J. DIFFERENTIAL GEOMETRY
4 (1970) 91-104

REPRESENTATIONS OF COMPACT GROUPS AND MINIMAL IMMERSIONS INTO SPHERES

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1. Let G be a compact group, K a closed subgroup of G, and $C(M)$ the space of all real-valued continuous functions on the homogeneous space $M = G/K$. Then G has a natural action on $C(M)$ given by $g \cdot f(p) = f(g^{-1}p)$, $f \in C(M)$, $g \in G$, $p \in M$. Let V be a (necessarily finite-dimensional) invariant irreducible subspace of $C(M)$. Then V may be given an inner product $\langle \ , \ \rangle$ by $\langle f, g \rangle = \int_M fg d\mu$, where the homogeneous measure $d\mu$ normalized in such a way that $\int_M d\mu = \dim V$; relative to $\langle \ , \ \rangle$, G acts orthogonally on V.

Definition. We say that V satisfies condition A if $f_1, \cdots, f_r$ form an orthonormal basis of V (in particular, $r = \dim V$), whenever $f_1, \cdots, f_r$ are linearly independent in V and $\sum_{i=1}^{r} f_i^2(p) = 1$ for all $p \in M$.

In this paper, we are concerned with the following question: For which homogeneous spaces M is condition A satisfied for all invariant irreducible subspaces of $C(M)$?

We shall restrict ourselves to the simplest homogeneous spaces, namely, the simply connected homogeneous spaces G/K, where (G, K) is a symmetric pair of compact type. We recall that for such a pair, G is a compact, semisimple Lie group with an involutive automorphism $s: G \to G$ which is such that K is left fixed by s, and K contains the component of the identity of the fixed point set of s. To ensure the simply connectedness of G/K, we assume further that G is connected, simply connected and that K is connected. In this situation, condition A is strangely rare. In fact, we prove the following:

Theorem 1. *Let $M = G/K$ be a homogeneous space such that (G, K) is a symmetric pair of compact type, G is connected and simply connected, and K is connected. Then condition A is satisfied for all invariant, irreducible subspaces of $C(M)$ if and only if M is the 2-dimensional sphere $S^2 = SU(2)/U(1)$.*

In § 2, we prove Proposition 1, which says that the invariant, irreducible subspaces of $C(S^2)$ satisfy condition A. In § 3, we prove Proposition 2, which

Communicated by S. S. Chern, November 25, 1968. The first author was a Guggenheim fellow partially supported by C. N. Pq. and N. S. F. GP-8623, and the second author was partially supported by N. S. F. GP-7499.

shows that some invariant, irreducible subspace of $SU(2)$ does not satisfy condition A, and also Proposition 3, which is a similar assertion for $M = G/K$, where (G, K) satisfy the hypothesis of Theorem 1, and $M \neq S^2$. Theorem 1 follows from Propositions 1 and 3.

The above question was motivated by a problem of differential geometry, namely, to determine all isometric, minimal immersions of a symmertic space M into the standard sphere. In § 4, we give an exposition of this problem and show how Proposition 1 of § 2 can be used to give an answer in the case $M = S^2$.

The paper is written with an eye for the differential geometer. § 4 can be read independently of § 3, and the use of the theory of representations of Lie groups in § 2 and 4 has been reduced to a minimum.

2. In this section, we prove Proposition 1, for which we need some preliminary lemmas.

Let G/K be a homogeneous space of a compact Lie group G, V be an invariant irreducible subspace of $C(G/K)$, and $\dim V = n$. We first remark that the choice of an orthonormal basis $h_1, \cdots, h_n$ for V determines an isometry of V with the Euclidean space R^n, and also a map $x: G/K \to R^n$ given by

$$x(gK) = (h_1(gK), \cdots, h_n(gK)) , \qquad g \in G .$$

Since G acts orthogonally on V, it is easily seen that

$$(1) \qquad \sum (h_i(gK))^2 = 1, \text{ for all } g \in G ,$$

and therefore $x(G/K)$ is contained in the unit sphere of R^n. It follows that we may choose $h_1, \cdots, h_n$ in such a way that $x(eK) = (1, 0, \cdots, 0)$ and then h_1 is a unit vector in V left fixed by the isotropy subgroup K.

Lemma 1. *Let S^{n-1} be the unit sphere of V. Then the following conditions are equivalent*:

(1) *V satisfies condition A,*

(2) *If $v \in S^{n-1}$ is left fixed by K, and $L: V \to V$ is linear and such that $L(G \cdot v) \subset S^{n-1}$, then L is orthogonal.*

Proof. Let $v \neq 0$ be left fixed by K, and choose an orthonormal basis $\{h_1, \cdots, h_n\}$ in V. We shall identify V with R^n through the isometry determined by this basis. Assume now condition A holds. The condition $L(G \cdot v) \subset S^{n-1}$ is equivalent to $\langle {}^tLLg \cdot v, g \cdot v \rangle = 1$ for all $g \in G$. If B is the non-negative square root of ${}^tLL$, this last condition is equivalent to

$$(2) \qquad \langle Bg \cdot v, Bg \cdot v \rangle = 1, \text{ for all } g \in G.$$

Now, let $T = (t_{ij})$ be an orthogonal matrix such that ${}^tTBT = D$ is diagonal, with non-zero entries $d_1, \cdots, d_r$, $d_i > 0$, $i = 1, \cdots, r$. Let $p_i = \sum t_{ij} h_j$, $j = 1, \cdots, n$, and let $f_i = d_i p_i$. Then a simple computation shows that (2) implies that $\sum (f_i(gK))^2 = 1$, for all $g \in G$. Since $f_1, \cdots, f_r$ are linearly independent, it follows from condition A that $r = n$, and $f_1, \cdots, f_n$ form an orthonormal basis. Hence D is orthogonal and $d_1 = \cdots = d_n = 1$. Therefore ${}^tLL = I$ and L is orthogonal.

The converse is straightforward, and the proof of Lemma 1 is complete.

Before stating Lemma 2, we need some algebraic notation to be used throughout the paper.

