J. DIFFERENTIAL GEOMETRY
4 (1970) 21-35

CONTACT RIEMANNIAN SUBMANIFOLDS

MASAFUMI OKUMURA

Introduction

In a previous paper [3] the author studied a submanifold of codimension 2, which inherits a contact Riemannian structure from the enveloping contact Riemannian manifold.

In the present paper, the author generalizes the results obtained in [3] to submanifolds of codimension greater than 2. In § 1 we recall first of all the definition of contact Riemannian manifolds and some identities which hold in such manifolds, and in § 2 we give some formulas which hold for submanifolds in a Riemannian manifold. After these preliminaries, § 3 contains some identities which hold for submanifolds in a contact Riemannian manifold. In § 4 we define the notion of contact Riemannian submanifolds in the same way as given in [3]. In § 5 we define an F-invariant submanifold and study the relations between contact Riemannian submanifolds and F-invariant submanifolds.

§ 6 is devoted to a condition for a submanifold to be a contact Riemannian manifold. In the last section, § 7, we introduce the notion of normal contact submanifolds in a normal contact manifold, and obtain a condition for a contact Riemannian manifold to be a normal contact manifold.

1. Contact Riemannian manifolds

A $(2n+1)$-dimensional differentiable manifold $\tilde{M}$ is said to have a *contact structure* and called a *contact manifold* if there exists a 1-form $\tilde{\eta}$, to be called the *contact form*, on $\tilde{M}$ such that

$$\tilde{\eta} \wedge (d\tilde{\eta})^n \neq 0 \tag{1.1}$$

everywhere on $\tilde{M}$, where $d\tilde{\eta}$ is the exterior derivative of $\tilde{\eta}$, and the symbol $\wedge$ denotes the exterior multiplication.

In terms of local coordinate $\{y^\kappa\}$ of $\tilde{M}$ the contact form $\tilde{\eta}$ is expressed as

$$\tilde{\eta} = \eta_\lambda dy^\lambda . \tag{1.2}$$

Since, according to (1.1), the 2-form $d\tilde{\eta}$ is of rank $2n$ everywhere on $\tilde{M}$, we can find a unique vector field ξ^κ on $\tilde{M}$ satisfying

Communicated by K. Yano, December 11, 1968.

$$\eta_\lambda \xi^\lambda = 1 \,, \qquad (d\tilde{\eta})_{\lambda\kappa} \xi^\kappa = 0 \,. \tag{1.3}$$

It is well known that there exists a positive definite Riemannian metric $\tilde{g}_{\lambda\mu}$ such that the (1, 1)-tensor $F_\lambda{}^\kappa$, defined by

$$2\tilde{g}_{\lambda\kappa} F_\mu{}^\kappa = (d\tilde{\eta})_{\mu\lambda} \,, \tag{1.4}$$

satisfies the conditions

$$F_\lambda{}^\kappa F_\mu{}^\lambda = -\delta_\mu^\kappa + \tilde{\eta}_\mu \tilde{\xi}^\kappa \,, \tag{1.5}$$

$$\tilde{\eta}_\kappa F_\lambda{}^\kappa = 0 \,, \tag{1.6}$$

$$\tilde{g}_{\lambda\mu} \tilde{\xi}^\mu = \tilde{\eta}_\lambda \,, \tag{1.7}$$

$$\tilde{g}_{\lambda\kappa} F_\nu{}^\lambda F_\mu{}^\kappa = \tilde{g}_{\nu\mu} - \tilde{\eta}_\nu \tilde{\eta}_\mu \,. \tag{1.8}$$

(S. Sasaki [4], Y. Hatakeyama [1]). The set $(F_\lambda{}^\kappa, \tilde{\xi}^\kappa, \tilde{\eta}_\lambda, \tilde{g}_{\lambda\kappa})$ satisfying (1.1), (1.3), (1.5) and (1.7) is called a *contact Riemannian* (or *metric*) *structure*, and the manifold with such a structure is called a *contact Riemannian* (or *metric*) *manifold*.

If in a contact Riemannian manifold the tensor, defined by

$$\begin{aligned} N_{\mu\lambda}{}^\kappa = {} & F_\mu{}^\nu(\partial_\nu F_\lambda{}^\kappa - \partial_\lambda F_\nu{}^\kappa) - F_\lambda{}^\nu(\partial_\nu F_\mu{}^\kappa - \partial_\mu F_\nu{}^\kappa) \\ & + \partial_\lambda \tilde{\xi}^\kappa \tilde{\eta}_\mu - \partial_\mu \tilde{\xi}^\kappa \tilde{\eta}_\lambda \,, \end{aligned} \tag{1.9}$$

where $\partial_\nu = \partial/\partial y^\nu$ vanishes everywhere on $\tilde{M}$, then the structure is said to be *normal*, and the manifold is called a *normal contact manifold* or a *Sasakian manifold*. In a normal contact manifold we have

$$\tilde{\nabla}_\mu \tilde{\eta}_\lambda = F_{\mu\lambda} \,, \tag{1.10}$$

$$\tilde{\nabla}_\mu F_{\lambda\kappa} = \tilde{\eta}_\lambda \tilde{g}_{\mu\kappa} - \tilde{\eta}_\kappa \tilde{g}_{\mu\lambda} \,, \tag{1.11}$$

where $\tilde{\nabla}$ denotes the covariant differentiation with respect to the Riemannian metric $\tilde{g}$. Conversely, if (1.11) holds, the manifold is a normal contact manifold (Y. Hatakeyama, Y. Ogawa, and S. Tanno [2]).

