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INFINITE DIMENSIONAL PRIMITIVE LIE ALGEBRAS

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Introduction

Transitivity questions in differential geometry can often be reduced to problems involving a certain type of "topologized" Lie algebra. In [6] we developed a structure theory for such algebras analogous to the classical Jordan-Hölder theory for groups and rings. (See paragraphs 4 and 5 in § 2 below.) The building blocks of this theory are the primitive algebras. In this paper we will study these algebras in detail. In particular we will sketch the "Cartan classification theorem" for primitive algebras over an algebraically closed field. In its broad outlines, our proof follows Weisfeiler [19].

Weisfeiler's proof is based on a remarkable theorem of Kac about infinite dimensional graded Lie algebras [11]. We will show how this theorem can be deduced by a simple completion trick from the following theorem proved in § 3 below: *An infinite dimensional linearly compact Lie algebra possesses at most one primitive subalgebra.* This theorem can be proved without assuming that the base field is algebraically closed, and the proof requires relatively little machinery (mainly some elementary results from commutative algebra).

This paper is organized as follows: The first section is a compendium of standard results on primitivity (included mainly for motivation).

§ 2 is a review of the material in [6]. In § 3 we prove our main theorems on primitivity (modulo some results on characteristics which are proved in the appendix). In § 4 we discuss primitivity for graded algebras and prove two important lemmas (Lemmas 4.2 and 4.3). In §§ 5, 6 we prove the theorem of Kac alluded to above. The rest of the paper is a sketch of the Cartan classification theorem, the main idea of which is the Weisfeiler trick of associating with every primitive Lie algebra a graded Lie algebra with the property that the term of degree zero acts irreducibly on the term of degree -1.

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1. Primitivity

Let G be a group and let S be a set on which G acts. Let "$\sim$" be an equivalence relation on S. We will say that "$\sim$" is invariant with respect to G

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if $a \sim b$ implies $ga \sim gb$ for all g in G. An example is the identity equivalence relation (any two distinct points are inequivalent); and another is the trivial equivalence relation (all points are equivalent). If these are the *only* invariant equivalence relations on S we will say that the action of G on S is primitive. We list below a few elementary facts about primitive group actions:

1. If G acts primitively on S there is a single orbit for the action since the equivalence relation:

$a \sim b \Longleftrightarrow a$ and b on the same orbit, is an invariant equivalence relation. Therefore if H is the isotropy group at some point, S can be identified with the coset space G/H such that the action of G on S is identical with the usual left coset action.

2. H is a maximal subgroup of G; for suppose G contains a proper subgroup H' sitting over H. Then there is a natural projection $\pi\colon G/H \to G/H'$, and the equivalence relation $x \sim y \Longleftrightarrow \pi(x) = \pi(y)$ is an invariant equivalence relation on G/H.

3. It is easy to see that the converse is true. If H is a maximal subgroup of G, the action of G on G/H is primitive. Therefore, the problem of determining primitive representations of G reduces to the problem of determining maximal subgroups.

4. Suppose we require that the representation of G on S be faithful. This amounts to requiring that the isotropy subgroup H contain no normal subgroup of G.

It may happen for certain groups that no such subgroups exist. (An example is the additive group of integers.)

5. Let G be a Lie group and $\mathscr{G}$ its Lie algebra. Let H be a closed connected subgroup of G. Then for the representation of G on G/H to be primitive, the subalgebra of $\mathscr{G}$ corresponding to H must be maximal. If in addition we require the representation to be faithful we must require that this maximal subalgebra contain no ideals of $\mathscr{G}$ except $\{0\}$. We will call such a subalgebra *primitive*.

6. Not all Lie algebras possess primitive subalgebras: A result of Morozov [15] says that if $\mathscr{G}$ possesses a primitive subalgebra, then $\mathscr{G}$ is either semisimple or is an abelian extension of a semisimple algebra. Moreover, if $\mathscr{G}$ is semisimple, then it is either simple or is of the form $\mathscr{G} = \mathscr{G}_1 \oplus \mathscr{G}_1$ where $\mathscr{G}_1$ is simple. If $\mathscr{G}$ is not simple, and possesses a primitive subalgebra, this subalgebra is unique up to conjugacy by inner automorphisms. If $\mathscr{G}$ is simple, however, then every maximal subalgebra is primitive, and there exist maximal subalgebras which are not conjugate. Those which are of the same rank as $\mathscr{G}$ were classified by Borel-de Siebenthal in [2]. The non-maximal rank ones were determined by Dynkin in [4]. (Dynkin's result involves an extremely deep theorem about the non-existence of inclusion relations among linear representations of the simple groups.) Some amplifications of the Borel-de Siebenthal results were obtained recently by Kobayashi and Nagano [13], Ochiai [16], and Golubitsky [5].

7. The notion of primitivity makes sense for pseudogroups as well as for groups. (An example of a primitive pseudogroup is the set of all diffeomorphisms of open sets of $\boldsymbol{R}^n$. For definitions concerning pseudogroups, see [18].)

Elie Cartan showed that there are very few primitive pseudogroups which are not associated with primitive actions of Lie groups. He gave a complete list of these in [20]. Besides the pseudogroup described above they include the pseudogroup of volume preserving transformations on $\boldsymbol{R}^n$, the pseudogroup of symplectic transformations on $\boldsymbol{R}^{2n}$, the pseudogroup of contact transformations on $\boldsymbol{R}^{2n+1}$, and complex versions of these pseudogroups.

8. The problem of determining the primitive pseudogroups can be reduced to a purely algebraic problem: in a certain class of infinite dimensional Lie algebras determine those which prossess primitive subalgebras. A precise formulation of this problem will be given below (§ 3). In [7] Quillen, Sternberg, and the author solved this problem over the field of complex numbers, and verified the results of Cartan described above. However, our proof was in a certain respect unsatisfactory: It required imbedding the given Lie algebra into a Lie algebra of holomorphic vector fields; and to do this we needed a rather complicated result from analysis, the Cartan-Kaehler theorem.

Recently B. Yu. Weisfeiler gave a purely algebraic proof of our result, using some remarkable theorems on infinite dimensional graded Lie algebras due to Kac [11]. In §§ 7–9 we will sketch a version of this argument, and we will prove Kac's results in § 6.

2. The linearly compact topology

We will review some basic definitions from our earlier paper [6]:

Let Δ be a field of characteristic zero. We will give Δ the discrete topology so that we can think of it as a topological field. Let W be a topological vector space over Δ. The following three properties turn out to be equivalent:

a) W is the projective limit of finite dimensional discrete spaces.

b) W is the topological dual of a discrete space.

c) W is the product of finite dimensional discrete spaces with the standard product topology.

If W has one of the above properties we will say that W is *linearly compact*. The basic facts about such spaces have been summarized in [6]. One useful fact is the following:

If W is linearly compact then a subspace of W is open if and only if it is closed and of finite codimension. The open subspaces form a system of neighborhoods for the origin.

Another useful fact is *Chevalley's principle*:

Let $W_1 \supset W_2 \supset \cdots$ be a sequence of closed subspaces of a linearly compact topological space and suppose $\bigcap W_i = \{0\}$. If U is a neighborhood of the origin then for some i_0, $W_{i_0} \subset U$.

By a *topological Lie algebra* over Δ we will mean a topological vector space W over Δ together with a bracket operation "[ ]": $W \times W \to W$ which is bilinear, continuous, antisymmetric, and satisfies Jacobi's identity. A topological Lie algebra will be called *linearly compact* if as a topological vector space it is linearly compact.

Example 1. Let L be any finite dimensional Lie algebra over Δ with the discrete topology. Then L is linearly compact.

Example 2. Let F_n be the ring of formal power series in n indeterminants over Δ, and $\tilde{L}_n$ be the Lie algebra of Δ-linear derivations of F_n. Every such derivation can be written as a "vector field" of the form $f_1\frac{\partial}{\partial x_1} + \cdots + f_n\frac{\partial}{\partial x_n}$, where each f_i is a formal power series in $x_1, \cdots, x_n$. Therefore as a vector space $\tilde{L}_n$ is isomorphic to n copies of F_n. If we give F_n its usual Krull topology, which is linearly compact, then $\tilde{L}_n$ acquires a linearly compact topology, and it is not hard to see that the bracket operation is continuous.

Example 3. Any closed subalgebra of the algebra above. This provides many examples of linearly compact algebras which are not finite dimensional. (See below.)

It turns out that the category of linearly compact Lie algebras is small enough for some interesting theorems to be true, and large enough to include most of the interesting examples which come up in differential geometry. For details we refer to our article "A formal model of transitive differential geometry" [9]. We list below some relevant facts about linearly compact Lie algebras, most of which are proved in [6].

1. If L is linearly compact the following two properties are equivalent.

(2.1)
a) There is a neighborhood of zero containing no ideals except $\{0\}$.
b) L satisfies the descending chain condition of closed ideals.

2. Let F_n be the ring of formal power series in n indeterminants over Δ, and $\tilde{L}_n$ be the Lie algebra of Δ-linear derivations of F_n. (See Example 2 above.) A subalgebra L of $\tilde{L}_n$ is called *transitive* if no ideals of F are invariant under L except $\{0\}$ and F_n itself. It is not hard to see that a transitive subalgebra of $\tilde{L}_n$ satisfies condition a) of (2.1). Conversely any algebra which satisfies either of the conditions (2.1) is isomorphic to a transitive subalgebra of $\tilde{L}_n$ for some n. (An elegant proof of this result can be found in Blattner [1].)

3. Suppose that L satisfies condition a) of (2.1), i.e., suppose there is a neighborhood $\mathcal{O}$ of zero in L containing no ideals except $\{0\}$. We can assume that $\mathcal{O}$ is a closed subspace of L of finite codimension. (See the remarks at the beginning of the section.)

Let $L^0 = \{x \in \mathcal{O} \mid [x, y] \in \mathcal{O}, \forall y \in L\}$. It is not hard to show that L^0 is also closed and of finite codimension; and it is also easy to see that it is a subalgebra. Since $\mathcal{O}$ contains no non-zero ideals, the same is true of L.

Definition 2.1. A closed subalgebra L^0 of L, which is of fininite codimension in L and contains no ideals except $\{0\}$, is called a *fundamental subalgebra* of L.

From the previous remarks we see that L has a fundamental subalgebra if and only if it satisfies the conditions (2.1).