Let W be an n-dimensional G-module with an inner product $\langle \ , \ \rangle$, relative to which G is orthogonal. If $v, w \in W$, we set $v \cdot w = 1/2(v \otimes w + w \otimes v)$, the symmetric product of v and w; in particular, we write $v^2 = v \cdot v$. We denote by W^2 the vector space generated by the symmetric products and make it into a G-module by

$$g \cdot (v \cdot w) = \frac{1}{2}(gv \otimes gw + gw \otimes gv) , \qquad g \in G, v, w \in W .$$

Using the inner product $\langle \ , \ \rangle$ we can identify V^2 with the space of all symmetric linear maps, defining map $v \cdot w$ by

$$(v \cdot w)(u) = \frac{1}{2}(\langle v, u \rangle w + \langle w, u \rangle v) , \qquad u, v, w \in W .$$

This identification may be used to define an inner product $(\ , \ )$ on V^2, setting $(x, y) = \text{trace } xy$, for $x, y \in W^2$. It is easily checked that

$$(3) \qquad g \cdot v^2 = g v^2 g^{-1} ,$$

and therefore G acts orthogonally on W^2 with respect to $(\ , \ )$.

The following relation will be useful. If $w \in W$ is a unit vector, and A is a symmetric linear map on W, then

$$(4) \qquad \langle Aw, w \rangle = \text{trace } Aw^2 = (A, w^2) .$$

This is easily proved by choosing an orthonormal basis $w = w_1, \cdots, w_n$ in W, and computing with coordinates.

The following lemma is a very convenient form of condition A.

Lemma 2. *Let V be an invariant, irreducible subspace of $C(G/K)$. Then V satisfies condition A if and only if for each unit vector $v \in V$, which is left fixed by K, the orbit $G \cdot v^2$ of v^2 spans V^2.*

Proof. Assume that $G \cdot v^2$ spans V^2, and let $L \colon V \to V$ be a linear map such $L(G \cdot v)$ is contained in the sphere of unit vectors of V. Then

$$\langle Lg \cdot v, Lg \cdot v \rangle = \langle g^{-1} \cdot {}^tLLg \cdot v, v \rangle = 1, \text{ for all } g \in G.$$

Using (3) and (4), we obtain that

$$(g^{-1} \cdot ({}^tLL), v^2) = ({}^tLL, g \cdot v^2) = 1, \quad \text{for all} \quad g \in G .$$

It follows that $({}^tLL - I, g \cdot v^2) = 0$, for all $g \in G$, which implies that ${}^tLL - I$

$= 0$ since $G \cdot v^2$ spans V^2. Hence L is orthogonal, and by Lemma 1, V satisfies condition A.

Conversely, assume that V satisfies condition A. Let $B \in V^2$ be such that $(B, g \cdot v^2) = 0$, for all $g \in G$. Then $(I + tB, g \cdot v^2) = 1$, for all $g \in G$ and all real t. Let $t > 0$ be such that $I + tB$ is positive definite, and L be the positive square root of $I + tB$. Then $\langle Lg \cdot v, Lg \cdot v \rangle = 1$; hence L is orthogonal by Lemma 1. Since L is symmetric and positive definite, $L = I$. It follows that $B = 0$ and therefore $G \cdot v^2$ spans V^2, which finishes the proof of Lemma 2.

We now assemble some facts on the representations of $SO(3)$, which will be used in the proof of Proposition 1.

Let $G = SO(3)$. It is known that the real irreducible representations V^k of G may be labeled by non-negative integers k, where $\dim V^k = 2k + 1$; V^k is essentially the G-module of real spherical harmonics of degree k on the sphere $SO(3)/SO(2)$ (see § 4, Example 1). Now, let $\mathfrak{g}$ be the complexified Lie algebra of G, with a basis $\{X, Y, H\}$ such that $\sqrt{-1}\,H$ is an element of the real Lie algebra of G and

$$[X, Y] = H, \quad [H, X] = 2X, \quad [H, Y] = -2Y .$$

Let W^{2k} be the complxification of V^k, looked upon as a G-module. Then it is known that there exists a basis $\{v_0, v_1, \cdots, v_{2k}\}$ of W^{2k} with the following properties [6, Chap. III, § 8]:

$$(5) \qquad X \cdot v_0 = 0, \quad X \cdot v_j = j(2k - j + 1)v_{j-1}, \quad j = 1, \cdots, 2k;$$

$$(6) \qquad Y \cdot v_j = v_{j+1}, \quad j = 0, 1, \cdots, 2k - 1, \quad Y \cdot v_{2k} = 0;$$

$$(7) \qquad H \cdot v_j = 2(k - j)v_j, \quad j = 0, 1, \cdots, 2k .$$

It follows from (7) that $\sqrt{-1}\,H \cdot v_k = 0$ and that the eigenspace of zero is one-dimensional, hence we may assume that $v_k \in V^k$.

Now, let $\Gamma = XY + YX + 1/2\,H^2$ (although we do not use it, we mention the fact that Γ is essentially the Casimir element of $\mathfrak{g}$). A straightforward computation with the above relations shows that the action of Γ on W^{2k} is given by

$$(8) \qquad \Gamma = 2k(2k + 1)I .$$

Let us consider the symmetric product representation $(W^{2k})^2$. It can be shown that as a $\mathfrak{g}$-module $(W^{2k})^2 = \sum_{j=0}^{k} W^{4k-4j}$. Let $P_j : (W^{2k})^2 \to W^{4k-4j}$ be the corresponding projection and set $\gamma_j = (4k - 4j)(2k - 2j + 1)$. Then, by (8), the tensor product action of Γ on $(W^{2k})^2$ is given by $\Gamma = \sum_0^k \gamma_i P_j$.

Lemma 3. *Let $w \in (W^{2k})^2$. Then $G \cdot w$ spans $(W^{2k})^2$ if and only if w, $\Gamma \cdot w$, $\cdots$, $\Gamma^k w$ are linearly independent.*

Proof. The matrix of $I, \Gamma, \cdots, \Gamma^k$ in terms of $P_0, P_1, \cdots, P_k$ is a Vandermonde matrix. It is easily checked that this matrix is non-singular, because $\gamma_i \neq \gamma_j$ for $i \neq j$. Thus $w, \Gamma w, \cdots, \Gamma^k w$ are linearly independent if and only if $P_0 w, P_1 w, \cdots, P_k w$ are non-zero. Since $G\cdot(P_j w), P_j w \neq 0$, clearly spans the irreducible W^{4k-4j}, the conclusion follows.