2. Submanifolds in a Riemannian manifold

Let M be an m-dimensional oriented differentiable manifold and ι be an immersion of M into an $(m+k)$-dimensional oriented Riemannian manifold $\tilde{M}$. In terms of local coordinates $(x^1, \cdots, x^m)$ of M and $(y^1, \cdots, y^{m+k})$ of $\tilde{M}$ the immersion ι is locally expressed by $y^\kappa = y^\kappa(x^1, \cdots, x^m)$, $\kappa = 1, \cdots, m+k$. If we put $B_i{}^\kappa = \partial_i y^\kappa$, $\partial_i = \partial/\partial x^i$, then $B_i{}^\kappa$ are m local vector fields in M spanning the tangent space at each point of M. A Riemannian metric g on M is naturally induced from the Riemannian metric $\tilde{g}$ on $\tilde{M}$ by the immersion in such a way that

$$g_{ji} = \tilde{g}_{\lambda\kappa} B_j{}^\lambda B_i{}^\kappa . \tag{2.1}$$

Since M and $\tilde{M}$ are both orientable, in each coordinate neighborhood U of $p \in M$, we can choose k fields of mutually orthogonal unit normal vectors $N_A{}^\kappa$ $(A = 1, \cdots, k)$ of M at each point of U in such a way that $(N_1{}^\kappa, \cdots, N_k{}^\kappa, B_i{}^\kappa)$ is positively oriented in $\tilde{M}$, provided that the frame $(B_i{}^\kappa, i = 1, \cdots, m)$ is so in M.

Let H_{Aji} $(A = 1, \cdots, k)$ be the second fundamental tensors, and L_{ABi} the third fundamental tensors of the immersion ι. Then we have the following Gauss and Weingarten equations:

$$\nabla_j B_i{}^\kappa = \sum_{A=1}^{k} H_{Aji} N_A{}^\kappa , \tag{2.2}$$

$$\nabla_j N_A{}^\kappa = -H_{Aj}{}^i B_i{}^\kappa + \sum_{B=1}^{k} L_{ABj} N_B{}^\kappa , \tag{2.3}$$

where ∇_j is the so-called van der Waerden-Bortolotti covariant differentiation, where $\nabla_j B_i{}^\kappa$ and $\nabla_j N_A{}^\kappa$ are defined respectively by

$$\nabla_j B_i{}^\kappa = \partial_j B_i{}^\kappa - \left\{ {h \atop j\ i} \right\} B_h{}^\kappa + \widetilde{\left\{ {\kappa \atop \lambda\ \mu} \right\}} B_j{}^\lambda B_i{}^\mu ,$$

$$\nabla_j N_A{}^\kappa = \partial_j N_A{}^\kappa + \widetilde{\left\{ {\kappa \atop \lambda\ \mu} \right\}} B_j{}^\mu N_A{}^\lambda \qquad (A = 1, \cdots, k) ,$$

$\left\{ {i \atop j\ k} \right\}$ and $\widetilde{\left\{ {\kappa \atop \lambda\ \mu} \right\}}$ being the Christoffel's symbols of M and $\tilde{M}$ respectively.

3. Submanifolds in a contact Riemannian manifold

Let $\tilde{M}$ be a $(2n + 1)$-dimensional contact Riemannian manifold with a contact Riemannian structure $(F_\lambda{}^\kappa, \tilde{\xi}^\kappa, \tilde{\eta}_\lambda, \tilde{g}_{\lambda\kappa})$ and M a $(2m + 1)$-dimensional submanifold in $\tilde{M}$. The transform $F_\lambda{}^\kappa B_i{}^\lambda$ of the tangent vector field $B_i{}^\kappa$ by $F_\lambda{}^\kappa$ can be represented as a sum of its tangential part and its normal part, that is,

$$F_\lambda{}^\kappa B_i{}^\lambda = f_i{}^h B_h{}^\kappa + \sum_A f_{Ai} N_A{}^\kappa . \tag{3.1}$$

In the same way, we can put

$$F_\lambda{}^\kappa N_A{}^\lambda = \underset{A}{h^i} B_i{}^\kappa + \sum_B h_{AB} N_B{}^\kappa , \quad A = 1, \cdots, 2(n - m) . \tag{3.2}$$

From these two equations we have

$$h_{i \atop A} = -f_{i \atop A}\ , \tag{3.3}$$

$$h_{AB} = -h_{BA}\ . \tag{3.4}$$

On the other hand, $\tilde{\xi}^\kappa$ being tangent to $\tilde{M}$ is expressed as a linear combination of $B_i{}^\kappa$ and $N_A{}^\kappa$. Hence we can put

$$\tilde{\xi}^\kappa = u^h B_h{}^\kappa + \sum_A u_A N_A{}^\kappa\ , \tag{3.5}$$

which implies

$$u_i = \tilde{\eta}_\kappa B_i{}^\kappa\ , \tag{3.6}$$

$$u_A = \tilde{\eta}_\kappa N_A{}^\kappa\ . \tag{3.7}$$

Transforming both members of (3.1) by $F_\lambda{}^\kappa$ and making use of (1.5), (3.1), (3.2), (3.3) and (3.5), we find

$$\begin{aligned}-B_i{}^\mu + u_i u^j B_j{}^\mu + \sum_B u_i u_B N_B{}^\mu = {} & (f_i{}^h f_h{}^j + \sum_A f_{i \atop A} f^j_{\ A})B_j{}^\mu \\ & + \sum_B (f_i{}^h f_{h \atop B} + \sum_A f_{i \atop A} h_{AB}) N_B{}^\mu\ ,\end{aligned}$$

which implies

$$f_i{}^h f_h{}^j = -\delta_i^j + u_i u^j + \sum_A f_{i \atop A} f^j_{\ A}\ , \tag{3.8}$$

$$f_i{}^h f_{h \atop A} = u_A u_i - \sum_B f_{i \atop B} h_{BA}\ . \tag{3.9}$$

Transforming again both members of (3.2) by $F_\lambda{}^\kappa$ and taking account of (1.5), (3.1), (3.2), (3.3) and (3.5), we obtain

$$\begin{aligned}u_A u^j B_j{}^\mu - N_A{}^\mu + \sum_B u_A u_B N_B{}^\mu = {} & -(f^i f_i{}^j + \sum_B h_{AB} f^j_{\ B})B_j{}^\mu \\ & + \sum_B (-f^i_{\ A} f_{i \atop B} + \sum_C h_{AC} h_{CB}) N_B{}^\mu\ ,\end{aligned}$$

which implies

$$f^i_{\ A} f_i{}^j = -\sum_B h_{AB} f^j_{\ B} - u_A u^j\ , \tag{3.10}$$

$$f^i_{\ A} f_{i \atop B} = \delta_{AB} - u_A u_B + \sum_C h_{AC} h_{CB}\ . \tag{3.11}$$