4. Suppose L satisfies the conditions (2.1). Then there exists a sequence of closed ideals

$$(*) \qquad L = I_0 \supset I_1 \supset \cdots \supset I_k = \{0\}$$

such that for all $0 \leq j \leq k$ one of the following alternatives holds.

a) There are no closed ideals of L properly containing I_{j+1} and properly contained in I_j.

b) $[I_j, I_j] \subset I_{j+1}$.

Though the above sequence is not unique the non-abelian quotients I_j/I_{j+1} occurring in the sequence are unique up to isomorphism (multiplicities counted).

5. Let I be a non-abelian quotient occurring in the sequence (*). Then as a Lie algebra I is isomorphic to a tensor product $R \hat{\otimes} F_n$, where F_n is the formal power series ring in n indeterminants and R is a simple non-abelian linearly compact Lie algebra. (Here the symbol $\hat{\otimes}$ denotes the tensor product in the category of linearly compact topological spaces, and is a completion of the usual algebraic tensor product.)

3. Primitivity for linearly compact Lie algebras

Let L be a linearly compact Lie algebra, and L^0 be a fundamental subalgebra of L. (See Definition 2.1 above.) We will say that L^0 is *primitive* if it is a maximal subalgebra of L.

We pointed out above that only rather special kinds of finite dimensional Lie algebras possess primitive subalgebras. (See paragraph 5 of § 1.) We will see below that the same thing is true for the infinite dimensional algebras. We begin by discussing some concepts required for the proof of our main results:

1. Let L be a linearly compact Lie algebra satisfying the descending chain condition on closed ideals, Θ be the collection of all closed subspaces of L which are invariant with respect to the group of continuous automorphisms of L, and α be a derivation of L. We will say that α is *exponentiable* if for all $H \in \Theta$, $\alpha(H) \subset H$. In particular we will say that an element x of L is exponentiable if the derivation $\operatorname{ad} x$ is exponentiable.

Proposition 3.1. *Let L^0 be a fundamental subalgebra of L, and α be a derivation of L which maps L^0 into L^0. Then α is exponentiable.*

Proof. We define a filtration on L as follows. Starting with L^0, we define:

$$L^i = \{x \in L^{i-1} \mid [x, y] \in L^{i-1}, \forall y \in L\}$$

for $i = 1, 2, 3$, etc. It is easy to verify that the L^i are closed and of finite codimension in L, and that the bracket of L^i with L^j is in L^{i+j} for all i and j.

The intersection of the L^i's is an ideal in L; however, L^0 is a fundamental subalgebra by assumption, so this intersection is zero. This means that the sequence $L, L^0, L^1, \cdots$ is a filtration of L in the sense of [6]. It is easy to verify inductively that α maps L^i into L^i for all i.

Now let

$$[\quad]_k : L/L^k \times L/L^k \to L/L^{k-1}$$

be the truncation of the bracket operation on L. Let G^k be the group of all linear mappings of L/L^k onto L/L^k, which preserve the bracket $[\quad]_k$ and the filtration on L/L^k. In [9] we proved the following:

Lemma. *There exists an integer k_0 such that for all $k > k_0$ every element ρ in G^k extends to a continuous automorphism $\bar{\rho}: L \to L$ with the property that $\bar{\rho}(L^i) = L^i$ for all i.* (See Theorem II on page 272.)

Let H be a closed subspace of L belonging to Θ. We will show that for all k sufficiently large, α maps $H + L^k$ into itself. Since H is invariant under all continuous automorphisms of $L, (H + L^k)/L^k$ is invariant under G^k by the lemma. However, G^k is an algebraic group whose Lie algebra contains the endomorphism induced on L/L^k by α. Therefore, $H + L^k$ is invariant under α. Since H is closed, $H = \bigcap (H + L^k)$ (see Chapter 1 of [6],); so $\alpha(H) \subset H$.

2. If L is a finite dimensional Lie algebra, every derivation is exponentiable. We will show that this is not always the case for the infinite dimensional algebras. To show this we need the following fact.

Proposition 3.2. *Let L be an infinite dimensional linearly compact Lie algebra satisfying the descending chain condition on closed ideals. Then there exists a proper closed subspace of L of finite codimension which is mapped into itself by every continuous automorphism of L.*

The proof of this proposition involves some properties of the "characteristic variety of L." We will discuss this concept and prove the above assertion in the appendix.

Our first main result is a corollary of Proposition 3.2.

Proposition 3.3. *Let L be a simple infinite dimensional linearly compact Lie algebra. Then L possesses a unique primitive subalgebra which consists of the exponentiable elements of L.*

Proof. Let L^0 be the set of exponentiable elements of L. It is clear that L^0 is a closed subalgebra of L. By Proposition 3.1, L^0 contains every fundamental subalgebra of L. Since L is simple, L^0 is itself a fundamental subalgebra providing it is not equal to L. Suppose $L^0 = L$; i.e., suppose every element of L is exponentiable.

By Proposition 3.2 there exists a proper closed subspace H of finite codimension in L, which is invariant under $\operatorname{ad} x$ for all x in L. This, however, implies that H is an ideal; and thus contradicts our assumption that L is simple and infinite dimensional. Therefore $L^0 \neq L$, and we are done.

Remark. Proposition 3.3 is not true if L is finite dimensional. (See paragraph 5 in § 1.)

3. The following proposition shows that if a linearly compact Lie algebra admits a primitive subalgebra, then it is a finite dimensional extension of a simple algebra.

Proposition 3.4. *Let M be an infinite dimensional linearly compact Lie algebra possessing a primitive subalgebra M^0, and L be the intersection of all the closed non-zero ideals of M. Then L has the following properties*:

a) *L is of finite condimension in M.*

b) *As a Lie algebra, L is simple and non-abelian.*

c) *The adjoint representation of M on L is faithful.*

d) *$L \cap M^0$ is the unique primitive subalgebra of L described in Proposition* 3.3.

Proof. We define a filtration on M just as we did in the proof of Proposition 3.1. Starting with M^0 we define

$$M^k = \{x \in M^{k-1} \mid [x, y] \in M^{k-1}, \forall y \in M\}$$

for $i = 1, 2, 3$, etc. Let I be an arbitrary closed non-zero ideal of M. Since $I + M^0$ is a subalgebra of M containing M^0, the primitivity of M^0 implies $I + M^0 = M$. Therefore, the homomorphism $M^0/(M^0 \cap I) \to M/I$ is bijective. M satisfies the d.c.c. on closed ideals; so M/I does also. Each M^k is a closed ideal of M^0, and $\cap M^k = \{0\}$; so for some k $M^k \subset M^0 \cap I$. This shows in particular that every closed ideal of M is of finite codimension. If L is the intersection of all non-zero closed ideals, it is also of finite codimension because of the d.c.c. on closed ideals in M. In particular, L is non-zero; so if we apply the previous argument to it, we get $L \supset M^k$ for large k.

Next we will show that M^0 does not contain non-zero ideals of L. Suppose J is an ideal of L contained in M^0. Then $[L, J] \subset J$ and $J \subset M^0 \Rightarrow [L, J] \subset M^0$; and $J \subset M^0 \Rightarrow [M^0, J] \subset M^0$. We showed above that $M = L + M^0$; so we get $[M, J] \subset M^0$ which implies $J \subset M^1$. If the same argument is repeated with M^0 replaced by M^1, M^2, etc., we get $J \subset M^2$, M^3, etc. and finally $J \subset \cap M^k = \{0\}$, which proves our assertion.

We will now show that L is non-abelian. If L were abelian, we would get $[L, M^k] = \{0\}$ for large k because, as we saw above, $L \supset M^k$ for large k. However, if this were to happen, M^k would be an ideal of L contained in M^0 contradicting what we just proved.

Let J be a closed ideal of L such that L/J is simple and non-abelian. (Such an ideal exists by Proposition 6.1 of [6].) For every integer $i > 0$ let J^i be the set of all $a \in L$ such that

$$\operatorname{ad}(x_1) \cdots \operatorname{ad}(x_k) a \in J$$

for all $k < i$ and all $x_1, \cdots, x_k \in M$. It is easy to verify that $[J^i, J^k] \subset J^{i+k}$.

In particular, since $J^0 = L, J^i$ is an ideal of L. The intersection $\bigcap J^i$ is an ideal of M; so this intersection is zero because by definition L is contained in every non-zero ideal of M. By Chevalley's principle, $J^i \subset M^0$ for some large i; and since we have shown that M^0 contains no non-zero ideals of $L, J^i = \{0\}$. This shows in particular that J is nilpotent.

In [6] we showed that the graded Lie algebra $\Sigma J^i/J^{i+1}$ is isomorphic to a tensor product $I/J \otimes S$ where S is a graded polynomial ring. (See Proposition 7.2.) This contradicts the nilpotency of J unless S is just the polynomial ring in zero variables (i.e. the ring of scalars). In this case, J is zero, and L is simple as a Lie algebra.

Consider the representation of M on L obtained by restricting the adjoint representation. The kernel of this representation is a closed ideal I' of M; so either $I' = \{0\}$ or $I' \subset L$. The second alternative is ruled out because L is non-abelian, so the adjoint representation on L is faithful.

We have proved all items of Proposition 3.4 except d). To prove d) let L^0 be the set of exponentiable elements of the Lie algebra L. It is obvious that every continuous automorphism of M maps L^0 into L^0, so $\operatorname{ad} x$ maps L^0 into L^0 whenever x is an exponentiable element of M. By Proposition 3.1, M^0 consists of exponentiable elements, so $[M^0, L^0] \subset L^0$. This proves that $M^0 + L^0$ is a subalgebra of M. Because of the primitivity, it is either equal to M or to M^0; and since $L^0 \supset L \cap M^0$, it is equal to M^0. Thus $L^0 \subset M^0$ and $M^0 \cap L = L^0$. This concludes the proof of d).

4. As a corollary of Proposition 3.4 we get the following strengthened form of Proposition 3.3.

Proposition 3.5. *Let M be an infinite dimensional linearly compact Lie algebra possessing a primitive subalgebra M^0. Then M^0 is the only primitive subalgebra of M, and consists of the exponentiable elements of M.*

Proof. Let M' be the set of exponentiable elements of M. M' contains M^0 by Proposition 3.1; so either $M' = M^0$ or $M' = M$. Let L be the ideal described in Proposition 3.4, and L^0 be the set of exponentiable elements of the Lie algebra L. Every continuous automorphism of M is also a continuous automorphism of L, and so it has to preserve L^0. Therefore, if x is an exponentiable element of M, then $\operatorname{ad} x$ maps L^0 into L^0. If M' were equal to M, then L^0 would be an ideal of M, contradicting the fact that L is the smallest non-zero ideal of M. So $M' = M^0$ and we are done.