Lemma 4. *$v_r^2, \Gamma\cdot v_r^2, \cdots, \Gamma^r v_r^2$ are linearly independent for $0 \leq r \leq k$.*

Proof. Set $C_j = j(2k - j + 1)$, $j=0, 1, \cdots, 2k$. By using (5), a straightforward computation shows that

$$\Gamma v_r^2 = \left(XY + YX + \frac{1}{2}H^2\right)v_r^2 \equiv 2C_r v_{r+1}\cdot v_{r-1} ,$$

modulo the space generated by v_r^2. We can also easily see from (5) that, for $t = 1, \cdots, r$,

$$\Gamma v_{r+t}\cdot v_{r-t} \equiv 2C_{r-t} v_{r+t+1}\cdot v_{r-t-1} ,$$

modulo the space spanned by $v_{r+t}\cdot v_{r-t}, v_{r+t-1}\cdot v_{r-t+1}, \cdots, v_r^2$. It follows by induction that

$$\Gamma^t v_r^2 \equiv 2^t C_r \cdots C_{r-t+1} v_{r+t}\cdot v_{r-t} ,$$

modulo the space spanned by $v_{r+t-1}\cdot v_{r-t+1}, \cdots, v_r^2$; furthermore, $2^t C_r \cdots C_{r-t+1} \neq 0$, for $t \leq r$. Since the vectors $v_{r+t}\cdot v_{r-t}$, $t = 0, 1, \cdots, r$, are linearly independent, the conclusion follows.

We recall that an irreducible G-module W is called a class one representation of the pair (G, K) if there exists a $w \in W$, $w \neq 0$, such that $k\cdot w = w$, for all $k \in K$.

We are now in a position to prove the main result of this section.

Proposition 1. *Let $M = SU(2)/U(1) = SO(3)/SO(2)$. Then all invariant irreducible subspaces of $C(M)$ satisfy condition A.*

Proof. As we saw earlier in this section, an invariant irreducible subspace V of $C(M)$ is a class one representation of the pair $(SO(3), SO(2))$. V is in particular a representation of $SO(3)$ and, using the notation of Lemmas 3 and 4, we may denote it by V^k, k an integer, $\dim V^k = 2k + 1$. By Lemma 4, with $r = k$, $v_k^2, \Gamma\cdot v_k^2, \cdots, \Gamma^k v_k^2$ are linearly independent and then, by Lemma 3, $G\cdot v_k^2$ spans $(W^{2k})^2$; hence it spans $(V^k)^2$. On the other hand, since $\sqrt{-1}H\cdot v_k = 0$ and $\sqrt{-1}H$ is real, the vector v_k is left fixed by the subgroup of $SO(3)$ corresponding to the subalgebra spanned by $\sqrt{-1}H$, namely, by $SO(2)$. Since the subspace of V^k left fixed by $SO(2)$ is Rv_k (see (7)), we may apply Lemma 2 to show that $V = V^k$ satisfies condition A, and hence complete the proof of Proposition 1.

3. In this section, we prove Propositions 2 and 3 (stated below), and therefore complete the proof of Theorem 1.

Proposition 2. *Let $G = SU(2)$. Then there exists an invariant irreducible subspace of $C(G)$, which does not satisfy condition A.*

Proof. Since $SU(2)$ is the universal covering of $SO(3)$, it clearly suffices to prove the statement of Proposition 2 for $G = SO(3)$. Let V^k, W^{2k}, $\{v_0, \cdots, v_{2k}\}$ and Γ be as in § 2. A typical element of V^k is of the form

$$w = \sum_{i=0}^{k-1} z_i v_i + x v_k + \sum_{i=0}^{k-1} (-1)^{k-i}(i!/(2k-i)!)\bar{z}_i v_{2k-i} ,$$

where $z_i \in C$, $i = 1, \cdots, k-1$, and $x \in R$. The proof will consist merely in checking that a k can be chosen such that the element

$$w = z_1 v_1 + (-1)^{k-1}(1/(2k-1)!)\bar{z}_1 v_{2k-1}$$

has the property that $G \cdot w^2$ does not span $(V^k)^2$, which by Lemma 2 gives the desired conclusion.

To see that, we first remark that for $0 \leq r \leq k$, from (7) we have $H \cdot v_r^2 = (4k - 4j)v_r^2$. Therefore $v_r^2 \in \sum_{j=0}^{r} W^{4k-4j}$, and hence $\prod_{j=0}^{r}(\Gamma - \gamma_j I)v_r^2 = 0$, where $\gamma_j = (4k - 4j)(2k - 2j + 1)$. It follows that $\prod_{i=0}^{k}(\Gamma - \gamma_i I)u = 0$ for all $u \in (W^{2k})^2$. Now

$$\Gamma v_0 v_{2k} = 2XY v_0 v_{2k} = 4k v_0 v_{2k} + 4k v_1 v_{2k-1} ,$$

and hence

$$(\Gamma - 4kI)v_0 v_{2k} = 4k v_1 v_{2k-1} .$$

Choose a positive integer s and let $k = s(2s + 1)$. If $p = k - s$ then $\gamma_p = 4k$. It follows from the above remark that

$$\prod_{i=0; i\neq p}^{k}(\Gamma - \gamma_i I)(\Gamma - 4kI)v_0 \cdot v_{2k} = 0 ,$$

and therefore

$$4k \prod_{i=0; i\neq p}^{k}(\Gamma - \gamma_i I)v_1 v_{2k-1} = 0 . \tag{9}$$

Clearly $p \geq 2$, and $v_{2k-1}^2 \in W^{4k} + W^{4k-4}$; thus

$$\prod_{i=0; i\neq p}^{k}(\Gamma - \gamma_i I)v_1^2 = 0 = \prod_{i=0; i\neq p}^{k}(\Gamma - \gamma_i I)v_{2k-1}^2 . \tag{10}$$

Since

$$w^2 = z_1^2 v_1^2 + \frac{(-1)^{k-1}}{(2k-1)!}|z_1|^2 v_1 \cdot v_{2k-1} + \frac{1}{((2k-1)!)^2}\bar{z}_1^2 v_{2k-1}^2 ,$$

we conclude from (9) and (10) that

$$\prod_{i=0; i\neq 0}^{k}(\Gamma - \gamma_i I)w^2 = 0 ,$$

hence $w^2, \Gamma \cdot w^2, \cdots, \Gamma^k w^2$ are not linearly independent. It follows from Lemma 3 that $G \cdot w^2$ does not span $(V^k)^2$, and the proof is finished.

