compact. By a well-known theorem of Garding [5], there exists a $\lambda > 0$ such that $A + \lambda I$ is strictly positive. Let $A_\lambda = A + \lambda I$, and write $F(f) = \frac{1}{2}(A_\lambda f, f)_0 + (J(f) - \frac{1}{2}\|f\|^2_0)$. It is easy to show that the map $f \to \|f\|^2_0$ satisfies Condition Γ, so that $J(f) - \frac{1}{2}\|f\|^2_0$ is a C^1 function which is bounded below and satisfies Condition Γ.

Remark. Cf. Eells [1, p. 786]. We note that the theorem he gives here does not apply in our case since, among other things, the map $f \to N^k_f f$ is not compact.

D. Examples. 1. The following are examples of functions which satisfy Condition Γ.

(i) If $l < k$, then $f \to \|f\|^2_l$ satisfies Condition Γ.

(ii) If F satisfies Condition Γ, and g is a C^1 function on $\boldsymbol{R}$, then $g \circ F$ satisfies Condition Γ.

(iii) If V is a C^1 function on Y, then $f \to \int_X V \circ f d\mu$ satisfies Condition Γ.

2. Let $X = S^1, A = -d^2/dt^2$, and $F(f) = \frac{1}{2}(Af, f)_0 - \int_{S^1} V \circ f dt$. It is easy to see that $\nabla^0 F(f) = -P^0_f(d^2 f/dt^2) - \nabla V(f) = -D^2 f/dt^2 - \nabla V(f)$. Hence, interpreting V as the potential of a conservative dynamical system, we get the following result: If Y is compact, then every homotopy class of maps from S^1 to Y contains at least one solution to the dynamical equation $D^2 f/dt^2 = -\nabla V$. For the case $V = 0$, we get the well-known theorem of Fet: If Y is compact, then every map $S^1 \to Y$ is homotopic to a geodesic.

3. The following example shows that the boundedness condition on J is necessary. Let $X = S^1, Y = \boldsymbol{R}, A = 1 - d^2/dt^2$. Let $F(f) = \frac{1}{2}(Af, f)_0 - \frac{1}{2}\|f\|^2_0$. Then $J(f) = -\frac{1}{2}\|f\|^2_0$ is not bounded below. Let $f_n(t) = n + n^{-4}\cos nt$ $(0 \leq t \leq 2\pi)$. Writing everything down in terms of Fourier series (so that one obtains a simple expression for G) it can be shown that $|F(f_n)| \leq$ constant, $\nabla^1 F(f_n) \xrightarrow{H^k} 0$, and that $\|f_m - f_n\| \geq |m - n|$. Hence F does not satisfy Condition C.

3. The curvature structure of H^k

A. Let $M \to H$ be a smooth isometric embedding of a (possibily infinite dimensional) riemannian manifold M into a hilbert space H, and for each $x \in M$, let P_x be the orthogonal projection $H \to T_x(M)$. Hereafter we delete the appearance of the variable x in P and dP. Let $\eta, \xi, \theta, \cdots$ denote vector fields on M. Then, generalizing some well-known facts about finite dimensional manifolds, the Riemannian affine connection ∇ and curvature form R are given by

$$\nabla_\xi \theta = P\theta_*(\xi) \ , \tag{10}$$

$$R(\eta, \xi)\theta = [\nabla_\xi, \nabla_\eta]\theta - \nabla_{[\xi,\eta]}\theta \ . \tag{11}$$