4. Primitivity for graded Lie algebras

In this section we will consider graded Lie algebras[1] of the form:

[1] By a graded Lie algebra we will mean an ordinary Lie algebra which is graded in such a way that the bracket of an element of degree k with an element of degree l is of degree $k + l$. Some authors use this term to mean an algebra with a bracket operation which satisfies the identity $[x, y] = (-1)^{k+l}[y, x]$ if x is of degree k and y of degree l.

$$\mathscr{G} = \sum_{-\infty}^{\infty} g^l \ . \tag{4.1}$$

We will assume each of the summands g^l is a finite dimensional vector space, and we will also assume that the summation on the left is finite, i.e., that there is a positive integer k such that $g^{-k} \neq \{0\}$ and $g^{-l} = \{0\}$ for $-l < -k$.

We will denote by $\mathscr{G}^0$ the subalgebra of $\mathscr{G}$ consisting of elements of degree greater than or equal to zero, i.e., the sum $\sum_{l=0}^{\infty} g^l$. We will also denote by $\mathscr{G}^+$ the subalgebra of $\mathscr{G}$ consisting of elements of positive degree, and by $\mathscr{G}^-$ the subalgebra consisting of elements of negative degree.

Definition 4.1. We will say that the graded Lie algebra $\mathscr{G}$ is *primitive* if $\mathscr{G}^0$ is a maximal graded subalgebra of $\mathscr{G}$, and contains no graded ideals of $\mathscr{G}$ except $\{0\}$.

1. We point out some simple consequences of Definition 4.1:

Proposition 4.1. *Let $\mathscr{G}$ be primitive.*

a) *Then g^{-1} generates $\mathscr{G}^-$.*

b) *If $a \in \mathscr{G}^0$ and $[g^{-1}, a] = 0$, then $a = 0$.*

Proof. Let $\mathfrak{h}$ be the graded subspace of $\mathscr{G}^-$ generated by g^{-1}. We will show that $[\mathfrak{h}, \mathscr{G}^0]$ is contained in $\mathfrak{h} + \mathscr{G}^0$. This is clearly true for elements of degree -1 in $\mathfrak{h}$. Suppose it is true for elements of degree $-l$. Let a be an element of degree $-l-1$ in $\mathfrak{h}$ of the form $[x, b]$ where b is an element in $\mathfrak{h}$ of degree $-l$, and let c be an element of $\mathscr{G}^0$. Then $[a, c] = [[x, b], c] = [[x, c], b] + [x, [b, c]]$. The first term on the right is clearly in $\mathfrak{h} + \mathscr{G}^0$, and the second term is in $\mathfrak{h} + \mathscr{G}^0$ by induction; so the sum is in $\mathfrak{h} + \mathscr{G}^0$, proving our assertion. Since $[\mathscr{G}^0, \mathfrak{h}]$ is in $\mathfrak{h} + \mathscr{G}^0$, $\mathfrak{h} + \mathscr{G}^0$ is a graded subalgebra of $\mathscr{G}$ containing $\mathscr{G}^0$, so it must be equal to $\mathscr{G}$ since $\mathscr{G}$ is primitive. This proves a).

To prove b) we note that if a is in $\mathscr{G}^0$ and $[x, a] = 0$ for all x in g^{-1}, then $[y, a] = 0$ for all y in $\mathscr{G}^-$ by part a). Let $\mathfrak{b}$ be the graded ideal of $\mathscr{G}^0$ generated by all elements whose bracket with g^{-1} is zero. It is easy to see that $\mathfrak{b}$ is also an ideal of $\mathscr{G}$. By the primitivity $\mathfrak{b} = \{0\}$, which implies part b).

Corollary. *Let i and j be positive integers, and suppose g^{-i} and g^j are non-zero. Then g^r is non-zero when $-i < r < j$.*

2. Let $\mathscr{G}$ be a graded Lie algebra. There is a rather simple way to associate a linearly compact Lie algebra with $\mathscr{G}$. Suppose we replace the sum (4.1) by the product:

$$\overline{\mathscr{G}} = \prod_{l=-\infty}^{\infty} g^l \ .$$

This is also a Lie algebra, with basically the same bracket operation as that on $\mathscr{G}$. $\overline{\mathscr{G}}$ can be topologized by giving each g^l its discrete topology and $\overline{\mathscr{G}}$ the standard product topology. With this topology $\overline{\mathscr{G}}$ is linearly compact. (See Chapter 1 of [6].) Note that $\mathscr{G}$ can be imbedded in $\overline{\mathscr{G}}$ as a dense subset by identifying homogeneous elements in $\mathscr{G}$ with those in $\overline{\mathscr{G}}$. There is also a way to recapture $\mathscr{G}$ from $\overline{\mathscr{G}}$ as follows:

Let $\overline{\mathscr{G}}^l = \prod_{i \geq l} g^i$. This is a closed subspace of finite codimension in $\overline{\mathscr{G}}$, and $[\overline{\mathscr{G}}^i, \overline{\mathscr{G}}^j]$ is contained in $\overline{\mathscr{G}}^{i+j}$, so the sequence $\cdots, \overline{\mathscr{G}}^l, \overline{\mathscr{G}}^{l+1}, \cdots$ is a filtration of $\overline{\mathscr{G}}$ in the sense of [6]. It is easy to see that $\mathscr{G}$ is the graded Lie algebra associated with this filtration.

We will need below the following:

Lemma 4.1. *If $\mathscr{G}$ is a primitive graded algebra, then $\overline{\mathscr{G}}^0$ is a primitive subalgebra of $\overline{\mathscr{G}}$ in the sense of* § 3.

Proof. Suppose $\bar{\mathfrak{b}}$ is an ideal of $\overline{\mathscr{G}}$ contained in $\overline{\mathscr{G}}^0$. Applying the "gradation functor" to $\bar{\mathfrak{b}}$ we get a graded ideal $\mathfrak{b}$ of $\mathscr{G}$ contained in $\mathscr{G}^0$. Since $\mathscr{G}$ is primitive, this ideal is zero; so $\overline{\mathscr{G}}^0$ is a fundamental subalgebra of $\overline{\mathscr{G}}$. Next suppose that $\bar{\mathfrak{a}}$ is a subalgebra of $\overline{\mathscr{G}}$ containing $\overline{\mathscr{G}}^0$. Applying the gradation functor to $\bar{\mathfrak{a}}$, we get a graded subalgebra $\mathfrak{a}$ of $\mathscr{G}$ which contains $\mathscr{G}^0$ and has the same codimension in $\mathscr{G}$ as $\bar{\mathfrak{a}}$ does in $\overline{\mathscr{G}}$. Since $\mathscr{G}$ is primitive, this subalgebra must be either $\mathscr{G}$ or $\mathscr{G}^0$. Therefore, $\bar{\mathfrak{a}}$ must be either $\overline{\mathscr{G}}$ or $\overline{\mathscr{G}}^0$.

3. We will use the results of the preceding paragraph and Proposition 3.5 to prove:

Proposition 4.2. *Let $\mathscr{G}$ be an infinite dimensional primitive graded Lie algebra, and let $\mathfrak{a}$ be a graded subalgebra of $\mathscr{G}$ which contains g^k for all k sufficiently large. Then either $\mathfrak{a}$ contains $\mathscr{G}^-$ or is contained in $\mathscr{G}^0$.*

Proof. Let $\bar{\mathfrak{a}}$ be the closure of $\mathfrak{a}$ in $\overline{\mathscr{G}}$. Then either $\bar{\mathfrak{a}}$ is a fundamental subalgebra of $\overline{\mathscr{G}}$ in which case it is contained in $\overline{\mathscr{G}}^0$ by Proposition 3.5; or $\bar{\mathfrak{a}}$ contains an ideal $\bar{\mathfrak{b}}$ of $\overline{\mathscr{G}}$. By applying the gradation functor to $\bar{\mathfrak{b}}$ we get a graded ideal $\mathfrak{b}$ contained in $\mathfrak{a}$. The sum $\mathfrak{b} + \mathscr{G}^0$ is a graded subalgebra of $\mathscr{G}$, so it is equal to $\mathscr{G}$ by the primitivity. Since $\mathfrak{b}$ is contained in $\mathfrak{a}$, $\mathfrak{a}$ contains $\mathscr{G}^-$.

4. The following pair of lemmas will be the main tool used in the classification argument in § 6.

Lemma 4.2. *Let $\mathscr{G}$ be an infinite dimensional primitive graded Lie algebra whose leading non-zero term is of degree $-k$, r and s be positive integers such that $r + s \leq k$, and a be a non-zero element of g^{-r}. Then $[a, g^{-s}] = g^{-r-s}$.*

Proof. Define a graded subspace,

$$\mathfrak{h} = \sum_{l=-k}^{\infty} h^l ,$$

of $\mathscr{G}$ as follows:

Let $h^l = g^l$ for $l \neq -r-s$, and let $h^{-r-s} = [a, g^{-s}]$. Let $\mathfrak{k}'$ be the normalizer of $\mathfrak{h}$ in $\mathscr{G}$, i.e., the set of all elements $x \in \mathscr{G}$ such that $[x, \mathfrak{h}]$ is contained in $\mathfrak{h}$, and let $\mathfrak{k} = \mathfrak{k}' \cap \mathfrak{h}$. It is easy to see that $\mathfrak{k}$ is a graded subalgebra of $\mathscr{G}$. Since $\mathfrak{h}$ contains all elements in $\mathscr{G}$ of degree ≥ 0, $\mathfrak{k}$ contains all elements of degree $\geq k$. We will show that $\mathfrak{k}$ also contains the element a. In fact if

$l \neq -s$, then $[a, g^l]$ is contained in h^{-r+l} since $h^{-r+l} = g^{-r+l}$. If $l = -s$, then $[a, g^{-s}] = h^{-r-s}$ by definition. Therefore $[a, \mathscr{G}]$ is in $\mathfrak{h}$, and a is in the normalizer of $\mathfrak{h}$. Since $g^{-r} = h^{-r}$, a is also in $\mathfrak{h}$, so a is in $\mathfrak{k}$ as asserted. Now $\mathfrak{k}$ contains non-zero elements of negative degree and all elements of sufficiently high degree, so $\mathfrak{k}$ contains $\mathscr{G}^-$ by Proposition 4.2. Since $\mathfrak{k}$ is contained in $\mathfrak{h}$, $\mathfrak{h}$ contains $\mathscr{G}^-$. In particular, $[a, g^{-s}] = g^{-r-s}$.