Before proving Proposition 3 below we need some notation and a few preliminary lemmas. As always (G, K) is a symmetric pair of compact type, with G connected and simply connected and K connected. Let $\mathfrak{g}_0$ be the Lie algebra of G, $\mathfrak{k}_0$ be the Lie algebra of K, and $\sigma: \mathfrak{g}_0 \to \mathfrak{g}_0$ be the involutive automorphism with $\mathfrak{k}_0$ as fixed point set. Let $\mathfrak{p}_0 = \{X \in \mathfrak{g}_0 \mid \sigma X = -X\}$ and let $\mathfrak{a}_0$ be a maximal abelian subsystem of $\mathfrak{p}_0$; the dimension of $\mathfrak{a}_0$ is called the rank of G/K. Let $\mathfrak{m}_0$ be maximal in $\mathfrak{k}_0$ relative to the conditions that $\mathfrak{m}_0$ be abelian and $[\mathfrak{m}_0, \mathfrak{a}_0] = 0$. Let $\mathfrak{h}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0$; then $\mathfrak{h}_0$ is a maximal abelian subalgebra of $\mathfrak{g}_0$ such that $\sigma\mathfrak{h}_0 = \mathfrak{h}_0$. Let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$, $\mathfrak{h}$ the complexification of $\mathfrak{h}_0$ in $\mathfrak{g}$, and Δ the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let $\mathfrak{h}_R = \sqrt{-1}\,\mathfrak{h}_0$. if $\alpha \in \Delta$, then $\alpha(\mathfrak{h}_R) \subset R$. Set $\mathfrak{h}_R^- = \sqrt{-1}\,\mathfrak{a}_0$, $\mathfrak{h}_R^+ = \sqrt{-1}\,\mathfrak{m}_0$; let $\{h_1, \cdots, h_p\}$ be a basis for $\mathfrak{h}_R^-$, and $\{h_{p+1}, \cdots, h_n\}$ be a basis for $\mathfrak{h}_R^+$. Order $\mathfrak{h}_R^*$ lexicographically with respect to the ordered basis $\{h_1, \cdots, h_n\}$ of $\mathfrak{h}_R$ and let $\prod = \{\alpha_1, \cdots, \alpha_n\}$ be the simple system with respect to this order. Finally, denote the Weyl group of Δ by $W(\Delta)$.

Now let $C(M; C)$ be the space of continuous complex-valued functions on $M = G/K$, and V an invariant irreducible complex subspace of $C(M; C)$. Then, there is a unique element $\varphi_V \in V$ such that $\varphi_V(K) = 1$ and $k\varphi_V = \varphi_V$, for all $k \in K$ [5, p. 416]; φ_V is called the zonal of V.

Lemma 5. *Let V be an invariant, irreducible complex subspace of $C(M, C)$, and assume that there exists an element $s \in W(\Delta)$ such that $s \mid \mathfrak{h}_R^- = -I$. Then the zonal φ_V of V is real-valued.*

Proof. Let $d\mu$ be the G-invariant volume element of M and define a Hermitian structure on $C(M; C)$ by $\langle f, g\rangle = \int_M f\bar{g}\,d\mu$, where $f, g \in C(M; C)$. Next, define a map $A: V \to C(M; C)$ by $Af(gK) = \langle g\cdot\varphi_V, f\rangle$, $g \in G$. Then A is linear unitary with respect to $\langle\ ,\ \rangle$. Furthermore

$$(Ag_0\cdot f)(gK) = \langle g\cdot\varphi_V, g_0\cdot f\rangle = Af(g_0^{-1}gK) = (g_0\cdot Af)(gK),$$

and hence AV is equivalent to V as a representation. Since $C(M; C)$ contains each irreducible subrepresentation exactly once [3, p. 15], $AV = V$. It follows that $\varphi_V(g\cdot K) = \langle g\varphi_V, \varphi_V\rangle$, and hence φ_V is a positive definite function [5, p. 412] as a function on G given by $\varphi_V(g) = \varphi_V(gK)$. Therefore $\overline{\varphi_V(gK)} = \varphi_V(g^{-1}K)$.

We remark that φ_V is entirely determined by its restriction $\varphi_V|_{\exp(\mathfrak{a}_0)\cdot K}$. In fact, from $M = \exp(\mathfrak{p}_0)\cdot K$, and $Ad(K)\cdot\mathfrak{a}_0 = \mathfrak{p}_0$ [5, p. 211], it follows that $M = K\exp\mathfrak{a}_0\cdot K$.

Now assume that there exists $s \in W(\Delta)$ such that $s \mid \mathfrak{h}_R^- = -I$. Then there exists a $k \in K$ such that $Ad(k)\mathfrak{h}_R^- = \mathfrak{h}_R^-$ and $Ad(k) \mid \mathfrak{h}_R^{-1} = -I$ [5, p. 249]. Joining these facts together, we obtain

$$\begin{aligned}\varphi_V(\exp H\cdot K) &= \varphi_V(k\exp H\cdot k^{-1}K) = \varphi_V(\exp Ad(k)H\cdot K)\\ &= \varphi_V(\exp(-H)\cdot K) = \overline{\varphi_V(\exp H\cdot K)}\ ,\end{aligned}$$

for all $\sqrt{-1}\,H \in \mathfrak{h}_R^-$, where $\varphi_V = \overline{\varphi_V}$, as we wished to prove.