From these last two relations one obtains

$$R(\eta, \xi)\theta = [dP(\xi)dP(\eta) - dP(\eta)dP(\xi)]\theta \ . \tag{12}$$

To derive this last result note that $dP = d(PP) = (dP)P + P(dP)$. Hence (i) $(dP)P = NdP$ and (ii) $(dP)N = PdP$ where $N = I - P$. Similarly, from the relation $P\theta = \theta$ one obtains (iii) $dP(\xi)\theta = N\theta_*(\xi)$ where θ_* denotes the differential of θ. We identify $T_x(M)$ with a subspace of H, so that θ_* is a map from $T(M)$ to $T(H)|M$. By abuse of notation, we let θ_* also represent to composition of the two maps $T(M) \to T(H)|M$ and $T(H)|M \to H$ where this latter map is the natural injection. Also, from (i) and (ii) we see that $dP(\xi)dP(\eta)\theta = dP(\xi)dP(\eta)P\theta = dP(\xi)NdP(\eta)P\theta = PdP(\xi)NdP(\eta)P\theta = PdP(\xi)dP(\eta)\theta$ so that the right hand side of (12) is actually a vector tangent to M. Now, from (10) and (iii) a direct calculation shows that (iv) $[\nabla_\xi, \nabla_\eta]\theta = P(dP(\xi)dP(\eta) - dP(\eta)dP(\xi))\theta + P((\theta_* \circ \eta)_*(\xi) - (\theta_* \circ \xi)_*(\eta))$ and that (v) $\nabla_{[\xi,\eta]}\theta = P\theta_*[\xi, \eta]$. If F is a function on H, we have $\theta_*[\xi, \eta]F = [\xi, \eta](F \circ \theta) = \xi(F_* \circ (\theta_* \circ \eta)) - \eta(F_* \circ \xi)) = F_{**} \circ (\theta_* \circ \eta)_*(\xi) - F_{**} \circ (\theta_* \circ \xi)_*(\eta)$. This shows that $\nabla_{[\xi,\eta]}\theta = P((\theta_* \circ \eta)_*(\xi) - (\theta_* \circ \xi)_* \eta)$, which with (iv) and (11) yields the desired result.

B. For later calculations we have to specialize these results to the finite dimensional case. Let the embedding $Y \to \boldsymbol{R}^q$ (see Paragraph 1A) be given locally by vector-valued functions $w = w(y^1, y^2, \cdots y^n)$ where $(y^1, y^2, \cdots, y^n)$ are local coordinates on Y. Let $w_i = \partial w / \partial y^i$, and let w_{ij} denote the coefficientes of the second covarient differential of w. For a tangent vector ξ we write $\xi = \xi^i(\partial / \partial y^i) = \xi^i w_i$ (summation convention), so that the second fundamental form of Y is given by the symmetric bilinear vector-valued form $B(\xi, \eta) = w_{ij}\xi^i\eta^j$. For each $y \in Y$, let P_y^0 represent the orthogonal projection $\boldsymbol{R}^q \to T_y(Y)$. Then for $v \in \boldsymbol{R}^q$, $P_y^0 v = \langle v, w^i \rangle w_i = \langle v, w_i \rangle w^i$ where indices are raised and lowered in the usual tensorial fashion via the metric tensor on Y. A direct calculation shows that

$$dP^0(\xi)v = \langle v, w_{ij} \rangle \xi^j w^i + \langle v, w^i \rangle w_{ij} \xi^j \ , \tag{13}$$

where $\langle , \rangle$ is the standard inner product on $\boldsymbol{R}^q$. Note that if P_f^0 is the operator discussed in Paragraph 1B, then $(P_f^0 v)(x) = P_{f(x)}^0 v(x)$.

Let S denote the curvature form on Y. Then from (12) and (13) one obtains the well-known result (one of the equations of Gauss and Codazzi, see [2]).

$$S(\eta, \xi)\eta, \xi = B(\eta, \eta), B(\xi, \xi) - B(\eta, \xi), B(\eta, \xi) \; . \tag{14}$$

C. We now apply these results to the embedding $H^k \to \hat{H}^k$. Our main result is the following

Theorem. *Let R denote the curvature form on H^k. Then for $\eta, \xi \in T_f H^k$,*

$$dP_f^k(\xi)\eta = N_f^k dP_f^0(\xi)\eta \; , \tag{15}$$

$$(R(\eta, \xi)\eta, \xi)_k = (N_f^k B(\eta, \eta), N_f^k B(\xi, \xi))_k - \|N_f^k B(\eta, \xi)\|_k^2 \; . \tag{16}$$

Hereafter we delete the appearance of the variable f in P^k, P^0, dP^k, dP^0. From (2) we have $dP^0 = d(P^k P^0) = (dP^k)P^0 + P^k dP^0$. Hence $(dP^k)P^0 = N^k(dP^0)$. Multiplying on the right by P^0 we get $(dP^k)P^0 = N^k(dP^0)P^0$, which proves (15).

Using (12), we have

$$\begin{aligned}(R(\eta, \xi)\eta, \xi)_k &= (dP^k(\xi)dP^k(\eta)\eta, \xi) - (dP^k(\eta)dP^k(\xi)\eta, \xi)_k \\ &\quad + (dP^k(\eta)\eta, dP^k(\xi)\xi)_k - (dP^k(\xi)\eta, dP^k(\eta)\xi)_k \; .\end{aligned}$$

But from (13) and (15) we get $dP^k(\eta)\eta = N^k dP^0(\eta)\eta = N^k B(\eta, \eta)$. Similarly, $dP^k(\xi)\eta = N^k B(\xi, \eta) = N^k B(\eta, \xi)$ (since B is symmetric) $= dP^k(\eta)\xi$, which proves (16).