Lemma 4.3. *Let $\mathscr{G}$ be an infinite dimensional primitive graded Lie algebra whose leading term is of degree $-k$, and r be a positive integer $\leq k$. Then the adjoint representation of g^0 on g^{-r} is irreducible. Moreover, if a is a non-zero element of g^{-r}, then either a spans g^{-r} or $g^{-r} = [a, g^0]$.*

Proof. The argument is similar to the preceding one. Let W be an invariant irreducible subspace of g^{-r} with respect to the adjoint representation of g^0. Define a graded subspace:

$$\mathfrak{h} = \sum_{l=-k}^{\infty} h^l$$

of $\mathscr{G}$ by setting $h^l = g^l$ for $l \neq -r$ and $h^r = W$. Let $\mathfrak{k}$ be the intersection of $\mathfrak{h}$ with its own normalizer. $\mathfrak{k}$ is a graded subalgebra of $\mathscr{G}$ containing all elements of degree $\geq k$, and it is also easy to see that $\mathfrak{k}$ contains W by an argument similar to the above one. Therefore $\mathfrak{k}$ contains $\mathscr{G}^-$ by Proposition 4.2. In particular, $W = g^{-r}$, and the representation of g^0 on g^{-r} is irreducible.

Now let a be a non-zero element of g^{-r}, and define a graded subspace $\mathfrak{h} = \Sigma h^l$ of $\mathscr{G}$ by setting $h^l = g^l$ for $l \neq -r$ and $h^{-r} = [a, g^0]$. Let $\mathfrak{k}$ be the normalizer of $\mathfrak{h}$ in $\mathscr{G}$. Then a is in $\mathfrak{k}$ and all elements of degree $\geq k$ are in $\mathfrak{k}$, so $\mathfrak{k}$ contains g^{-r} by Proposition 4.2. Since $g^0 = h^0$ and g^{-r} is in the normalizer of $\mathfrak{h}$, $[g^0, g^{-r}]$ is contained in h^{-r}. But $h^{-r} = [a, g^0]$; so $[g^0, g^{-r}]$ is contained in $[g^0, a]$. Since the representation of g^0 on g^{-r} is irreducible, either $[g^0, g^{-r}]$ is zero and g^{-r} is one-dimensional, or $[g^0, g^{-r}] = g^{-r}$ and $g^{-r} = [a, g^0]$ as asserted.

5. Transitive linear representations of finite dimensional Lie algebras

Let g be a finite dimensional Lie algebra over an algebraically closed field Δ of characteristic zero, and V be a finite dimensional vector space over Δ on which g acts as a Lie algebra of linear transformations. We will say that the representation of g on V is transitive if, for every pair of non-zero elements v and w in V there exists an element in g which maps v onto w. It is clear that a transitive representation is irreducible; therefore, as a Lie algebra, g is either semi-simple or semi-simple with a one-dimensional center.

We will show:

Proposition 5.1. *If the representation of g on V is faithful and transitive, the semisimple part of g is simple.*

Proof. Let $g_1, \cdots, g_n$ be the simple components of the semi-simple part of g. We can write V as a tensor product:

$$V = V_1 \otimes \cdots \otimes V_k ,$$

and we can write the representation of the semisimple part of g as a tensor product of irreducible representation of the g_i on the V_i. Let G_i be the algebraic Lie group associated with the representation of g_i on V_i, and G be the product of the G_i's. The transitivity of the representation of g on V implies that the non-zero orbits of G are open in the Zariski topology. (See Chevalley [3].) Therefore since V is connected, there is just one non-zero orbit. We will say that an element v in V is decomposable if it can be written in the form: $v = v_1 \otimes v_2 \otimes \cdots \otimes v_k$, where v_i is in V_i. If two elements of V are conjugate via an element of G, and one is decomposable, then the other is also; so by the remark above every element of V is decomposable. This implies that all of the V_i except one are one dimensional. (Suppose, for example, that V_1 and V_2 were of dimension ≥ 2. Let v_1 and w_1 be two linearly independent vectors in V_1, and v_2 and w_2 two linearly independent vectors in V_2. Then $v_1 \otimes w_1 + v_2 \otimes w_2$ is not decomposable.)

Therefore, all g_i except one must be zero; and the semi-simple part of g is simple as asserted.

Proposition 5.2. *Suppose the representation of g on V is faithful and transitive. Let ν^+ and ν^- be the maximal and minimal weights of V. Then there exists a root α such that $\nu^+ - \nu^- = \alpha$.*

Proof. Let w^+ be a weight vector corresponding to the maximal weight, and w^- a weight vector corresponding to the minimal weight. Then there exists an element A in g such that $Aw^- = w^+$. Let $H_1, \cdots, H_r, E_{\alpha_1}, \cdots, E_{\alpha_s}$ be the standard basis for g, where $H_1, \cdots, H_r$ are a basis for the Cartan subalgebra of g, and E_{α_i} spans the root space corresponding to the root α_i. We can write A as a linear combination: $A = \sum c_i H_i + \sum d_i E_{\alpha_i}$. Applying A to w^- we get

$$w^+ = \{\sum c_i \nu^-(H_i)\} w^- + \sum d_i E_{\alpha_i}(w^-) .$$

Each of the individual terms in the sum on the right lies in a different weight space; so all of the terms but one are zero. The first term must be zero since w^- and w^+ are linearly independent, so $w^+ = d_i E_{\alpha_i}(w^-)$ for some i. Hence $\nu^+ = \nu^- + \alpha_i$, proving the assertion.

Corollary. *The simple part of the algebra g is either $sl(n)$ in which case the representation of g on V is its standard n dimensional representation, or $sp(2n)$ in which case the representation of g on V is its standard $2n$ dimensional representation.*

Proof (due to Sternberg [8]*).* We first observe that since $\nu^+ - \nu^-$ is the maximal weight of $V \otimes V^*$ and g is contained in $V \otimes V^*$, α is the maximal root of g. Suppose g is of rank $n - 1$, and let $\lambda_1, \cdots, \lambda_{n-1}$ be the maximal weights of the fundamental representations of g. Since ν^+ and $-\nu^-$ are both maximal weights of g (on V and V^* respectively), we conclude that $\nu^+ = \sum_{i=1}^{n-1} a_i\lambda_i$ and $-\nu^- = \sum_{i=1}^{n-1} b_i\lambda_i$ where the a's and b's are non-negative integers with $\sum a_i \geq 1$ and $\sum b_i \geq 1$.

Now for all simple algebras except A_{n-1} and C_{n-1} the maximal root is a λ_i for some i, so it is impossible for α to be written in the form $\nu^+ - \nu^- = \sum (a_i + b_i)\lambda_i$ with $\sum (a_i + b_i) = 2$. In the case of A_{n-1} the maximal root is $\lambda_1 + \lambda_{n-1}$, so the only possibility is $\nu^+ = \lambda_1$ and $-\nu^- = \lambda_{n-1}$ or visa versa. These two representations are both equivalent to the standard representation of $sL(n)$. For C_{n-1} the maximal root is $2\lambda_1$, and so $\nu^+ = -\nu^- = \lambda_1$ which gives the representation of C_{n-1} as the standard representation of $sp(2n)$.

6. A theorem of Kac

Combining Lemmas 4.2 and 4.3 with the results of the preceding section we will prove:

Proposition 6.1. *Let $\mathscr{G} = \sum g^i$ be an infinite dimensional primitive graded Lie algebra over an algebraically closed field of characteristic zero. Then*

a) *g^0 is simple or simple with a one dimensional center.*

b) *The simple part of g^0 is either $sl(n)$ or $sp(2n)$.*

c) *The adjoint representation of the simple part of g^0 on g^{-1} is either the standard n dimensional representation of $sl(n)$, or the standard $2n$ dimensional representation of $sp(2n)$.*

d) *Either $g^{-2} = \{0\}$ or g^{-2} is one dimensional.*

e) *$g^i = \{0\}$ for $i < -2$.*

Proof. The adjoint representation of g^0 on g^{-1} is faithful by Proposition 4.1 and transitive by Lemma 4.3, so a), b) and c) are a consequence of the results proved in § 5. To prove d) and e) suppose $\dim g^{-2} \neq 1$. Then, by Lemma 4.3, the representation of g^0 on g^{-2} is transitive. However, since $[g^{-1}, g^{-1}] = g^{-2}$, this representation can be identified with an irreducible subrepresentation of $\Lambda^2 g^{-1}$. Suppose the simple part of g^0 is $sl(n)$. Since the representation of g^0 on g^{-1} is the standard n dimensional representation, $\Lambda^2 g^{-1}$ is irreducible and, with one exception, inequivalent to the standard representation. (The one exception is $n = 3$ in which case $\Lambda^2 = \Lambda^{n-1}$.) Therefore, with one possible exception, $g^{-2} = \{0\}$.

Next suppose that the simple part of g^0 is $sp(2n)$. Then $\Lambda^2 g^{-1}$ decomposes into two irreducible subspaces, one of which is one dimensional and the other $n(2n - 1) - 1$ dimensional. By comparing dimensions one sees that neither of these representations is equivalent to the standard one, so again $g^{-2} = \{0\}$.

When the simple part of g^0 is $sl(3)$ we need a slightly more complicated argument. In this case either g^{-2} is zero or $g^{-2} = \Lambda^2 g^{-1}$. Suppose the latter. Let μ^+ be the maximal weight of the representation of g^0 on g^{-2}, and ν^- the minimal weight of the representation of g^0 on g^{-1}. The representation of $sl(3)$ on $\Lambda^2 g^{-1}$ is the transpose of the representation on g^{-1}, so $\nu^- = -\mu^+$. Let v^+ be a weight vector corresponding to the μ^+, and v^- a weight vector corresponding to ν^-. Applying Lemma 4.2 with $v^- = a$ we get an element x in g^{-1} such that $[x, v^-] = v^+$. We can assume x is a weight vector corresponding to a weight β; so we get $\mu^+ = \nu^- + \beta$ or $\beta = 2\nu^-$. However, β is a weight of the standard representation and ν^- is its minimal weight, so this is impossible. Hence $g^{-2} = \{0\}$. This excludes the case $g^0 = sl(3)$.