Corollary. *If M is of rank one, then all the zonals are real.*

Proof. Let $\alpha \in \prod$ be such that $\alpha(\mathfrak{h}_R^-) \neq 0$. Then the Weyl reflection S_α about the hyperplane $\alpha = 0$ is equal to $-I$ in $\mathfrak{h}_R^-$.

Before stating the next lemma, we need a little more notation. Let $\mathfrak{g}_0$ act on $C(M; C)$ by

$$(X\cdot f)(m) = \frac{d}{dt}f(\exp(-tX)\cdot m)|_{t=0}\,, \qquad m \in M\,.$$

If V is an invariant irreducible subspace of $C(M; C)$ then $\mathfrak{g}\cdot V \subset V$. For each $\mu \in \mathfrak{h}^*$ (the complex dual of $\mathfrak{h}$) let $V_\mu = \{f \in V \mid h\cdot f = \mu(h)\cdot f \text{ for all } h \in \mathfrak{h}\}$. Let $V = \sum V_\mu$. If $V_\mu \neq \{0\}$, then $\mu(\mathfrak{h}_R) \subset R$ (cf. [6. p. 113]). Let λ_V be the largest λ such that $V_\lambda \neq \{0\}$, with respect to the given lexicographic order on $\mathfrak{h}_R^*$; λ_V is called the highest weight of V. If W is another irreducible invariant subspace of $C(M, C)$ with highest weight λ_V then $W = V$ (see Cartan [3. p. 15]). We note that if V and W are irreducible invariant subspaces of $C(M, C)$ then there is an irreducible subspace U of $C(M, C)$ such that $\lambda_U = \lambda_V + \lambda_W$. In fact, let $f \in V$ (resp. $g \in W$) be such that $h\cdot f = \lambda_V(h)\cdot f$ (resp. $h\cdot g = \lambda_W(h)\cdot g$), for each $h \in \mathfrak{h}$. If $q = f\cdot g$ then $h\cdot q = (\lambda_V + \lambda_W)(h)\cdot q$, and the linear span U of $G\cdot q$ is the desired representation. There are elements $\lambda_1, \cdots, \lambda_p$ of $\mathfrak{h}_R^*$ such that $\lambda_i = \lambda_{V_i}$ for V_i an irreducible invariant subspace of $C(M, C)$, and if V is an irreducible invariant subspace of $C(M, C)$ then $\lambda_V = \sum n_i\lambda_i$ with n_i non-negative integers (see Cartan [3, pp. 22-23]). It is convenient to label the invariant irreducible subspace V of $C(M, C)$ by its highest weight λ, that is, $V = V^\lambda$.

Lemma 6. *Let V be a real class one representation of (G, K) and let $v \in V$ be such that $K\cdot v = v$. Let W be the linear span of $G\cdot v^2$ in V^2. Then each irreducible subrepresentation of W is of class one and W contains such a representation at most twice. Furthermore, if (G, K) satisfies the assumption of Lemma 5, then W contains each irreducible subrepresentation exactly once.*

Proof. We first remark that if U is a real blass one representation of (G, K) and $N = \{u \in U \mid K\cdot u = u\}$, then $\dim N \leq 2$. This follows from the fact that the complexification U_C of U either is irreducible, in which case $\dim N = 1$, or can be written as $U_C = U_1 \oplus U_2$, with U_1 contragradient to U_2. In the latter case, $\varphi_{U_1} = \bar{\varphi}_{U_2}$, hence $\varphi_{U_1} + \varphi_{U_2}$ and $\sqrt{-1}\,\varphi_{U_1} + \varphi_{U_2}$ generates N, and thus $\dim N \leq 2$, which proves our claim.

Now, $W = \sum W_i$, W_i irreducible. Thus $v^2 = \sum w_i \in W_i$, $w_i \in W_i$, and W_i is the linear span of Gw_i. It follows that w_i is left fixed by K and thus W_i is of class one. By our previous remark $\dim N_i \leq 2$, where $N_i = \{w \in W_i \mid Kw = w\}$.

Let us assume that $\dim N_i = \dim N_j = 1$ and that W_i is equivalent to $W_j \neq W_i$. Then w_i and w_j transform in exactly the same manner as $w_i + w_j$, and therefore the linear span of $G(w_i + w_j)$ is equivalent to W_i and W_j and contains $w_i + w_j$, a contradiction showing that $W_i = W_j$.

Assume now that $\dim N_i = \dim N_j = \dim N_k = 2$, and that W_i is equivalent to W_j and W_k, and that W_i, W_j, W_k are distinct. Then w_i, say, must transform in the same manner as some combination of w_j and w_k, say, $w_j + bw_k$. Therefore, the linear span U of $G(w_i + w_j + bw_k)$ is irreducible and $U + W_k$ contains $w_i + w_j + w_k$. This is a contradiction and shows that W_i, W_j, W_k are not distinct.

From the above considerations it follows that W contains each irreducible subrepresentation at most twice. Moreover, if (G, K) satisfies the assumption of Lemma 5, then $\dim N_i = 1$ for all i. Therefore each irreducible subrepresentation appears at most once, and this completes the proof of the lemma.

We now state and prove Proposition 3 in a form slightly more precise that it was announced in the introduction.