On the other hand, conditions (1.6) and (1.3) can be rewritten respectively as

$$F_\lambda{}^\kappa \tilde{\xi}^\lambda = F_\lambda{}^\kappa (u^i B_i{}^\lambda + \sum_A u_A N_A{}^\lambda) = 0\ ,$$

$$\tilde{\eta}_\lambda \tilde{\xi}^\lambda = (u^i B_{i\kappa} + \sum_A u_A N_{A\kappa})(u^j B_j{}^\kappa + \sum_B u_B N_B{}^\kappa) = 1\ ,$$

from which we easily have

$$u^i f_i{}^h = \sum_A u_A f_A{}^h \,, \tag{3.12}$$

$$u^i \underset{A}{f}_i = -\sum_B u_B h_{BA} \,, \tag{3.13}$$

$$u^i u_i = 1 - \sum_A u_A{}^2 \,. \tag{3.14}$$

Let $\tilde{M}$ be a normal contact manifold. Differentiating (3.1) covariantly and making use of (1.11), (3.2) and (3.5), we obtain

$$\begin{aligned} u_i B_j{}^\kappa - g_{ji}(u^h B_h{}^\kappa + \sum_A u_A N_A{}^\kappa) + \sum_A H_{Aji}(-f_A{}^h B_h{}^\kappa + \sum_B h_{AB} N_B{}^\kappa) \\ = \nabla_j f_i{}^h B_h{}^\kappa + \sum_A (f_i{}^h H_{Ajh} N_A{}^\kappa + \nabla_j \underset{A}{f}_i N^\kappa - \underset{A}{f}_i H_{Aj}{}^h B_h{}^\kappa \\ + \sum_B \underset{B}{f}_i L_{BAj} N_A{}^\kappa) \,, \end{aligned}$$

which implies

$$\nabla_j f_{ih} = u_i g_{jh} - u_h g_{ji} - \sum_A (\underset{A}{f}_h H_{Ahi} - \underset{A}{f}_i H_{Ajh}) \,, \tag{3.15}$$

$$\nabla_j \underset{A}{f}_i = -u_A g_{ji} + \sum_B (H_{Bji} h_{BA} - \underset{B}{f}_i L_{BAj}) - f_i{}^h H_{Ajh} \,. \tag{3.16}$$

Differentiating (3.2) covariantly and making use of (1.11), (3.1), (3.2) and (3.5), we have

$$\begin{aligned} u_A B_j{}^\kappa - H_{Aj}{}^h (f_h{}^i B_i{}^\kappa + \sum_B \underset{B}{f}_h N_B{}^\kappa) + \sum_B L_{ABj}(-f_B{}^i B_i{}^\kappa + \sum_C h_{BC} N_C{}^\kappa) \\ = -\nabla_j f_A{}^h B_h{}^\kappa - \sum_B (f_A{}^i H_{Bji} - \nabla_j h_{AB}) N_B{}^\kappa \\ + \sum_B h_{AB}(-H_{Bj}{}^i B_i{}^\kappa + \sum_C L_{BCj} N_C{}^\kappa) \,, \end{aligned}$$

which implies

$$\nabla_j f_A{}^i = -u_A \delta_j^i + H_{Aj}{}^h f_h{}^i + \sum_B (h_{AB} H_{Bj}{}^i - L_{ABj} f_B{}^i) \,,$$

$$\nabla_j h_{AC} = f_A{}^i H_{Cji} - \underset{C}{f}{}^i H_{Aji} + \sum_B (L_{ABj} h_{BC} - L_{BCj} h_{AB}) \,. \tag{3.17}$$

Differentiating (3.5) covariantly and using (1.10) which holds in a normal contact manifold, we find

$$\begin{aligned} f_j{}^i B_i{}^\kappa + \sum_A \underset{A}{f}_j N_A{}^\kappa = \nabla_j u^i B_i{}^\kappa + \sum_A u^i H_{Aji} N_A{}^\kappa \\ + \sum_A \{\nabla_j u_A N_A{}^\kappa + u_A(-H_{Aj}{}^i B_i{}^\kappa + \sum_B L_{BAj} N_B{}^\kappa)\} \,, \end{aligned}$$

which implies

$$\nabla_j u^i = f_j{}^i + \sum_A u_A H_{Aj}{}^i , \tag{3.18}$$

$$\nabla_j u_A = \underset{A}{f_j} - u^i H_{Aji} - \sum_A u_B L_{BAj} . \tag{3.19}$$

4. Contact Riemannian submanifolds

Let $\tilde{M}$ be a $(2n+1)$-dimensional contact Riemannian manifold, and M a $(2m+1)$-dimensional orientable differentiable submanifold in $\tilde{M}$. We define a 1-form u on M by

$$u = u_i dx^i = \tilde{\eta}_\lambda B_i{}^\lambda dx^i , \tag{4.1}$$

in terms of the contact form $\tilde{\eta} = \tilde{\eta}_\lambda dy^\lambda$.