Now suppose g^{-2} is one dimensional. We will show that g^{-3} is zero or one dimensional. In fact, let a be a non-zero element of g^{-1}. Then $[a, g^{-2}] = g^{-3}$ by Lemma 4.2, so dimension g^{-3} is less than one or equal to one as claimed.

Let g' be the simple part of g^0, and b be a non-zero element of g^{-2}. By Lemma 4.2, $[b, g^{-1}] = g^{-3}$. Since g' acts trivially on g^{-2}, the representations on g^{-1} and g^{-3} are intertwined. This is impossible if $\dim g^{-3} = 1$, so $\dim g^{-3} = 0$. By the corollary to Proposition 4.1, $g^{-k} = \{0\}$ for all $k \geq 3$. This concludes the proof of Proposition 6.1.

Corollary. *If* $\dim g^{-2} = 1$, *then* g^0 *is* $sp(2n)$ *plus a one dimensional center.*

Proof. Let g' be the simple part of g^0. Since $[g', g^{-2}] = 0$, the Lie bracket on g^{-1} defines an antisymmetric bilinear form on g^{-1} invariant with respect to g', so g' clearly has to be $sp(2n)$. If g^0 did not have a center, then we would have $[g^0, g^{-2}] = 0$. We will show that this would imply $[g^k, g^{-2}] = 0$ for all k. In fact, suppose we have shown this for $k - 1$. Then $[g^{-1}, [g^k, g^{-2}]]$ is contained in $[g^{-1}, g^{k-1}]$ which is zero by induction. By Proposition 4.1, $[g^{-2}, g^k] = 0$. This implies that g^{-2} is an ideal in $\mathscr{G}$, and contradicts the primitivity of $\mathscr{G}$. q.e.d.

Using the above results one can prove the following theorem:

a) There is just one primitive infinite dimensional graded algebra with $g^0 = sl(n)$.

b) There are just 2 primitive infinite dimensional graded algebras with $g^0 = gl(n)$.

c) There is just one primitive infinite dimensional graded algebra with $g^0 = sp(2n)$.

d) There is just one primitive infinite dimensional graded algebra with $g^0 = sp(2n) + \{cI\}$ and $g^{-2} = \{0\}$.

e) There is just one primitive infinite dimensional graded algebra with $g^0 = \mathrm{sp}\,(2n) + \{cI\}$ and $\dim g^{-2} = 1$.

Thus there are six classes of primitive infinite dimensional graded algebras in all. These correspond to the six classes or primitive infinite groups discovered by Cartan.

The proof of this theorem can be found in [18]. It is not hard, but the details are a little messy. To indicate the idea of the proof we will carry out the details for a):

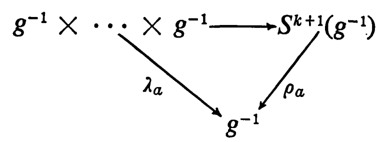
In this case $g^{-2} = \{0\}$; so $[g^{-1}, g^{-1}] = \{0\}$. For every positive integer k we will define an injective linear mapping:

$$\rho\colon g^k \to \operatorname{Hom}(S^{k+1}(g^{-1}), g^{-1})$$

as follows. Let a be an element of g^k, and $x_1, \cdots, x_{k+1}$ be elements of g^{-1}. Set

$$\lambda_a(x_1, \cdots, x_{k-1}) = \operatorname{ad} x_1 \operatorname{ad} x_2 \operatorname{ad} x_3 \cdots \operatorname{ad} x_{k+1} a \ .$$

Since $\operatorname{ad} x_i \operatorname{ad} x_j = \operatorname{ad} x_j \operatorname{ad} x_i$, λ_a is a $k+1$ linear mapping of $\overbrace{g^{-1} \times g^{-1} \times \cdots \times g^{-1}}^{k+1}$ into g^{-1} which is symmetric in its $k+1$ variables. Therefore, by a universal property of the symmetric product there exists a unique mapping ρ_a in $\operatorname{Hom}(S^{k+1}(g^{-1}), g^{-1})$ which makes the diagram



commute. It is clear that $\lambda_{a+b} = \lambda_a + \lambda_b$; so $\rho_{a+b} = \rho_a + \rho_b$. We define

$$\rho\colon g^k \to \operatorname{Hom}(S^{k+1}(g^{-1}), g^{-1})$$

by setting $\rho(a) = \rho_a$. It is not hard to show from Proposition 4.1 that ρ is injective. Moreover, the representation of g^0 on g^k commutes with the representation of g^0 on $\operatorname{Hom}(S^{k+1}(g^{-1}), g^{-1})$; so to identify g^k we must look for invariant subspaces of $\operatorname{Hom}(S^{k+1}(g^{-1}), g^{-1})$. This is not hard to do, since the representation of g^0 on g^{-1} is just the standard n dimensional representation of $sl(n)$. It turns out that there are just two invariant subspaces, one of which is isomorphic to $S^k(g^{-1})$, and the other isomorphic to elements of trace zero in $\operatorname{Hom}(S^{k+1}(g^{-1}), g^{-1})$. It is easy to exclude the first possibility because of the fact that the g^k's have to form a Lie algebra, so there is just one possibility left.

To conclude the proof of a) we must exhibit a Lie algebra with $g^0 = sl(n)$. Such an algebra is the algebra of formal vector fields:

$$f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}, f_n \in F_n$$

satisfying the "divergence condition":

$$\sum \frac{\partial f_i}{\partial x_i} = 0 \ .$$

(See Example 2 in § 2.)

Remarks. 1) The statement of the above theorem appeared in a short note of Kac in [11]. The datails are due to appear in [12].

2) Kac also gave a classification for the graded simple algebras whose gradations are infinite in both directions. There does not appear to be any relation between this result and the theory of primitive algebras developed in § 3.

7. The Weisfeiler filtration

Let L be a linearly compact Lie algebra over an algebraically closed field of characteristic zero, and L^0 be a primitive subalgebra of L. The adjoint representation of L^0 on L induces a finite dimensional representation L^0 on L/L^0. Let W be an irreducible subspace of L/L^0 for this representation, and H be the preimage of W in L with respect to the projection $L \to L/L^0$. We define an increasing sequence of subspaces in L as follows. We set $L^{-1} = H$ and, by induction, we define

$$L^{-(r+1)} = [L^{-1}, L^{-r}] + L^{-r} .$$

We will show that for some $k, L^{-k} = L$. Since H is of finite codimension in L, the sequence L^{-1}, L^{-2}, etc... stabilizes for some k; so for some k, $[L^{-1}, L^{-k}]$ is contained in L^{-k}. However, since L^{-k} is generated by L^{-1}, this implies that $[L^{-k}, L^{-k}]$ is contained in L^{-k}. Since L^0 is primitive, $L^{-k} = L$ proving our assertion.

Next we define a decreasing sequence of subspaces in L^0 as follows. We set

$$L^1 = \{x \in L^0 \mid [y, x] \in L^0, \forall y \in L^{-1}\} ,$$

and by induction we define

$$L^r = \{x \in L^{r-1} \mid [y, x] \in L^{r-1}, \forall y \in L^{-1}\} .$$

It is easy to see that L^r is a closed subspace of L of finite codimension. (See Chapter 2 of [6].) We will show that $\bigcap_{r=0}^{\infty} L^r = \{0\}$. It is clear that this intersection is invariant when bracketed with elements of L^{-1}. However, as we saw above, L^{-1} generates L; so this intersection is an ideal. Since L^0 is a fundamental subalgebra the intersection is zero as claimed.

Next we will show that for all $-k \leq i, i \leq \infty$, $[L^i, L^j]$ is contained in L^{i+j}. When i and j are negative, this is obvious by definition. It is also obvious, when i is negative and j is positive since $[L^{-1}, L^r]$ is contained in L^{r-1}. To prove it for i and j positive we observe that for $i > 0, L^i$ is the set of all $a \in L$ with the property that $\operatorname{ad} x_1 \circ \operatorname{ad} x_2 \circ \cdots \circ \operatorname{ad} x_l a \in L^0$ for all $0 \leq l \leq k$ and all sequences $x_1, \cdots, x_l$ in L. This fact and the generalized Leibnitz rule imply

$$[L^i, L^j] \subset L^{i+j} \qquad \text{for all } i, j \geq 0 .$$

We summarize the remarks above in the following proposition:

Proposition 7.1. *The sequence $L^{-k}, L^{-k+1}, \cdots, L^0, L^1, \cdots$ is a Lie algebra filtration in the sense of* [6] *with the property that L^{-1}/L^0 is irreducible for the adjoint action of L^0.*

Let $\mathscr{G} = \sum_{r=-k}^{\infty} g^r$ be the graded Lie algebra associated with the above filtration. We list a few of its properties:

(7.1)
a) g^{-1} generates $\mathscr{G}^-$.
b) If $y \in \mathscr{G}^0$ and $[x, y] = 0$ for all $x \in g^{-1}$, then $y = 0$.
c) The adjoint representation of g^0 on g^{-1} is irreducible.

(Property a) follows from the identity $L^{-(r+1)} = L^{-r} + [L^{-1}, L^{-r}]$, property b) from the definition of L^i for positive i, and property c) from the definition of L^{-1}.)

We will use these properties of $\mathscr{G}$ to get information about L. We will consider separately the case where g^0 is non-semisimple and the case where g^0 is semisimple, since the first case turns out to be somewhat easier to handle.

We recall that in § 4 we showed every graded algebra $\mathscr{G}$ can be imbedded as a dense subset in a linearly compact algebra $\overline{\mathscr{G}}$. We will call $\overline{\mathscr{G}}$ the linear compactification of $\mathscr{G}$.

Proposition 7.2. *If g^0 is not semisimple, then L is isomorphic to the linear compactification of $\mathscr{G}$.*

Corollary 1. *$\mathscr{G}$ is a primitive graded algebra.*

Corollary 2. *If L is infinite dimensional, it is isomorphic to the linear compactification of one of the graded algebras listed at the end of* § 6.

Proof. The representation of g^0 on g^{-1} is faithful by condition b) of (7.1) and irreducible by condition c); so g^0 is semisimple plus a one dimensional center. This means we can find an element z in the center of g^0 such that ad z is -1 times the identity mapping on g^{-1}. We will show that ad z is i times the identity mapping on g^i for all i. Assume this is true for $g^{-r}, r \geq 1$. Every element of g^{-r-1} is the sum of elements of the form $[x, y]$ with $x \in g^{-1}$ and $y \in g^{-r}$. Applying z to such an element we get

$$\begin{aligned}(\operatorname{ad} z)([x, y]) &= [(\operatorname{ad} z)(x), y] + [x, (\operatorname{ad} z)(y)] \\ &= -[x, y] - r[x, y] \\ &= -(r+1)[x, y] ;\end{aligned}$$

so the assertion is true for all terms of negative degree by induction.