Proposition 3. *Let (G, K) be a symmetric pair of compact type, G connected and simply connected, and K connected. Assume that $G/K = M$ is not a two-dimensional sphere S^2. Then there exists an invariant, irreducible subspace of $C(M)$, which does not satisfy condition A. Furthermore, if M has rank one and $M \neq S^2$, then there exists a number $N > 0$ such that if V is an invariant, irreducible subspace of $C(M)$ and $\dim V \geq N$, then V does not satisfy condition A.*

Proof. We first show that there are invariant irreducible subspaces of $C(S^2 \times S^2)$, which do not satisfy condition A. Observe that $S^2 \times S^2$ corresponds also to the symmetric pair $(G = SO(3) \times SO(3), K = SO(2) \times SO(2))$ and let V^k be the $(2k+1)$-dimensional real irreducible representation of $SO(3)$. Let $V^k \otimes V^m$ be the tensor product representation of $SO(3) \times SO(3)$, and denote by $v_k \in V^k$, $v_m \in V^m$ the unit vectors which are left fixed by $SO(2)$. Then $v_k \otimes v_m$ is a unit vector left fixed by $SO(2) \times SO(2)$ in $V^k \otimes V^m$; it follows easily from Lemma 5 that such a vector is unique up to a sign. Furthermore every class one representation of (G, K) is of the form $V^k \otimes V^m$. By Lemma 6, the linear span $W_{k,m}$ of $G\cdot(v_k \otimes v_m)^2$ contains each irreducible representation exactly once. It is easy to see from our results in § 2 that

$$W_{k,m} = \sum_{i=0}^{m}\sum_{j=0}^{k} V^{2k-2j} \otimes V^{2m-2i} .$$

Now

$$\dim W_{k,m} = (2k+1)(k+1)(2m+1)(m+1) ,$$
$$\dim (V^k \otimes V^m)^2 = \frac{1}{2}(2k+1)(2m+1)\{(2k+1)(2m+1)+1\} .$$

Therefore,

$$\dim (V^k \otimes V^m)^2 - \dim W_{k,m} = (2k+1)(2m+1)km .$$

Thus, if k and m are positive, $G\cdot(v_k \otimes v_m)^2$ does not span $(V^k \otimes V^m)^2$, which by Lemma 2 proves our claim.

We may now assume that the symmetric space M is irreducible and $M \neq S^2$.

Let $\langle\ ,\ \rangle$ be the Killing inner product of $\mathfrak{h}_R^*$ (the real dual of $\mathfrak{h}_R^*$), and let $\Delta_i^+ = \{\alpha \in \Delta \mid \alpha > 0$ and $\langle \alpha, \lambda_i \rangle \neq 0\}$, $i = 1, \cdots, p$. Suppose that, for some i, Δ_i^+ consists of one element. Then $\Delta_i^+ = \{\alpha_j\}$, for some j, $1 \leq j \leq n$, and $\alpha_j + \alpha_k \notin \Delta$ for any $k = 1, \cdots, n$. The condition of irreducibility on M implies then that $n \leq 2$. If $n = 1$, then $G = SU(2)$; since the only possible symmetric pair $(SU(2), U(1))$ corresponds to the sphere S^2, this case is excluded. If $n = 2$, then $G = SU(2) \times SU(2)$. For such a G, the only possible symmetric pairs correspond to $K = U(1) \times U(1)$ and $K = \{(g, g) \mid g \in SU(2)\}$; the first case has already been considered, and in the second case $\langle \alpha_1, \lambda_1 \rangle \neq 0$, $\langle \alpha_2, \lambda_1 \rangle \neq 0$. By Proposition 1, it follows that we may assume that the number of elements k_i in Δ_i^+ satisfies $k_i \geq 2$.

Let V^λ be the invariant irreducible subspace of $C(M, C)$ with $\lambda = q \sum \lambda_i$, $q \geq 0$, q an integer. Then V^λ is self dual and thus the zonal of V^λ is real. Hence V^λ is the complexification of the real irreducible G-module $V^\lambda \cap C(M)$. Let V^μ be a complex irreducible class one subrepresentation of V_C^2 with highest weight μ. Then $\mu = \sum r_i \lambda_j$ with $r_j \geq 0$, r_i an integer, $i = 1, \cdots, p$. We now find an upper bound for r_i, $i = 1, \cdots, p$.

Since $\{\alpha_1, \cdots, \alpha_n\}$ is abas is for $\mathfrak{h}_R^*$, $\lambda_i = \sum_{j=1}^n a_{ji}\alpha_j$, $i = 1, \cdots, p$. It is easy to see that $a_{ji} \geq 0$, $i = 1, \cdots, p$, $j = 1, \cdots, n$. (In fact, $\langle \alpha_i, \alpha_j \rangle \leq 0$ if $i \neq j$. Thus, if $\xi_1, \cdots, \xi_n$ is the Gram-Schmidt orthonormalization of $\alpha_1, \cdots, \alpha_n$, then $\xi_i = \sum_{j=1}^i t_{ji}\alpha_j$ and $t_{ji} \geq 0$. Further $\langle \lambda_i, \xi_j \rangle = b_{ji} \geq 0$, $\lambda_i = \sum b_{ji}\xi_j = \sum_{j,k} b_{ji}t_{kj}\alpha_k$, and $a_{ki} = \sum_j t_{kj}b_{ji} \geq 0$). Moreover, the matrix (a_{ji}) is of rank p. Now $2\lambda - \mu = \sum m_i\alpha_i$ with $m_i \geq 0, m_i$ an integer (cf. Jacobson [6, p. 215]). Hence $2q \sum_i a_{ji} \geq \sum_i a_{ji}r_i$ for $j = 1, \cdots, n$. This implies, in particular, that $2q(\sum_{ij} a_{ij}) \geq \sum_{ji} a_{ji}r_i$. Set $c = \sum_{ij} a_{ji}$, $p_i = \sum_j a_{ji}$, $i = 1, \cdots, p$. Then since (a_{ji}) is of rank p, $c > 0$, $p_i > 0$, $i = 1, \cdots, p$. Let r be an integer such that $c/p_i \leq r$ for $i = 1, \cdots, p$; then $r_i \leq 2rq$, $i = 1, \cdots, p$.