Definition 4.1. Let g_{ji} be the induced Riemannian metric of M, and u the 1-form defined by (4.1). If there exists a pair of positive constants t and c such that $\eta = tu$ and $G_{ji} = cg_{ji}$ constitute a contact Riemannian structure on M, then we call the submanifold M a *contact Riemannian submanifold* of $\tilde{M}$.

Since (η, G) is a contact metric structure in a contact Riemannian submanifold M, the linear mapping $\phi_j{}^i: T(M) \to T(M)$ and the vector field ξ^i defined respectively by

$$2\phi_j{}^h G_{hi} = \partial_j \eta_i - \partial_i \eta_j , \quad \eta_i = G_{ji}\xi^j \tag{4.2}$$

satisfy the conditions

$$\eta_i \xi^i = 1 , \tag{4.3}$$

$$\phi_j{}^i \xi^j = 0 , \qquad \eta_i \phi_j{}^i = 0 , \tag{4.4}$$

$$\phi_j{}^h \phi_h{}^i = -\delta_j^i + \eta_j \xi^i . \tag{4.5}$$

Directly from Definition 4.1 we have

Proposition 4.2. *Let M be a contact Riemannian submanifold in $\tilde{M}$, and $'M$ a contact Riemannian submanifold in M. Then $'M$ is a contact Riemannian submanifold in $\tilde{M}$.*

Proposition 4.3. *Let M be a contact Riemannian submanifold of $\tilde{M}$, and $'M$ a submanifold of M. If $'M$ is a contact Riemannian submanifold of $\tilde{M}$, then $'M$ is also a contact Riemannian submanifold of M.*

Proposition 4.4. *Let M be a contact Riemannian submanifold of a contact Riemannian manifold $\tilde{M}$. If the dimension of M is greater than the codimension of M in $\tilde{M}$, then we have*

$$\phi_j{}^i = f_j{}^i , \tag{4.6}$$

$$u^i = \xi^i . \tag{4.7}$$

Proof. From the definitions of ξ^i, η_i, G_{ji} we have

$$\xi^j = G^{ji}\eta_i = \frac{t}{c}g^{ji}u_i = \frac{t}{c}u^j , \tag{4.8}$$

from which

$$1 = \eta_j\xi^j = tu_j\frac{t}{c}u^j = \frac{t^2}{c}u_iu^i , \tag{4.9}$$

$$u_iu^i = c/t^2 . \tag{4.10}$$

On the other hand, the two equations

$$2f_{ji} = 2B_j{}^\lambda B_i{}^\kappa F_{\lambda\kappa} = B_j{}^\lambda B_i{}^\kappa(\tilde{\nabla}_\lambda\tilde{\eta}_\mu - \tilde{\nabla}_\mu\tilde{\eta}_\lambda) = \nabla_ju_i - \nabla_iu_j ,$$
$$2\phi_{ji} = \partial_j\eta_i - \partial_i\eta_j = t(\nabla_ju_i - \nabla_iu_j)$$

imply $f_{ji} = (1/t)\phi_{ji}$ and hence

$$f_j{}^h = g^{hi}f_{ji} = \frac{c}{t}G^{hi}\phi_{ji} = \frac{c}{t}\phi_j{}^h . \tag{4.11}$$

Since $f_j{}^h$, $\phi_j{}^h$ satisfy (3.8) and (4.5) respectively, (4.11) together with (4.10) implies

$$-\delta_j^h + u^hu_j + \sum_A \underset{A}{f^h}\underset{A}{f_j} = \frac{c^2}{t^2}\left(-\delta_j^h + \frac{t^2}{c}u_ju^h\right) . \tag{4.12}$$

We assume now that there is a point p in M, at which the $2(n-m)+1$ vectors u^i, $f_A{}^i$ $(A = 1, \cdots, 2(n-m))$ are linearly dependent. Then we can find a vector $v^i(p)$ orthogonal to the subspace spanned by u^i and $f_A{}^i$ $(A = 1, \cdots, 2(n-m))$, since M is of dimension greater than $2(n-m)$. Transforming this vector $v^i(p)$ by (4.12), we get $v^h(p) = (c/t)^2v^h(p)$, that is, $(c/t)^2 = 1$, which together with (4.8) and (4.11) implies the Proposition.

Next we suppose that u^i and $f_A{}^i$ $(A = 1, \cdots, 2(n-m))$ are linearly independent at any point of M. Then (3.12), (4.4) and (4.8) imply $\sum_A u_Af_A{}^h = f_j{}^hu^j = (c/t)\phi_j{}^h(c/t)\xi^j = 0$. Since $f_A{}^h$'s are linearly independent, we have, in this case,

$$u_A = 0 \qquad (A = 1, \cdots, 2(n-m)) , \tag{4.13}$$

which and (3.1) give

$$u^i\underset{A}{f_i} = 0 . \tag{4.14}$$

Transforming $f_A{}^j$ by (4.12), we have

$$-\underset{A}{f^h} + \sum \underset{B}{f_j} \underset{B}{f^j} \underset{A}{f^h}{}_{B} = -\frac{c^2}{t^2} \underset{A}{f^h} \tag{4.15}$$

because of (4.14). Substituting (3.11) into (4.15) we get $\sum_{B,C} h_{AC} h_{CB} \underset{B}{f^h} = -(c/t)^2 \underset{A}{f^h}$ implying

$$\sum_C h_{AC} h_{CB} = -\frac{c^2}{t^2} \delta_{BA} , \tag{4.16}$$

and consequently

$$\sum_{A,C} h_{AC} h_{CA} = -\frac{c^2}{t^2} \sum_A \delta_{AA} = -2(n-m)\frac{c^2}{t^2} . \tag{4.17}$$

Furthermore, from (4.11) we obtain $(c/t)^2 \phi_i{}^h \phi_h{}^j = -\delta_i^j + u_i u^j + \sum_A \underset{A}{f_i} \underset{A}{f^j}$, which yields

$$-2m\frac{c^2}{t^2} = -2m - 1 + u_i u^i + 2(n-m) + \sum_{A,C} h_{AC} h_{CA}$$

because of (3.11). On the other hand, $u_A = 0$ and (3.14) imply $u_i u^i = 1$. Thus we have, from the equation obtained above,

$$-2m\frac{c^2}{t^2} = 2(n-2m) + \sum_{A,C} h_{AC} h_{CA} . \tag{4.18}$$

Combining (4.17) and (4.18), we have $(t/c)^2 = 1$, which completes the proof.