Next note that the statement is true for g^0 since z is in the center of g^0. Assume the statement is true for $g^r, r \geq 0$. We will prove it for g^{r+1}. Let y be an element of g^{r+1}, and x an element of g^{-1}. We get

$$(\mathrm{ad}\, z)([x,y]) = r[x,y] = [(\mathrm{ad}\, z)(x), y] + [x, (\mathrm{ad}\, z)(y)]$$
$$= -[x,y] + [x, (\mathrm{ad}\, z)(y)] \;;$$

so

$$[x, (\mathrm{ad}\, z)(y)] = [x, (r+1)y] \qquad \text{for all } x \in g^{-1} .$$

By part b) of (7.1), $(\mathrm{ad}\, z)(y) = (r+1)y$. This proves our assertion for all integers r.

Now let $\hat{z}$ be an element in L^0 representing z. We will show that L decomposes into an (infinite) direct sum of finite dimensional subspaces $\hat{g}^r$, $-k \leq r \leq \infty$, such that on $\hat{g}^r$, $\mathrm{ad}\, \hat{z}$ is r times the identity mapping.

Let s be a large positive integer. $\mathrm{ad}\, \hat{z}$ induces a linear mapping on the finite dimensional vector space L/L^s which we can decompose into its primary subspaces (corresponding to distinct eigenvalues). It is clear that these eigenvalues are just the integers between $-k$ and $s-1$. Let $\hat{g}^i$ be the subspace corresponding to the i-th eigenvalue. $\mathrm{ad}\, \hat{z}$ on $\hat{g}^i$ is conjugate to $\mathrm{ad}\, z$ on g^i, so it is equal to i times the identity mapping. Therefore, on L/L^s we get the required decomposition. By letting s tend to infinity and and using the fact that L is complete in the filtration topology we get the required decomposition on all of L.

If x is in $\hat{g}^r$ and g is in $\hat{g}^s$, then $[x,y]$ is in $\hat{g}^{r+s}$ by Jacobi's identity; so the graded algebras $\sum g^i$ and $\sum \hat{g}^i$ are isomorphic. It is clear that L is the linear compactification of $\sum \hat{g}^i$; so this concludes the proof of Proposition 7.2. q.e.d.

The result above is also true when g^0 is semisimple, providing L is infinite dimensional; however, the proof requires a slightly more sophisticated argument. This argument will take up the next two sections.

8. A lemma of Kobayashi-Nagano-Weisfeiler

A special case of the following lemma was proved by Kobayashi and Nagano in [13]. The lemma in its general form is due to Weisfeiler (private communication).

Lemma 8.1. *Let* $\mathscr{G} = \sum_{i=-k}^{l} g^i$ *be a finite dimensional graded Lie algebra over an algebraically closed field of characteristic zero. Assume*:

(8.1)
- i) g^{-1} *generates* $\mathscr{G}^-$.
- ii) *If* $a \in \mathscr{G}^0$ *and* $[x,a] = 0$ *for all* $x \in g^{-1}$, *then* $a = 0$.
- iii) *The adjoint representation of* g^0 *on* g^{-1} *is irreducible.*
- iv) g^{-k} *and* g^l *are non-zero*; $k, l > 0$.

Then $\mathscr{G}$ *is simple,* $l = k$, *and the Killing form identifies* g^{-r} *with the dual of* g^r *for all* $0 < r \leq k$.

Proof. Let $\mathfrak{r}$ be the radical of $\mathscr{G}$. We will show that $\mathfrak{r}$ is graded. Let $\mathfrak{r}'$ be the graded ideal in $\mathscr{G}$ associated with $\mathfrak{r}$. It is easy to see that $[\mathfrak{r}', \mathfrak{r}']$ is contained in the graded ideal associated with $[\mathfrak{r}, \mathfrak{r}]$. The same is true of the higher derived algebras, so $\mathfrak{r}'$ is solvable and therefore contained in $\mathfrak{r}$. However, $\dim \mathfrak{r}' = \dim \mathfrak{r}$; so $\mathfrak{r}' = \mathfrak{r}$ proving our assertion.

Next we will show that $\mathfrak{r} \cap g^{-1} = \{0\}$. Suppose this intersection were not zero. By i) and iii) this implies that $\mathfrak{r}$ contains $\mathscr{G}^-$. We will show inductively that if g^{-1} is in $\mathfrak{r}$ then g^{-1} is in the i-th derived algebra $\mathfrak{r}^{(i)}$. Suppose g^{-1} is in $\mathfrak{r}^{(i-1)}$. Since $\mathfrak{r}^{(i-1)}$ is an ideal, $[g^{-1}, g^1]$ is in $\mathfrak{r}^{(i-1)}$. This bracket is non-zero by ii); so $\mathfrak{r}^{(i-1)} \cap g^0 \neq \{0\}$. Therefore, $\mathfrak{r}^{(i)} \cap g^{-1} \neq \{0\}$ by ii); and so g^{-1} is contained in $\mathfrak{r}^{(i)}$ by iii). Since $\mathfrak{r}^{(i)}$ is zero for large i, we get a contradiction; so $\mathfrak{r} \cap g^{-1} = \{0\}$ as claimed.

This argument also shows that $\mathfrak{r} \cap \mathscr{G}^0 = \{0\}$, for if not then by ii) there would be non-zero elements in $\mathfrak{r} \cap g^{-1}$.

By the Levi theorem there exists a semi simple subalgebra $\mathfrak{h}$ in $\mathscr{G}$ such that $\mathscr{G} = \mathfrak{r} \oplus \mathfrak{h}$. Let $\mathfrak{h}'$ be the graded subalgebra of $\mathscr{G}$ obtained from $\mathfrak{h}$ by applying the gradation functor to the "reversed filtration";

$$L = L^{-l} \supset L^{-l+1} \supset \cdots \supset L^k = \{0\}\ ,$$

where $L^i = \sum_{r=-k}^{-i} g^r$. It is clear from the above remarks that $\mathfrak{h}'$ contains g^{-1} and $\mathscr{G}^0$, so $\mathfrak{h}'$ contains $\mathscr{G}$ by i). However, the dimension of $\mathfrak{h}$ is the same as that of $\mathfrak{h}'$, so $\mathfrak{h} = \mathscr{G}$ and $\mathfrak{r} = \{0\}$. This proves that $\mathscr{G}$ is semisimple.

Let σ be the mapping of $\mathscr{G}$ into $\mathscr{G}$, which preserves the gradation and on g^i is just i times the identity mapping. It is easy to check that σ is a derivation. Since $\mathscr{G}$ is semisimple, σ is an inner derivation. It follows that every ideal in $\mathscr{G}$ is graded since the distinct eigenspaces of σ are just the graded subspaces of $\mathscr{G}$.

Since $\mathscr{G}$ is semisimple we can write it as a direct sum of its simple ideals:

$$\mathscr{G} = \mathscr{G}_1 \oplus \cdots \mathscr{G}_s\ , \qquad \mathscr{G}_i\text{: simple}\ .$$

Clearly g^{-1} is contained in just one of these summands. Suppose g^{-1} is contained in $\mathscr{G}_1$. Then $\mathscr{G}^-$ is contained in $\mathscr{G}_1$ by property i), so $\mathscr{G}_2, \mathscr{G}_3$, etc. must consist entirely of non-negative elements. However, property i) implies that if an ideal contains non-zero positive elements, then it contains non-zero negative elements; so $\mathscr{G}_2 = \mathscr{G}_3 = \cdots = \mathscr{G}_s = \{0\}$, and $\mathscr{G}$ is simple.

Finally, suppose $l \neq k$, for example suppose $l > k$. Let x be an element of g^l, and y an element of g^i. Then $\operatorname{ad} x \operatorname{ad} y$ is a graded mapping of $\mathscr{G}$ which maps terms of degree j onto terms of degree $i + l + j$. Since $i + l > 0$ for all i, this mapping is nilpotent and $\operatorname{tr} \operatorname{ad} x \operatorname{ad} y = 0$. This shows that with respect to the Killing form, x is orthogonal to all of $\mathscr{G}$. By Cartan's criterion this cannot happen, so $l = k$ as asserted. The same argument shows that g^i and g^{-i} are put into duality by the Killing form; we leave the details to the reader.

Corollary. *Let $\mathscr{G}$ be a graded Lie algebra satisfying the axioms* (8.1). *Then g^0 is not semisimple.*

Proof. Let σ be the "degree derivation" introduced above, i.e., σ maps g^i into g^i and on g^i is i times the identity mapping. Since $\mathscr{G}$ is simple, σ is an inner derivation; and since σ is degree preserving, it must be an inner derivation of the form $\operatorname{ad} z$ where z is in g^0. Since σ is zero on g^0, z is in the center of g^0, and hence g^0 is not semisimple.

Remark. There is a $1 - 1$ correspondence between graded Lie algebras satisfying Axioms (8.1) and non-semisimple primitive subalgebras of simple Lie algebras. For details we refer to Golubitsky [5].

9. The Cartan classification theorem[2]

In this section we will finish our sketch of the Cartan classification theorem. For this we will need several lemmas.

Lemma 9.1. *Let $\mathscr{G} = \sum_{-k}^{\infty} g^i$ be a graded Lie algebra satisfying the Weisfeiler conditions* (7.1) *and, in addition, the following two conditions*:

i) *g^1 generates $\mathscr{G}^+$.*

ii) *No non-zero graded ideals of $\mathscr{G}$ are contained in $\mathscr{G}^-$.*

Then $\mathscr{G}$ is primitive.