Let W be the complex linear span of $G \cdot v^2$ in V_C^2. The dimension of V^μ is given by

$$\dim_C V^\mu = \prod_\alpha \frac{\langle \mu + \delta, \alpha \rangle}{\langle \delta, \alpha \rangle}, \quad \alpha > 0, \quad \alpha \in \Delta,$$

where $\delta = \dfrac{1}{2} \sum \alpha$, $\alpha \in \Delta$, $\alpha > 0$ (cf. [6, p. 257]). We set $\sum k_i = k$ and

$$\prod_\alpha \frac{\langle \lambda_i, \alpha \rangle}{\langle \delta, \alpha \rangle} = d_i, \quad \alpha \in \Delta, \alpha > 0,$$

for notational convenience. By the above and Lemma 6,

$$\dim_C W \leq 4(2qr + 1)^p \prod_\alpha \left(2qr \sum_{i=1}^p \frac{\langle \lambda_i, \alpha \rangle}{\langle \delta, \alpha \rangle} + 1\right)$$

$$= 2^{p+2+k} r^{p+k} q^{k+p} \prod_{i=1}^p d_i + \text{terms of lower degree in } q.$$

On the other hand, if $\dim_C V^\lambda = S$ then

$$\dim_C V_C^2 = S(S+1)/2 = \frac{1}{2} q^{2k}(\prod_{i=1}^{p} d_i)^2 + \text{terms of lower degree in } q.$$

Since $k_i \geq 2$ for $i = 1, \cdots, p, 2k > k + p$. Thus if q is sufficiently large then $\dim_C W < \dim_C V_C^2$. This proves the first assertion of Proposition 3. If rank $M = p = 1$ then by the corollary to Lemma 5 every invariant irreducible subspace V of $C(M)$ is of the form $V^{q\lambda_1} \cap C(M)$. Since $\dim_C V^{q\lambda_1} < \dim V^{(q+1)\lambda_1}$, the proposition is proved.

4. In this section we will show how Proposition 1 is related to a problem in differential geometry. For completeness, we recall some known facts.

Let M be an n-dimensional compact Riemannian manifold, and Δ the Laplace-Beltrami operator on M. Let $x: M \to R^{m+1}$ be an isometric immersion of M into a Euclidean space R^{m+1},

$$x(p) = (f_1(p), \cdots, f_{m+1}(p)), \qquad p \in M, \tag{11}$$

such that $\Delta x + \lambda x = 0$, where λ is a real number and Δx means $(\Delta f_1, \cdots, \Delta f_{m+1})$. It is then easy to prove [8, Th. 3] that λ is positive, $x(M)$ is contained in the m-sphere $S_r^m \subset R^{m+1}$ of radius $r = \sqrt{n/\lambda}$, and, as an immersion into S_r^m, x is minimal.

For completeness, we sketch a proof of the above fact, using moving frames. Let $e_1, \cdots, e_n, e_{n+1}, \cdots, e_{m+1}$ be a local orthonormal frame in R^{m+1} such that, restricted to M, $e_1, \cdots, e_n$ are tangent vectors and $e_{n+1}, \cdots, e_{m+1}$ are normal vectors. Let $h_{i\alpha j}$ be the coefficients of the second quadratic (fundamental) form in the direction e_α, $\alpha = n+1, \cdots, m+1$, and $i, j = 1, \cdots, n$, and set $H = (1/n)\sum_{\alpha i} h_{i\alpha i} e_\alpha$, the mean curvature vector of x. A simple computation shows that $\Delta x = nH$, and hence $x = -(n/\lambda)H$. It follows that $\langle x, dx\rangle = 0$, and therefore $|x|^2 = \text{constant} = r^2$. Thus $x(M) \subset S_r^m \subset R^{m+1}$. Now, let the last vector of the frame be given by $e_{m+1} = x/r$. It follows that if H^* is the component of H in the subspace generated by $e_{n+1}, \cdots, e_m$, then $H^* = 0$. That is, the mean curvature of x, as an immersion into S_r^m, is zero, which is the definition of minimal immersion into S_r^m. Furthermore, since the mean curvature $(1/n)\sum_i h_{i,m+1,i}$ of the sphere $S_r^m \subset R^{m+1}$ is $1/r$, we obtain $H = -x/r^2$. It follows that $r^2 = n/\lambda$ and $\lambda > 0$, which completes the proof. The above proof also shows that if $x: M^n \to S_r^m$ is minimal, then $\Delta x = -(n/r^2)x$, a remark that we shall use later in this section.

For the rest of this section we assume that M is a homogeneous space G/K of a compact Lie group G such that the linear action of K on the tangent space of the coset K is irreducible. G/K will be given a homogeneous Riemannian metric denoted by g. Let $\lambda \neq 0$ be a real number such that there exists a solution of

$$\Delta f + \lambda f = 0 \tag{12}$$

It is known that the vector space V_λ of solutions of (12) is finite dimensional [5, p. 424]. G acts on V_λ as in § 1, and V_λ is an invariant subspace of $C(M)$. Let $W \subset V_\lambda$ be an invariant non-zero subspace. Choose an inner product for W as in § 1. Then an orthonormal basis $\{f_1, \cdots, f_{m+1}\}$ of W determines a map $x: M \to R^{m+1}$ by (11), with $\sum_i f_i^2 = 1$. Since G acts orthogonally on W, the symmetric tensor $\bar{g} = \sum_i df_i \cdot df_i$ on M is invariant by G and, by the irreducibility of the action of K, we have that $\bar{g} = cg$, $c > 0$.

We now change the metric g of M to $\bar{g} = cg$ and denote by $\bar{M}$ the space M with this new metric. The Laplacian of $\bar{M}$ is given by $\bar{\Delta} = (1/c)\Delta$. Thus $x: \bar{M} \to S_1^m$ becomes an isometric immersion satisfying $\bar{\Delta}x = \bar{\lambda}x$, where $\bar{\lambda} = \lambda/c$. It follows that x is a minimal immersion into a sphere of radius $r = \sqrt{n/\bar{\lambda}}$. Since $r = 1$, we conclude that $c = \lambda/n$, which determines $\bar{g}$. Since the homogeneous metric g of G/K is determined up to a factor, it is easily seen that this process determines $\bar{g}$ uniquely.[1]

We remark that $x(M)$ is not contained in a hyperplane of R^{m+1} and that a change of orthonormal basis in W gives another isometric minimal immersion of $\bar{M}$, which differs from the first one by a rigid motion.

If G/K is a symmetric space of rank one, the functions which satisfy (12) will be called spherical functions.