Corollary 4.5. $G_{ji} = (u_r u^r)^{-1} g_{ji}$, $\eta_i = (u_r u^r)^{-1} u_i$.

5. *F*-invariant submanifolds

F-invariant submanifolds of a contact Riemannian manifold are recently studied in [5]. In this section we show that any *F*-invariant submanifold is a contact Riemannian submanifold.

Definition 5.1. Let $\tilde{M}$ be a $(2n+1)$-dimensional contact Riemannian manifold. A $(2m+1)$-dimensional submanifold M of $\tilde{M}$ is called an *F-invariant submanifold* if the tangent space of M is invariant under the action of $F_\lambda{}^\kappa$.

Proposition 5.2. *Let M be a $(2m+1)$-dimensional submanifold of a contact Riemannian manifold $\tilde{M}$. In order that M be an F-invariant submanifold it is necessary and sufficient that*

$$\sum_C h_{AC} h_{CB} = -\delta_{AB} . \tag{5.1}$$

Proof. We first assume M to be F-invariant, and then by (3.1) show that

$$F_\lambda{}^\kappa B_i{}^\lambda = f_i{}^h B_h{}^\kappa , \qquad F_\lambda{}^\kappa N_A{}^\lambda = \sum_B h_{BA} N_B{}^\kappa ,$$

or equivalently $f_A{}^i = 0$ $(A = 1, \cdots, 2(n-m))$. Consequently, we have $u_i u_A = 0$ because of (3.9). If there is a point p on M, where $u_i(p) = 0$, then (3.8) implies $f_j{}^i f_i{}^h = -\delta_j^h$, which means that the tangent space at p is even-dimensional, contradicting our assumption. Hence we have $u_A = 0$ in M. Therefore we have $\sum_C h_{AC} h_{CB} = -\delta_{AB}$ by virtue of (3.11). Next, we assume that M is a submanifold of $\tilde{M}$ satisfying the condition (5.1). Then, by means of (3.11), we have $f_A{}^i f_{Bi} + u_A u_B = 0$, and therefore $\sum_A f_A{}^i f_{Ai} + u_A{}^2 = 0$. Thus we get $f_A{}^i = 0$, $u_A = 0$, which show that M is F-invariant.

Proposition 5.3. *If M is a $(2m+1)$-dimensional F-invariant submanifold of $\tilde{M}$. Then M is necessarily a contact Riemannian submanifold of $\tilde{M}$.*

Proof. Since M is F-invariant, as seen in the proof of Proposition 5.2 we have $f_A{}^i = 0$, $u_A = 0$ $(A = 1, \cdots, 2(n-m))$. Therefore, (3.8) and (3.14) imply $f_i{}^h f_h{}^j = -\delta_i^j + u_i u^j$, $u_i u^i = 1$. If we now put $\eta = u$, $G_{ji} = g_{ji}$ then we find

$$\begin{aligned} \nabla_j \eta_i - \nabla_i \eta_j &= \nabla_j u_i - \nabla_i u_j = \nabla_j(\tilde{\eta}_\kappa B_i{}^\kappa) - \nabla_i(\tilde{\eta}_\kappa B_j{}^\kappa) \\ &= B_i{}^\kappa B_j{}^\lambda \tilde{\nabla}_\lambda \tilde{\eta}_\kappa - B_i{}^\lambda B_j{}^\kappa \tilde{\nabla}_\lambda \tilde{\eta}_\kappa + \sum_A (H_{Aji} N_A{}^\kappa - H_{Aij} N_A{}^\kappa)\tilde{\eta}_\kappa \\ &= B_i{}^\kappa B_j{}^\lambda (\tilde{\nabla}_\lambda \tilde{\eta}_\kappa - \tilde{\nabla}_\kappa \tilde{\eta}_\lambda) = 2 f_{ji} , \end{aligned}$$

which means that the (η, G) is a contact Riemannian structure on M. Thus the proof is complete.

6. Conditions for a submanifold to be a contact Riemannian submanifold

In this section we states a condition for a submanifold M in a contact Riemannian manifold $\tilde{M}$ to be a contact Riemannian submanifold. Since for this purpose we have to use Proposition 4.4 so that we always assume in this section that the dimension of M is greater than the codimension of M in $\tilde{M}$. First we have

Proposition 6.1. *Let $\tilde{M}$ be a $(2n+1)$-dimensional contact Riemannian manifold. In order that a submanifold M in $\tilde{M}$ be a contact Riemannian submanifold it is necessary and sufficient that the relations*

$$u_r u^r = const. \neq 0 , \tag{6.1}$$

$$f_j{}^i f_h{}^j = -\delta_h^i + (u_r u^r)^{-1} u_h u^i \tag{6.2}$$

be both valid.

Proof. Let M be a contact Riemannian submanifold of $\tilde{M}$. Then from Proposition 4.4 it follows that $f_j{}^i = \phi_j{}^i$ and consequently

$$f_i{}^h f_h{}^j = \phi_i{}^h \phi_h{}^j = -\delta_i^j + \eta_i \xi^j = -\delta_i^j + t u_i u^j . \tag{6.3}$$

On the other hand, we have $\eta_i \xi^i = t u_i \xi^i = t u_i u^i = 1$, which implies

$$u_i u^i = \frac{1}{t} = \text{const.} \,. \tag{6.4}$$

Combining (6.3) and (6.4), we get (6.1) and (6.2).