Proof. Let $\mathfrak{b}$ be a graded subalgebra of $\mathscr{G}$ containing $\mathscr{G}^0$. If $\mathfrak{b}$ contains g^{-1}, then by condition a) of (7.1), b contains $\mathscr{G}^-$ and so is equal to $\mathscr{G}$. Therefore, we can assume $\mathfrak{b} \cap g^{-1} = \{0\}$. Suppose $\mathfrak{b}$ contains no non-zero elements of degree $-i + 1$, but does contain non-zero elements of degree $-i$. Let a be such an element. Then $[a, g^{+1}] = 0$; so, by condition i), $[a, \mathscr{G}^+] = 0$. Let $\mathfrak{a}$ be the set of all elements a in $\mathscr{G}^-$ with the property that $[a, \mathscr{G}^+] = 0$. We will show that if a is in $\mathfrak{a}$ and $x_1, \cdots, x_k$ are in $\mathscr{G}$, then $\operatorname{ad} x_1 \cdots \operatorname{ad} x_k\, a$ is in $\mathscr{G}^-$. The proof will be by induction on k, the case $k = 1$ being obvious. Suppose this statement is true for $k - 1$. The expression $\operatorname{ad} x_1 \cdots \operatorname{ad} x_k a$ is obviously in $\mathscr{G}^-$ if the x_i are all in $\mathscr{G}^-$ or g^0. This expression is also in $\mathscr{G}^-$ if x_k is in $\mathscr{G}^+$. (In fact it is zero.) If one of the x_i's is in $\mathscr{G}^+$, but not the last one, we can move it into the last place by applying Jacobi's identity several times. This will introduce commutator terms of the form: $\operatorname{ad} y_1 \cdots \operatorname{ad} y_{k-1} a$ which are in $\mathscr{G}^-$ by induction; so this proves our assertion.

It is clear by what we have just shown that $\mathfrak{a}$ generates a non-zero ideal contained in $\mathscr{G}^-$. This contradicts our second hypothesis, so $\mathfrak{b} = \mathscr{G}^0$, and $\mathscr{G}$ is primitive.

Lemma 9.2. *Let $\mathscr{G} = \sum_{-k}^{\infty} g^i$ be an infinite dimensional graded Lie algebra satisfying the Weisfeiler conditions* (7.1) *and, in addition, the following two*

[2] In this section all algebras are defined over a base field which is algebraically closed and of characteristic zero.

conditions:

i) g^1 *generates* $\mathscr{G}^+$.

ii) g^0 *is semisimple.*

Then:

a) $g^i = \{0\}$ *for all* $i < -2$.

b) g^{-2} *is either zero or one dimensional.*

c) g^{-2} *is the center of* $\mathscr{G}$, *and* $\mathscr{G}/g^{-2}$ *is primitive.*

Proof. Let $\mathfrak{b}$ be the largest graded ideal of $\mathscr{G}$ contained in $\mathscr{G}^-$. By conditions b) and c) of (7.1), $\mathfrak{b}$ contains no non-zero elements of degree -1. The quotient $\mathscr{G}/\mathfrak{b}$ is a graded algebra satisfying the conditions (7.1) and the two conditions of Lemma 9.1; so $\mathscr{G}/\mathfrak{b}$ is primititive. Since g^0 is semisimple, the leading non-zero term of the graded algebra $\mathscr{G}/\mathfrak{b}$ is of degree -1 by the corollary to Proposition 6.1; so $\mathfrak{b} = g^{-k} + g^{-k+1} + \cdots + g^{-2}$. We will show that $g^{-i} = 0$ if $i > 2$. If not, g^{-k} and g^{-k+1} are non-zero and in $\mathfrak{b}$, for $k > 2$.

Let $\mathfrak{h}$ be the graded subspace of $\mathscr{G}$ consisting of all $a \in \mathscr{G}$ with the property that $[g^{-k}, a] = [g^{-k+1}, a] = 0$. Since $\mathfrak{b}$ is an ideal and contains no non-zero positive elements, $[\mathfrak{b}, g^i] = 0$ for $i \geq k$; so $\mathfrak{h}$ contains non-zero elements of positive degree. It is easy to see that $[g^{-1}, \mathfrak{h}] \subset \mathfrak{h}$, so $\mathfrak{h}$ contains g^{-1} by property b) of (7.1). This means that $[g^{-1}, g^{-k+1}] = 0$.

On the other hand, $[g^{-1}, g^{-k+1}] = g^{-k}$ by property a) of (7.1); so we get a contradiction. This shows that $g^i = \{0\}$ for $i < -2$.

Since g^{-2} is an ideal in $\mathscr{G}$, $[g^{-2}, g^k] = 0$ for $k \geq 1$. In particular, $[g^{-1}, [g^{-2}, g^1]] = [g^{-2}, [g^{-1}, g^1]] = 0$. Since g^0 is simple and $[g^{-1}, g^1] \neq 0, [g^{-1}, g^1] = g^0$ and hence $[g^{-2}, g^0] = 0$. This shows that g^{-2} is in the center of $\mathscr{G}$.

We can identify g^{-2} with a subspace of $\wedge^2 g^{-1}$. Since $[g^0, g^{-2}] = 0$, every element of g^{-2} defines an invariant bilinear form on g^{-1}. Since g^0 acts irreducibly on g^{-1}, there can be only one such bilinear form up to scalar multiples; so $\dim g^{-2} = 0$ or 1. This concludes the proof of Lemma 9.2. q.e.d.

We will now show that the first hypothesis of Lemma 9.2 is superfluous.

Lemma 9.3. *The conclusions of Lemma* 9.2 *are valid without the assumption that* g^1 *generates* $\mathscr{G}^+$ *(providing we still assume* g^0 *is semisimple).*

Proof. Let $\mathfrak{a}$ be the graded subalgebra of $\mathscr{G}^+$ generated by g^1. It is easy to see that if x is in g^{-1} and y is a term of degree grater than 1 in $\mathfrak{a}$, then $[x, y]$ is in $\mathfrak{a}$. This implies that the sum: $\mathscr{G}^- \oplus g^0 \oplus \mathfrak{a}$ is a subalgebra of $\mathscr{G}$. We will show that this subalgebra is infinite dimensional. In fact, if it were finite dimensional, it would satisfy all the hypotheses of the Kobayashi-Nagano-Weisfeiler lemma, so g^0 would be non-semisimple. (See the corollary to Lemma 8.1.) Since g^0 is semisimple, $\mathscr{G}^- \oplus g^0 \oplus \mathfrak{a}$ is infinite dimensional.

Now $\mathscr{G}^- \oplus g^0 \oplus \mathfrak{a}$ satisfies all the hypotheses of Lemma 9.2, so $g^{-i} = 0$ for $i \leq 2, g^{-2}$ is one dimensional or $\{0\}$ and g^{-2} is in the center of $\mathscr{G}^- \oplus g^0 \oplus \mathfrak{a}$. To conclude the proof we have to show that g^{-2} is in the center of $\mathscr{G}$. (It will follow that $\mathscr{G}/g^{-2}$ is primitive since g^0 acts irreducibly on g^{-1}.) Since g^{-2} is in the center of $\mathcal{G}^- \oplus g^0 \oplus \mathfrak{a}$, $[g^{-2}, g^0] = [g^{-2}, g^1] = 0$. We will show by induction that $[g^{-2}, g^i] = 0$ for all $i \geq 0$. Assume this is true for g^{i-1}. Let x be an element of g^{-1}, y an element of g^i and z an element of g^{-2}. Then $[x, [y, z]] = [[x, y], z] + [y, [x, z]] = 0$ by induction. Since this is true for all x, $[y, z] = 0$ by condition b) of (7.1). Thus g^{-2} is in the center of $\mathcal{G}$ as asserted. q.e.d.

We will now prove the main result of this section.

Proposition 9.1. *Let L be an infinite dimensional linearly compact Lie algebra, L^0 be a primitive subalgebra of L,*

$$L = L^{-k} \supset \cdots \supset L^0 \supset L^1 \supset \cdots$$

be the Weisfeiler filtration of L, and $\mathcal{G} = \sum_{i=-k}^{\infty} g^i$ be the graded Lie algebra associated with this filtration. Then $\mathcal{G}$ is primitive.

Proof. We have already proved this for the case where g^0 is non-semisimple, so we can assume g^0 is semisimple. By Lemma 9.3 either $\mathcal{G}$ is primitive, or g^{-2} is in the center of $\mathcal{G}$ and is one dimensional. Let z be a vector in L which projects onto a basis vector of g^{-2}.

We will show that we can find a vector x in L^{-1} such that $z + x$ is in the normalizer of L^0. Since z projects onto an element of degree -2 which is in the center of $\mathcal{G}$, $[z, L^0] \subset L^{-1}$ and $[z, L^1] \subset L^0$. We define a linear mapping $c\colon g^0 \to g^{-1}$ as follows: If a is in g^0, we let a' be a representative of a in L^0, and let $c(a)$ be the image of $[z, a']$ in g^{-1}. Because of the above conditions on z this map is well defined and does not depend on the choice of a'. We will show that the mapping c satisfies the "cocycle condition." Let a and b be elements of g^0, and a' and b' be elements of L^0 representing them. Then by Jacobi's identity, $[z, [a', b']] = [[z, a'], b'] + [a', [z, b']]$; hence $c([a, b]) = [a, c(b)] - [b, c(a)]$, which is precisely the cocycle condition for the adjoint representation of g^0 on g^{-1}. Since g^0 is simple there exists an element v in g^{-1} such that $c(a) = [a, v]$ for all $a \in g^0$ by Whitehead's lemma. (See Jacobson [10, Chapter III].) Let x be a representative of v in L^{-1}. Then for all a' in L^0, $[z, a'] = [a', x]$, modulo elements of L^0. Therefore, $z + x$ is in the normalizer of L^0. This implies that the subspace of L spanned by L^0 and $z + x$ is a subalgebra, contradicting the primitivity of L^0.

We conclude that $g^{-2} = \{0\}$ and that $\mathcal{G}$ is primitive as claimed.

Corollary. *L is isomorphic to the linear compactification of one of the graded algebras listed at the end of* §6.

The proof of this corollary involves the following type of question: Given a filtered Lie algebra and the corresponding graded Lie algebra, when are the two algebras isomorphic? (We ignore for the moment questions of completeness and linear compactness.) D. Rim has developed an elegant technique for handling this type of question [17]. He shows that the filtered algebra can be regarded as a "deformation" of the graded algebra, and applies standard techniques of deformation theory to show that a graded algebra is indeformable if certain "infinitesmal deformations" (represented by cohomology classes in $H^1(gr\ L, gr\ L)$) vanish.

Our situation is somewhat special. We know the corollary is true for four of the six algebras listed in §6 (by Proposition (7.1)) and the remaining two algebras are of the form: $\mathscr{G} = \mathrm{g}^{-1} \oplus \mathrm{g}^0 \oplus \cdots$. For graded algebras of this type the infinitesmal deformations can be effectively computed just by knowing the representation of g^0 on g^{-1}, which, in our case, is just the standard representation of $sl(n)$ or the standard representation of $sp(2n)$. Some straightforward computations show that both of the graded algebras in question are indeformable.