Example 1. Let $M = SO(n+1)/SO(n)$ be the sphere with metric of constant curvature one. M may be realized as the unit sphere $S_1^n \subset R^{n+1}$ of a Euclidean space R^{n+1}. It can be proved that a spherical harmonic f on M is the restriction to S_1^n of a homogeneous polynomial $P(x_0, \cdots, x_n)$ defined in R^{n+1} which satisfies $\sum_{i=0}^n \partial^2 P/\partial x_i^2 \equiv 0$; such a polynomial is said to be harmonic, and the degree of P is called the order k of f. The eigenvalue λ of f and the dimension of V_λ are explicitly determined by k [7, pp. 39,4]. It follows that an orthonormal basis of the vector space V_λ, $\lambda = \lambda(k)$, of the spherical harmonics of order k gives a minimal isometric immersion $x: S_r^n \to S_1^m \subset R^{m+1}$ of an n-sphere S_r^n of radius r into S_1^m, where $m + 1 = \dim V_\lambda$, and $r = \sqrt{\lambda/n}$; r is determined by the fact that the metric $\bar{g}$ in S_r^n is $(\lambda/n)g$, where g is the metric of S_1^n.

Example 2. Let $M = SU(d+1)/U(d) = P^d(C)$ be the complex projective space with the metric g of constant holomorphic curvature equal to one. Let $(z_0, \cdots, z_d) \in C^{d+1}$, $z_i \in C$, $i = 0, \cdots, d$, and consider $P^d(C)$ as the quotient space of the sphere $\sum_i z_i \bar{z}_i = 1$ by the equivalence relation $z_i \sim z_i e^{i\theta}$. A polynomial $P(z_0, \cdots, z_d, \bar{z}_0, \cdots, \bar{z}_d)$, homogeneous of degree k in both z_i and $\bar{z}_i$, is called harmonic if

$$\sum_i \partial^2 P/\partial z_i \partial \bar{z}_i \equiv 0.$$

From the homogeneity condition, it is clear that the restriction f of P to the

1 The result of this paragraph has been derived independently by J. Tirao of the University of California, Berkeley by using different methods, in the case when (G, K) is a symmetric pair of compact type.

sphere $\sum_i z_i \bar{z}_i = 1$ is actually defined on $P^d(C)$. It is possible to prove [4, p. 294] that, for a given degree k, the set of all such f will form an invariant irreducible subspace V of $C(P^d(C))$. It follows that $V = V_\lambda$ is the vector space of spherical functions on M, corresponding to a certain eigenvalue λ. Therefore for some multiple $\bar{g}$ of the metric g we obtain an isometric minimal immersion of $P^d(C)$ into $S_1^m \subset R^{m+1}$, $m + 1 = \dim V_\lambda$; the metric $\bar{g}$ and the dimension m are determined by the degree k. It can be proved that, for $d \neq 1$, these immersions are imbeddings [4, p. 310] and they include, for instance, the so-called Segre varieties.

Suppose now that we are given an isometric minimal immersion $x: M \to S_1^m \subset R^{m+1}$ of $M = G/K$, with some homogeneous metric g, such that $x(M)$ is not contained in a hyperplane of R^{m+1}, and let x be given by (11). Then, from the remark in the beginning of this section it follows that $\Delta f_i + n f_i = 0$, $i = 1, \cdots, m + 1$, where n is the dimension of M. Thus $f_1, \cdots, f_{m+1}$ is a linearly independent set of vectors belonging to the vector space V_λ of the solutions of (12), with $\lambda = n$ and the property that $\sum_i (f_i)^2 = 1$.

Rigidity conjecture. With the above notation, if G/K is a symmetric space of rank one, then $f_1, \cdots, f_{m+1}$ form an orthonormal basis of V_λ; in particular, $m + 1 = \dim V_\lambda$.

Assuming the truth of the conjecture, it follows that the immersion x is, up to a rigid motion, the one already described by the spherical harmonics of eigenvalue λ. This would give a complete description of all isometric minimal immersions of symmetric spaces of rank one into spheres.

Proposition 1 of this paper shows that the above conjecture is true for the two dimensional sphere and gives the following

Corollary of Proposition 1. *Let $x: S_r^2 \to S_1^m \subset R^{m+1}$ be an isometric minimal immersion of a 2-sphere of radius r into the unit m-sphere $S_1^m \subset R^{m+1}$ such that $x(S_r^2)$ is not contained in a hyperplane of R^{m+1}, and let $x(p) = (g_1(p), \cdots, g_{m+1}(p))$, $p \in S_r^2$. Then $g_1, \cdots, g_{m+1}$ form an orthonormal basis for the spherical harmonics of order k on S_1^2, $m = 2k$ and $r = [k(k + 1)/2]^{1/2}$.*

This result is probably already contained in [1] and, as Calabi pointed out to us, it also follows from his main theorem in [2]. In fact, it is proved in [2, p. 123] that the main theorem implies $m = 2k + 1$. Since, up to a rigid motion, any such immersion x has components $g_i = \lambda_i f_i$, $i = 1, \cdots, m + 1$, where $f_1, \cdots, f_{m+1}$ form an orthonormal basis for the spherical harmonics $V_{\lambda(k)}$ of degree k, it follows that $\sum_i \lambda_i^2 f_i^2 = \sum_i f_i^2 = 1$ and $\sum_i \lambda_i^2 df_i \cdot df_i = \sum_i df_i df$. Assume that λ_1 is the smallest of the λ_i. If $\lambda_1 < 1$, it is easily seen that the functions $c_j f_i$, $j = 2, \cdots, m + 1$, $c_j = [(\lambda_j^2 - \lambda_1^2)/(1 - \lambda_1^2)]^{1/2}$, give an isometric minimal immersion into S_1^{m-1}, which is a contradiction. Therefore $\lambda_1 \geq 1$, hence $\lambda_1 = \cdots = \lambda_{m+1} = 1$, and the functions g_i form an orthonormal basis of $V_{\lambda(k)}$.

We remark that condition A is stronger than the rigidity conjecture. Therefore Proposition 1 is not equivalent to the above corollary, and the bearing of

Theorem 1 on the present problem is to show that it is impossible to prove the rigidity conjecture for anything but the 2-sphere, relying on the constancy of the sum of the squares.

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