Before continuing we note the following relation (which will not be used in the sequel):

$$dP^k = (N^k + P^k G)dP^0(P^k + AN^k) \; . \tag{17}$$

Proof. Applying d to the relation (4) $P^k GN^0 = 0$, we obtain $(dP^k)GN^0 = P^k G dP^0$. In the derivation of (15) we obtained $(dP^k)P^0 = N^k dP^0$. Hence $(dP^k)(P^0 + GN^0) = (N^k + P^k G)dP^0$. But from (2) and (4) we have $(P^0 + GN^0) \cdot (P^k + AN^k) = I$.

D. Examples. *General Remarks*: As mentioned in the Introduction, one might hope that the functor $Y \to H^k(X, Y)$ preserves the property of having Riemannian sectional curvature of definite sign. We shall show by specific examples (the loop spaces of spheres and cylinders) that this is not the case. For computations we use (16), and we must therefore be able to compute $\|N_f^k u\|_k$ for a general $u \in \hat{H}^k$. Now $N_f^k u = u - P_f^k u$, and setting $v = P_f^k u = \tilde{G}_f P_f^0 A u$ (from (6)), and multiplying on the left by $P_f^0 A$, we obtain

$$\begin{aligned}&P_f^k u = v, \text{ where } v \text{ is the unique element of } H^k \\ &\qquad \text{satisfying } P_f^0 v = v \text{ and } P_f^0 A v = P_f^0 A u \; .\end{aligned} \tag{18}$$

The relations (18) can be obtained in another way: They are the Euler-

Lagrange equations for variational problem $\|N_f^k u\|_k = \inf\{\|u - \xi\|_k \mid \xi \in T_f H^k\}$.

There is one special case in which these computations are especially easy; viz., the case $f =$ constant; for if f is constant, then, from the relation $P_f^0 u = \langle u, w^i\rangle w_i$ (see Paragraph B), we see that $P_f^0 A = AP_f^0$. Therefore, from the remarks following (2), we get the following statement:

(19) If f is constant, then $P_f^k = P_f^0$.

In the computations below we use the notation of Paragraph B. We let $\{e_1, e_2, e_3\}$ be the standard basis for $\boldsymbol{R}^3$ and write (a, b, c) for $ae_1 + be_2 + ce_3$.

Example 1. $X = S^1$, $Y = S^2 =$ unit sphere $x^2 + y^2 + z^2 = 1$, $A = 1 - d^2/dt^2$, $f =$ constant. Then $B(\eta, \xi) = -\langle \xi, \eta\rangle w(f)$. Using (16), (19), and integrating by parts, we get

$$(R(\eta, \xi)\eta, \xi)_1 = \int \{\langle \eta, \eta\rangle\langle \xi, \xi\rangle - \langle \eta, \xi\rangle^2\}dt$$
$$+ \int \left\{\frac{d}{dt}\langle \eta, \eta\rangle \frac{d}{dt}\langle \xi, \xi\rangle - \left|\frac{d}{dt}\langle \xi, \eta\rangle\right|^2\right\} dt \ .$$

Let $f(t) \equiv (0, 0, 1)$; $\eta = \eta_1 e_1, \eta_1 =$ constant; $\xi = \xi_1 e_1 + \xi_2 e_2$. Then

$$(R(\eta, \xi)\eta, \xi)_1 = \eta_1^2 \int \left\{\xi_2^2 - \left|\frac{d\xi_1}{dt}\right|^2\right\} dt \ .$$

Hence we see that $(R(\eta, \xi)\eta, \xi)_1$ may be positive, negative, or zero.