Conversely, if (6.1) and (6.2) are both valid, putting

$$\eta_i = (u_r u^r)^{-1} u_i \,, \qquad G_{ji} = (u_r u^r)^{-1} g_{ji} \,,$$

we have

$$\begin{aligned} \eta_i \xi^i &= (u_r u^r)^{-1} u_i G^{ik} \eta_k = (u_r u^r)^{-1} u_i u^i = 1 \,, \\ f_i{}^j f_h{}^i &= -\delta_h^j + (u_r u^r)^{-1} u_h u^j = -\delta_h^j + \eta_h \xi^j \,. \end{aligned}$$

Thus $(f_j{}^i, \eta_i, G^{ji}\eta_j, G_{ji})$ is an almost contact Riemannian structure on M. By virtue of (6.1) and (1.4) we now have

$$\begin{aligned} \nabla_j \eta_i - \nabla_i \eta_j &= (u_r u^r)^{-1} (\nabla_j u_i - \nabla_i u_j) \\ &= (u_r u^r)^{-1} (\nabla_j (B_i{}^\lambda \tilde{\eta}_\lambda) - \nabla_i (B_j{}^\lambda \tilde{\eta}_\lambda)) \\ &= (u_r u^r)^{-1} (B_i{}^\lambda B_j{}^\mu \tilde{\nabla}_\mu \tilde{\eta}_\lambda - B_i{}^\mu B_j{}^\lambda \tilde{\nabla}_\mu \tilde{\eta}_\lambda + \sum_A (H_{Aji} - H_{Aij}) N_A{}^\kappa \tilde{\eta}_\kappa) \\ &= (u_r u^r)^{-1} B_i{}^\lambda B_j{}^\mu (\tilde{\nabla}_\mu \tilde{\eta}_\lambda - \tilde{\nabla}_\lambda \tilde{\eta}_\mu) = 2(u_r u^r)^{-1} B_j{}^\mu B_i{}^\lambda F_{\mu\lambda} \\ &= 2(u_r u^r)^{-1} f_{ji} = 2 G_{ih} f_j{}^h \,, \end{aligned}$$

which shows that (η, G) is a contact Riemannian structure on M.

Proposition 6.2. *Let $\tilde{M}$ be a contact Riemannian manifold. In order that a submanifold M in $\tilde{M}$ be a contact Riemannian submanfold, it is necessary and sufficient that the following relations be both valid*:

$$u_r u^r = const. \,, \tag{6.5}$$

$$f_A{}^i = -(u_r u^r)^{-1} \sum_B u_B h_{BA} u^i \,. \tag{6.6}$$

Proof. Let M be a contact Riemannian submanifold in $\tilde{M}$. Then from Proposition 6.1, we have (6.5). On putting

$$f_A{}^i = P_A u^i + P_A{}^i \qquad (A = 1, \cdots, 2(n-m)) \,, \tag{6.7}$$

where $P_A{}^i$ are vectors orthogonal to u^i, if we transvect (6.7) with u_i, we get $f_A{}^i u_i = u_i u^i P_A$, which together with (3.13) implies

$$P_A = (u_r u^r)^{-1} f_A{}^i u_i = -(u_r u^r)^{-1} \sum_B u_B h_{BA} \,. \tag{6.8}$$

Substituting (6.8) into (6.7), we have

$$f_A{}^i = -(u_ru^r)^{-1} \sum_B u_B h_{BA} u^i + P_A{}^i , \tag{6.9}$$

which implies $f_A{}^i f_{Bi} = (u_ru^r)^{-1} \sum_{C,D} u_D h_{DA} u_C h_{CB} + P_A{}^i P_{Bi}$ and consequently

$$\sum_A f_A{}^i f_{Ai} = (u_ru^r)^{-1} \sum_{A,B,C} u_B h_{BA} u_C h_{CA} + \sum_A P_A{}^i P_{Ai} . \tag{6.10}$$

On the other hand, since M is a contact Riemannian submanifold, from (3.9) we have $u^i f_i{}^h f_{Ah} = (u_iu^i)u_A - \sum_B f_{Bi} u^i h_{BA} = 0$. Substituting (3.13) into the above equation, we get

$$(u_iu^i)u_A = - \sum_{B,C} u_C h_{CB} h_{BA} . \tag{6.11}$$

Then a combination of (6.10) and (6.11) gives

$$\sum_A f_A{}^i f_{Ai} = \sum_A (u_A{}^2 + P_A{}^i P_{Ai}) . \tag{6.12}$$

However, by virtue of (3.8) we obtain $\sum_A f_A{}^i f_{Ai} = f_{ji} f^{ij} + 2m + 1 - u_iu^i$, which reduces to

$$\sum_A f_A{}^i f_{Ai} = 1 - u_iu^i = \sum_A u_A{}^2 \tag{6.13}$$

because of (3.14) since M is a contact Riemannian submanifold. Comparing (6.12) with (6.13), we have $\sum_A P_A{}^i P_{Ai} = 0$, that is, $P_A{}^i = 0$ $(A = 1, \cdots, 2(n - m))$. Hence we obtain (6.6).