A more pedestrian proof of the corollary can be found in [14] or [18].

Appendix

Let L be a linearly compact Lie algebra defined over the field Δ, and A be an open subalgebra of L. We recall that A is a fundamental subalgebra of L if it contains no ideal of L except $\{0\}$. In [6] we proved:

Proposition 1. *L possesses a fundamental subalgebra if and only if it satisfies the* d.c.c. *on closed ideals.* (See Theorem 3.1.)

Remark. It is clear that every continuous automorphism of L maps fundamental subalgebras onto fundamental subalgebras.

Given L and a fundamental subalgebra A, we will define a filtration on L as follows: Let A^i be the set of all $a \in L$ for which $\mathrm{ad}(x_1) \cdots \mathrm{ad}(x_k)a \in A$, for all $k<i$, and all $x_1, \cdots, x_k \in L$. We will set $A^i = L$ for $i = 0$ and $A^0 = A$.

Lemma 1. *The A^i have the following properties*:

a) *A^i is an open subalgebra of L.*

b) $[A^i, A^j] \subset A^{i+j}$.

c) *$A^i \supset A^{i+1}$ for all i.*

d) *$\bigcup A^i = L$ and $\bigcap A^i = \{0\}$.*

Proof. Part a) follows by induction using the fact that A^i is the kernel of the adjoint representation of A^0 on L/A^{i-1}. Part b) is a rather easy consequence of Jacobi's identity. Part c) and the first part of d) are obvious, and the second part of d) follows from the fact that $\bigcap A^i$ is an ideal and therefore is zero because A is a fundamental subalgebra.

From the A^i we can construct a graded Lie algebra $\sum A^i/A^{i+1}$, $-\infty < i < \infty$, which we will denote by $\mathscr{L}_A$ (since the construction depends on A). All the terms of degree < -1 in $\mathscr{L}_A$ are zero since $A^i = L$ for $i < 0$. Let V be the set of terms of degree -1. If we bracket an element of V by another element of V we get zero since the bracket is of degree -2; so V is a finite dimensional abelian subalgebra. Let $S(V)$ be the universal enveloping algebra of V. Since V is abelian, $S(V)$ is just the ring of polynomials over V. (See Jacobson [10, Chapter V].)

Now let $\mathscr{L}_A^*$ be the graded dual space of $\mathscr{L}_A$. This is an $\mathscr{L}_A$ module since $\mathscr{L}_A$ acts on it by the transpose of its adjoint action. Therefore, it is a V module and also an $S(V)$ module. In [6] we proved:

Proposition 2. *$\mathscr{L}_A^*$ is a finitely generated $S(V)$ module.* (See Chapter 3, Proposition 3.2.)

Let $I_A \subset S(V)$ be the annihilator ideal of the $S(V)$ module $\mathscr{L}_A^*$, Δ' be an extension field of Δ, and $\mathscr{V}^0(\Delta', A)$ be the variety of zeroes of I_A in $V^* \otimes \Delta'$. Since $V = L/A$, there is a projection map $L \to V$ which dualizes to give an injection mapping: $V^* \otimes \Delta' \to L^* \otimes \Delta'$. We will denote by $\mathscr{V}(\Delta', A)$ the image of $\mathscr{V}^0(\Delta', A)$ in $L^* \otimes \Delta'$.

Now suppose B is another fundamental subalgebra of L. We can duplicate the above construction using B instead of A. Our main result is

Theorem 1. $\mathscr{V}(\Delta', A) = \mathscr{V}(\Delta', B)$.

Proof. The proof involves several steps. We will first prove

Lemma 2. $\mathscr{V}(\Delta', A) = \mathscr{V}(\Delta', A^K)$.

Proof. Let $V = L/A^0$, $V' = L/A^K$, and $W = A^0/A^K$. Since $[A^0, A^K] \subset A^K$, W is contained in the annihilator of $\mathscr{L}_{A^K}^*$ regarded as an $S(V')$ module. Therefore, $\mathscr{L}_{A^K}^*$ can be regarded as an $S(V)$ module. Moreover, it is clear from the above remark that we will get the same result whether we compute its characteristics regarding it as an $S(V)$ module or regarding it as an $S(V')$ module. As a graded $S(V)$ module, $\mathscr{L}_{A^K}^*$ is identical with $\mathscr{L}_A^*$ except for a finite number of terms, and it is not hard to see that this implies that the zero varieties are the same.

Lemma 3. *Let $0 \to S \to T \to U \to 0$ be an exact sequence of modules over a commutative ring R, and I_S, I_T and I_U be the annihilator ideals of S, T, and U. Then $I_S \cdot I_U \subset I_T \subset I_S \cap I_U$.*

Proof. LTR.

Lemma 4. *Let A and B be fundamental subalgebras of L with the following two properties:*

a) $A \supset B \supset A^1$.

b) $[A, B] \subset B$.

Then $\mathscr{V}(\Delta', A) = \mathscr{V}(\Delta', B)$.

Proof. Let $V = L/A$ and $V' = L/B$. Since $[A, B] \subset B$, we can regard $\mathscr{L}_B^*$ as an $S(V)$ module, and its set of characteristics will be the same whether we regard $\mathscr{L}_B^*$ as an $S(V)$ module or as an $S(V')$ module[3].

From the inclusions $A \supset B \supset A^1$ we get $A^K \supset B^K \supset A^{K+1}$ for all K just from the definitions of these objects. Let $\mathscr{S}^K = (A^K/B^K)^*$ and $\mathscr{T}^K = (B^K/A^{K+1})^*$. The direct sums $\sum \mathscr{S}^K$ and $\sum \mathscr{T}^K$ are $S(V)$ modules, and are related by the following pair of exact sequences:

[3] See the proof of Lemma 2.

$$0 \to \sum_{K=0} \mathscr{T}^{K-1} \to \sum_{K=0} (B^{K-1}/B^K)^* \to \sum_{K=0} \mathscr{S}^K \to 0 ,$$

$$0 \to \sum_{K=0} \mathscr{S}^K \to \sum_{K=0} (A^K/A^{K+1})^* \to \sum_{K=0} \mathscr{T}^K \to 0 ,$$

where the middle term in the first sequence is $\mathscr{L}_B^*$, and the middle term in the second sequence is a truncation of $\mathscr{L}_A^*$.

Applying Lemma 3 to these sequences and comparing the middle terms, we get $\mathscr{V}(\Delta', A) = \mathscr{V}(\Delta', B)$ as claimed.[4]

Using Lemma 1 and Lemma 3, we will now prove the theorem.

Let A and B be fundamental subalgebras of L. Replacing B by $B \cap A$ if necessary we can assume $B \subset A$. Let $C_K = B \cap A^K$. Then $(C_K)^1 = B^1 \cap A^{K+1} \subset B \cap A^{K+1} = C_{K+1}$. Hence, we have $C_K \supset C_{K+1} \supset (C_K)^1$. We obviously have $[C_K, C_{K+1}] \subset C_{K+1}$ for all $K \geq 0$; therefore, we can apply Lemma 3 with A replaced by C_K and B by C_{K+1}. We get $\mathscr{V}(C_K, \Delta') = \mathscr{V}(C_{K+1}, \Delta')$ for all $K \geq 0$. When $K = 0$, $C_K = B$ and by Chevelley's principle $C_K = A^K$ for large K. So we get $\mathscr{V}(B, \Delta') = \mathscr{V}(A^K, \Delta')$ for large K; and the second set is equal to $\mathscr{V}(A, \Delta')$ by Lemma 1.

Since $\mathscr{V}(A, \Delta')$ is the same for all A we will just denote it by $\mathscr{V}(\Delta')$.

Definition 2. We will call $\mathscr{V}(\Delta')$ the *characteristic variety* of L with respect to the extension field Δ'.

We will prove one property of this set:

Theorem II. *If L is finite dimensional, then $\mathscr{V}(\Delta') = \{0\}$ for all extension fields Δ' of Δ. However, if L is infinite dimensional, there exist finite algebraic extensions Δ' of Δ for which $\mathscr{V}(\Delta') \neq \{0\}$.*

Proof. Let A be a fundamental subalgebra of L, and let us compute $\mathscr{V}(\Delta')$ using the filtration A, A^1, A^2, etc. Let $V = L/A$, and let I_A be the annihilator ideal of the $S(V)$ module $\mathscr{L}_A^*$.

Suppose that for every finite algebraic extension Δ' the zero variety of I_A in $V^* \otimes \Delta'$ consists just of $\{0\}$. Then by the Hilbert nullstellensatz, I_A has to contain $S^K(V)$ for all K greater than some K_1. Since $\mathscr{L}_A^*$ is a finitely generated $S(V)$ module, we can assume it is generated by its terms of degree $\leq K_0$. Then all terms of degree $\geq K_0 + K_1$ are zero, so L is finite dimensional. Conversely, if L is finite dimensional, then $A^{K_0} = \{0\}$ for some K_0, and in this case $I_A \supset S^K(V)$ for all $K > K_0$. So for all extension fields Δ' of Δ the zero variety of I_A in $V^* \otimes \Delta'$ is $\{0\}$.

We will now prove the result needed in § 3.

Proposition 3.2. *Let L be an infinite dimensional* LCT *Lie algebra satisfying the* d.c.c. *on its closed ideals. Then there exists a proper open subspace of L which is invariant with respect to every continuous automorphism of L.*

Proof. Let φ be a continuous automorphism of L. It is a corollary of

[4] The idea behind this argument was suggested to the author by Shlomo Sternberg.

Theorem 1 that $\varphi^t: L^* \otimes \Delta' \to L^* \otimes \Delta'$ preserves the characteristic variety of L. Now choose a finite extension field Δ' of Δ so that $\mathscr{V}(\Delta') \neq \{0\}$. We can regard $\mathscr{V}(\Delta')$ as a collection of Δ-linear mappings of L into Δ'. (Regard Δ' as a finite dimensional vector space over Δ.) Let H be the intersection of the kernels of these mappings. H is open since it contains every fundamental subalgebra of L, so it is a subspace of the required kind.

Remark. There are various ways to generalize the construction of $\mathscr{V}(\Delta')$. For example, if I is a closed ideal of L, one can define a characteristic variety $\mathscr{V}(\Delta', I)$ by considering the filtration $I \cap A^K$, $K = 0, 1, \cdots$, where the A^K's are as above. These generalized characteristic varieties will be discussed in the Harvard thesis of Colin Godfrey.

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