Example 2. $X = S^1$, $Y =$ cylinder $x^2 + y^2 = 1, A = 1 - d^2/dt^2, f =$ constant. We describe the cylinder by the parametric equations $w(\theta, z) = (\cos \theta, \sin \theta, z)$. Let $w_1 = \partial w/\partial\theta = (\text{-}\sin \theta, \cos \theta, 0)$ and $w_2 = \partial w/\partial z = (0, 0, 1)$. For tangent vectors ξ, η write $\xi = \xi_1 w_1 + \xi_2 w_2, \eta = \eta_1 w_1 + \eta_2 w_2$. Note that $P^0 v = \langle v, w_1\rangle w_1 + \langle v, w_2\rangle w_2$. Now, in general, $w_{ij} = \partial w_i/\partial y^j - \Gamma_{ij}^k w_k$ where Γ_{ij}^k are the Christoffel symbols; in our case $\Gamma_{ij}^k = 0$. Hence

(i) $B(\eta, \xi) = \xi_1\eta_1(\partial^2 w/\partial\theta^2) = -\xi_1\eta_1 (\cos \theta, \sin \theta, 0)$.

It turns out that, for $f =$ constant,

$$(R(\eta, \xi)\eta, \xi)_1 = -\int |\eta_1(d\xi_1/dt) - \xi_1(d\eta_1/dt)|^2 dt \ .$$

Hence the sectional curvature may be negative or zero. In the next example we shall show that for $f \neq$ constant the sectional curvature may be positive. Hence we have an example of a manifold Y of zero curvature such that the curvature of $H^1(S^1, Y)$ is indefinite.

Example 3. X, Y and A as above; $f(t) = (\cos t, \sin t, 0), 0 \leq t \leq 2\pi$. Let $u(t) = \phi(t)\ \partial^2 w/\partial\theta^2 = -\phi(t) (\cos t, \sin t, 0)$ be a general element of $\hat{H}$ satisfying

$P^0_f u = 0$. We want to compute $\|N^1_f u\|_1$. Writing $v = v_1 w_1 + v_2 w_2$ equations (18) reduce to

$$\text{(ii)} \qquad \begin{aligned} \frac{d^2 v_1}{dt^2} - v_1 &= -2\frac{d\phi}{dt}\,, \\ \frac{d^2 v^2}{dt^2} - v_2 &= 0\,. \end{aligned}$$

Hence $v_2 = 0$, since $v_2 = v_2(t)$ is periodic in t. The remaining equation in (ii) can be solved explicity by the use of Fourier series. Writing $\phi(t) = \sum \phi_n e^{int}$ where here, as always, all sums run from $-\infty$ to $+\infty$, it turns out that $(1/2\pi)\|P^1_f u\|_1^2 = 8 \sum (n^2/(n^2+2))\,|\phi_n|^2$, and that $(1/2\pi)\;\|N^1_f u\|_1^2 = \sum J_n |\phi_n|^2$ where $J_n = 2(n^4+4)/(n^2+2)$, $-\infty \leq n \leq +\infty$. Itfollows that if u^1 is another element of $\hat{H}$ satisfying $P^0_f u^1 = 0$, then

$$\text{(iii)} \qquad (N^k_f u, N^k_f u^1)_1 = \sum J_n \phi_n \bar{\phi}'^1_n\,.$$

Referring to (i) of Example 2, we see that $B(\eta, \xi) = -gh(\cos\theta, \sin\theta, 0)$ where $g = \langle \eta, w_1 \rangle, h = \langle \xi, w_1 \rangle$. Writing R for $(1/2\pi)(R(\eta, \xi)\eta, \xi)_1$, we obtain

$$\text{(iv)} \qquad R = \sum J_n \{(g^2)_n (h^2)_n - |(gh)_n|^2\}\,,$$

where $(g^2)_n, (h^2)_n$ and $(gh)_n$ are the Fourier coefficients of the indicated functions. Using the convolution law $(gh)_k = \sum_n \bar{g}_{n-k} \bar{h}_n = \sum_n g_n \bar{h}_{n+k}$, we get

$$\text{(v)} \qquad R = \sum_k J_k \left\{ \left(\sum_n g_{n-k} \bar{g}_n \right) \left(\sum_n h_{n-k} \bar{h}_n \right) - \left| \sum_n g_{n-k} \bar{h}_n \right|^2 \right\}.$$

This is the general expression for the sectional curvature. Let $g(t) \equiv 1/(2\pi)$. Then going back to (iv) and using the relation $(h^2)_0 = \sum |h_n|^2$, we get

$$\text{(vi)} \qquad R = J_0 \sum |h_n|^2 - \sum J_n |h_n|^2\,.$$

If $h(t) = e^{it} - e^{-it}$, we have $R = 2J_0 - J_{-1} - J_1 = 2(J_0 - J_1) > 0$. Hence this sectional curvature is positive.

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