Conversely, if the submanifold satisfies (6.5) and (6.6), according to (3.8) we get

$$\begin{aligned} f_i{}^h f_h{}^j &= -\delta_i^j + u_iu^j + \sum_A f_{Ai} f_A{}^j \\ &= -\delta_i^j + u_iu^j + (u_ru^r)^{-2} \sum_{A,B,C} u_B h_{BA} u_C h_{CA} u_i u^j . \end{aligned} \tag{6.14}$$

Since f_{ji} is skew symmetric, the condition (6.6) implies $f_i{}^h u^i f_{Ah} = (u_iu^i)u_A - \sum_B f_{Bi} u^i h_{BA} = 0$ because of (3.9). Substituting (3.13) into the above equation, we get

$$\sum_{B,C} u_C h_{CB} h_{BA} = -(u_iu^i)u_A . \tag{6.15}$$

Therefore (6.14) reduces to

$$\begin{aligned} f_i{}^h f_h{}^j &= -\delta_i^j + u_iu^j + (u_ru^r)^{-1} \sum_B u_B{}^2 u_i u^j \\ &= -\delta_i^j + (u_ru^r)^{-1}(u_ru^r + \sum_B u_B{}^2) u_i u^j \\ &= -\delta_i^j + (u_ru^r)^{-1} u_i u^j . \end{aligned}$$

Thus the conditions stated in Proposition 6.1 are satisfied, and the proof is complete.

7. Contact Riemannian submanifolds in a normal contact manifold

Let $\tilde{M}$ be a normal contact manifold. In this section we define the notion of a normal contact submanifold M in $\tilde{M}$. After deriving a condition for M to be a normal contact submanifold in $\tilde{M}$, we show that any $(2m+1)$-dimensional F-invariant submanifold M in $\tilde{M}$ is a normal contact submanifold.

Definition 7.1. Let $\tilde{M}$ be a normal contact manifold, and M a contact Riemannian submanifold in $\tilde{M}$. If the induced contact structure of M in $\tilde{M}$ is normal, the submanifold M is called a *normal contact submanifold.*

Proposition 7.2. *Let M be a normal contact submanifold in $\tilde{M}$, and $'M$ a normal contact submanifold in M. Then $'M$ is a normal contact submanifold in $\tilde{M}$.*

Proof. Since M and $'M$ are normal contact submanifolds respectively in $\tilde{M}$ and M, there exist two pairs of positive constants (t, c) and (t', c'). Then, as we have seen in § 4, $'M$ becomes a contact Riemannian submanifold in $\tilde{M}$ with respect to the pair $(t't, c'c)$. We denote these contact metric structures on M in $\tilde{M}$ and on $'M$ in M respectively by (η_i, G_{ji}) and (η_a, G_{ba}), and denote the contact metric structure on $'M$ in $\tilde{M}$ by $('\eta_a, 'G_{ba})$. Then we have

$$\begin{aligned} 2'\phi_{ba} &= \partial_b{}'\eta_a - \partial_a{}'\eta_b = tt'B_b{}^\lambda B_a{}^\kappa(\partial_\lambda\tilde{\eta}_\kappa - \partial_\kappa\tilde{\eta}_\lambda) \\ &= tt'B_b{}^jB_j{}^\lambda B_a{}^iB_i{}^\kappa(\partial_\lambda\tilde{\eta}_\kappa - \partial_\kappa\tilde{\eta}_\lambda) = tt'B_b{}^jB_a{}^i(\partial_ju_i - \partial_iu_j) \\ &= t'B_b{}^jB_a{}^i(\partial_j\eta_i - \partial_i\eta_j) = \partial_b\eta_a - \partial_a\eta_b = 2\phi_{ba}\ , \end{aligned}$$

and therefore

$$\begin{aligned} '\nabla_c{}'\phi_{ba} &= \nabla_c\phi_{ba} = \eta_bG_{ca} - \eta_aG_{cb} \\ &= t'c'(\eta_iB_b{}^iG_{jh}B_c{}^jB_a{}^h - \eta_hB_a{}^hG_{ji}B_c{}^jB_b{}^i) \\ &= tt'cc'(\tilde{\eta}_\lambda B_b{}^\lambda\tilde{g}_{\mu\kappa}B_c{}^\mu B_a{}^\kappa - \tilde{\eta}_\kappa B_a{}^\kappa\tilde{g}_{\mu\lambda}B_c{}^\mu B_b{}^\lambda) \\ &= '\eta_b{}'G_{ca} - '\eta_a{}'G_{ba}\ , \end{aligned}$$

which proves by virtue of (1.11) that the structure $('\eta_b, 'G_{ab})$ is normal.

Proposition 7.3. *Let M be a contact Riemannian submanifold of a normal contact manifold $\tilde{M}$, and suppose that the dimension of M is greater than the codimension of M in $\tilde{M}$. In order that M be a normal contact submanifold in $\tilde{M}$ it is necessary and sufficient that*

$$\sum_A P_AH_{Aji} = Hg_{ji} + Ku_ju_i \tag{7.1}$$

hold, where

(7.2) $$P_A = -(u_r u^r)^{-1} \sum_B u_B h_{BA} ,$$

and H and K are suitable scalar functions defined on M.

Remark. As it is easily checked, the left hand member of (7.1) is independent of the choice of the unit normal vectors to M.

Proof of Proposition 7.3. Let M be a normal contact submanifold in $\tilde{M}$. Then by the definition of normality we have

$$N_{ji}{}^h = f_j{}^r(\nabla_r f_i{}^h - \nabla_i f_r{}^h) - f_i{}^r(\nabla_r f_j{}^h - \nabla_j f_r{}^h) + \eta_j \nabla_i u^h - \eta_i \nabla_j u^h = 0$$

because of Proposition 4.4. Substituting (3.15) and (3.18) into the above equation and taking account of (4.4), (6.6), Proposition 4.4 and Corollary 4.5, we find

(7.3) $$\begin{aligned} N_{ji}{}^h &= f_j{}^r u_i(\delta_r^h + \sum_A P_A H_A{}^h{}_r) - f_i{}^r u_j(\delta_r^h + \sum_A P_A H_A{}^h{}_r) \\ &\quad + (u_r u^r)^{-1}\{(f_i^h + \sum_A u_A H_A{}^h{}_i)u_j - (f_j^h + \sum_A u_A H_A{}^h{}_j)u_i\} = 0 . \end{aligned}$$

On the other hand, we know that the vector field ξ^i is a Killing vector field if the contact Riemannian structure is normal. Thus, from (3.18) and (4.7), we have

(7.4) $$\sum_A u_A H_{Aji} = 0 .$$

Substituting (7.4) into (7.3) and taking account of (3.14), we obtain

$$N_{ji}{}^h = \{\sum_A \underset{A}{P} H_{Ar}{}^h - (u_r u^r)^{-1} \sum_A u_A{}^2 \delta_r^h\}(f_j{}^r u_i - f_i{}^r u_j) = 0 ,$$

and therefore $\sum_A P_A H_{Aji} = (u_r u^r)^{-1} \sum_A u_A^2 g_{ji} + K u_j u_i$, which proves the necessity of the given condition.

Conversely, suppose that in a contact Riemannian submanifold M in $\tilde{M}$ the condition (7.1) holds. Differentiating

(7.5) $$\underset{A}{f_i} = \underset{A}{P} u_i$$

covariantly, we get $\nabla_j f_{Ai} = \nabla_j P_A u_i + P_A \nabla_j u_i$. Substituting (3.16) and (3.18) into the above equation, we find

$$\begin{aligned} - u_A g_{ji} + \sum_B (H_{Bji} h_{BA} - \underset{B}{f_i} L_{BAj}) - f_i{}^h H_{Ajh} \\ = \nabla_j P_A u_i + \underset{A}{P}(f_{ji} + \sum_B u_B H_{Bji}) , \end{aligned}$$

which together with (7.5) implies

$$- \sum_A P_A u_A g_{ji} + \sum_{B,A} (H_{Bji} h_{BA} P_A - \underset{B\,A}{PPu_i L_{BAj}}) - \sum_A P_A H_{Ajh} f_i{}^h$$
$$= \sum_A u_i P_A \nabla_j P_A + \sum_A \underset{A}{P^2}(f_{ji} + \sum_B u_B H_{Bji}) \, .$$

Transvecting this with f^{ji} and making use of (7.1), we get $- f^{ji} f_i{}^h (H g_{jh} + K u_j u_h) = 2m \sum_A P_A{}^2$ from which $H = \sum_A P_A{}^2$. Therefore (7.1) reduces to

$$\sum_A P_A H_{Aji} = \sum_A P_A^2 g_{ji} + K u_j u_i \, . \tag{7.6}$$

Substituting (7.6) into the left hand member of (7.3), we find

$$\begin{aligned} N_{ji}{}^h &= (f_j{}^h u_i - f_i{}^h u_j)(1 + \sum_A \underset{A}{P^2} - (u_r u^r)^{-1} \\ &= (u_r u^r)^{-1} (u_r u^r + u_r u^r \sum_A P_A^2 - 1)(f_j{}^h u_i - f_i{}^h u_j) \, . \end{aligned} \tag{7.7}$$

On the other hand, (7.2) and (6.11) imply

$$\sum_A P_A{}^2 = (u_r u^r)^{-2} \sum_{B,C} u_B h_{BA} u_C h_{CA} = (u_r u^r)^{-1} \sum_C u_C^2 \, .$$

Thus, from (3.14) and (7.7) it follows that $N_{ji}{}^h = 0$, which completes the proof of the sufficiency.

Corollary 7.4. *Let M be a contact Riemannian submanifold in a normal contact manifold $\tilde{M}$. If M is a totally geodesic or a totally umbilical submanifold in $\tilde{M}$, then M is a normal contact submanifold.*

As we have mentioned in the previous paper [3], every totally umbilical submanifold M in a normal contact manifold $\tilde{M}$ is not a normal contact submanifold. In [3] we have proved that a normal contact submanifold of codimension 2 in a normal contact manifold of constant curvature is either an F-invariant submanifold or a totally umbilical submanifold. However, if the codimension is greater than 2 we cannot prove this fact, because by Proposition 7.2, for example, an F-invariant submanifold $'M$ in a totally umbilical submanifold M in $\tilde{M}$ is also a normal contact submanifold in $\tilde{M}$. In general, a normal contact submanifold in a normal contact manifold is neither F-invariant nor totally umbilical.

Proposition 7.5. *An F-invariant submanifold in a normal contact manifold is a normal contact submanifold.*

Proof. Since the submanifold is F-invariant, it follows that $f_A{}^i = 0$, $u_A = 0$ $(A = 1, \cdots, 2(n-m))$. Consequently we have $u_i u^i = 1$ because of (3.14). Substituting these into the left hand member of (7.3), we find

$$N_{ji}{}^h = (1 - (u_r u^r)^{-1})(f_j{}^h u_i - f_i{}^h u_j) = 0 \, ,$$

which completes the proof.

Bibliography

[1] Y. Hatakeyama, *On the existence of Riemann metrics associated with a 2-form of rank* 2r, Tôhoku Math. J. **14** (1962) 162–166.
[2] Y. Hatakeyama, Y. Ogawa & S. Tanno, *Some properties of manifolds with contact metric structure,* Tôhoku Math. J. **15** (1962) 42–48.
[3] M. Okumura, *On contact metric immersion,* Kōdai Math. Sem. Rep. **20** (1968) 389–409.
[4] S. Sasaki, *On differentiable manifolds with certain structures which are closely related to almost contact structure.* I, Tôhoku Math. J. **12** (1960) 459–476.
[5] K. Yano & S. Ishihara, *Invariant submanifold of almost contact manifold,* Kōdai Math. Sem. Rep. **21** (1969) 350–364.

Saitama University, Japan