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CRITICAL POINTS OF THE DISPLACEMENT FUNCTION OF AN ISOMETRY

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Introduction

Given a Riemannian manifold M and a group of isometries of M it is natural to study the fixed point set of this group. This problem was considered by S. Kobayashi in [9], [10], and by R. Bott in [2], in the case where the group is a 1-parameter group of isometries. In [4], Kobayashi shows that if $\{g_t\}$ is such a group, then the fixed point set of $\{g_t\}$ is a totally geodesic submanifold of even codimension. In fact, his proof shows that the fixed point set of any group of isometries is a totally geodesic submanifold. The fixed point set of the 1-parameter group $\{g_t\}$ is just the set of zeros of the associated Killing vector field X, and in [7] and [8] R. Hermann considers the more general problem of the critical points of the function $|X|^2$ giving the square of the length of X. He shows that these critical points are exactly the points lying on geodesic orbits of $\{g_t\}$. Moreover, he shows that if M has curvature $K \leq 0$, then the set of critical points of $|X^2|$ is convex (that is, any geodesic segment between two critical points lies in the critical set).

We consider the still more general situation of a single isometry f, and look at the critical point set Crit (f) of the function δ_f^2, where $\delta_f(x) =$ distance $(x, f(x))$. It is evident that Crit (f) contains the fixed points of f.

In Chapter I we let M be any Riemannian manifold and $f: M \to M$ an isometry whose displacement δ_f is small enough so that f takes each point into the complement of its cut locus. We say such an isometry has "small displacement." The main theorems are:

(1.2.1) Theorem. *Let $f: M \to M$ be an isometry of small displacement and $x \in M$. Then $x \in$ Crit (f) if and only if f preserves the unique minimizing geodesic between x and $f(x)$.*

(1.3.4) Theorem. *Let M have curvature $K \leq 0$, and assume $f: M \to M$ is an isometry of small displacement. Then*

(i) Crit (f) *is a totally geodesic submanifold possibly with boundary,*

(ii) δ_f *takes its absolute minimum on* Crit (f).

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(iii) *If* $\operatorname{Fix}(f) = \emptyset$ *then* $\operatorname{Crit}(f)$ *is connected*; *if* $\operatorname{Fix}(f) \neq \emptyset$ *then* $\operatorname{Crit}(f) = \operatorname{Fix}(f)$.

(iv) *If* M *is simply connected then* $\operatorname{Fix}(f)$ *is connected.*

(v) *If* $K < 0$ *and* $\operatorname{Fix}(f) = \emptyset$, *then* $\operatorname{Crit}(f)$ *is either empty or consists of a single geodesic.*

Moreover, we show that if $f \in I^0(M) =$ identity component of the isometry group of M, and $f = f_1$, where $\{f_t\}$ is a 1-parameter group of isometries with associated Killing vector field X, then $\operatorname{Crit}(|X|^2) = \bigcap_{n=1}^{\infty} \operatorname{Crit}(f_{1/n!})$, so that our results in a sense generalize those of R. Hermann in [8].

In Chapter II we restrict to Riemannian homogeneous spaces and principally to symmetric spaces. The main theorem is:

(2.7.1) Theorem. *Let* M *be a simply connected Riemannian symmetric space with* $M = M_0 \times M_1 \times \cdots M_k$, *where* M_0 *is a Euclidean space and the* M_i, $1 \leq i \leq k$, *are irreducible. If* $g \in I^0(M) = I^0(M_0) \times \cdots \times I^0(M_k)$, *and the components* g_i *of* g *which act on the compact* M_i *are sufficiently close to the identity, then the components of* $\operatorname{Crit}(g)$ *are the orbits* $Z^0_{I^0(M)}(g) \cdot x$, *where* x *is any point in the component, and* $Z^0_{I^0(M)}(g)$ *is the identity component of the centralizer of* g *in* $I^0(M)$. (*Here* $I^0(M) =$ *identity component of the isometry group of* M).

If the isometry g is sufficiently near the identity, it lies on a unique 1-parameter group $\{g_t\}$ of isometries, with associated Killing vector field X. If M is symmetric, we show that $\operatorname{Crit}(|X|^2) = \operatorname{Crit}(g_t)$ for any $t \in (0, 1]$. We then obtain an explicit formula for the Hessian of the function $|X|^2$, and show that $\operatorname{Crit}(|X|^2)$ is a non-degenerate critical sub-manifold in the sense of R. Bott [1] if M is either of non-compact type, or if M is of compact type and X is a regular element of the Lie algebra of the isometry group.

Notation. We adopt the notation used in the book of Kobayashi-Nomizu [11] for Riemannian manifolds, and refer to the books of S. Helgason [6] and J. A. Wolf [15] for the basic facts about symmetric spaces and Lie groups. In a homogeneous space $M = G/K$ we assume we have a fixed direct sum decomposition $\boldsymbol{G} = \boldsymbol{K} + \boldsymbol{m}$, where $\boldsymbol{G}$ is the Lie algebra of G, $\boldsymbol{K}$ the Lie algebra of K, and $\boldsymbol{m}$ a complementary subspace satisfying $\operatorname{ad}(K)\boldsymbol{m} \subset \boldsymbol{m}$. This is a *reductive homogeneous space*. We assume M has an invariant Riemannian metric B^*, and let B be its restriction to $\boldsymbol{m} \times \boldsymbol{m}$, where $\boldsymbol{m}$ is naturally identified with the tangent space of M at K. Then we say B^* is a *normal metric* if $B([X, Z]_{\boldsymbol{m}}, Y) + B(X, [Y, Z]_{\boldsymbol{m}}) = 0$ for $X, Y, Z \in \boldsymbol{m}$. A normal metric induces a Riemannian connection of type (A1) in the notation of Nomizu [12], and this connection is characterized by the fact that its geodesics are the translates $gx(s)$, where $x(s) = (\exp sT) \cdot T$ and $g \in G$, $T \in \boldsymbol{m}$.

Chapter I. The general case

(1.1) M will always be a complete connected Riemannian manifold with metric g and Riemannian connection ∇. Let ρ be the distance on M induced by g and defined by: $\rho(x, y) = \inf_p \{\text{length } p \mid p \text{ is a piecewise smooth path from } x \text{ to } y\}$. "Smooth" and "differentiable" will always mean C^∞, and $T(M)$ denotes the tangent bundle of M. Because of completeness, $\exp: T(M) \to M$ is defined, surjective, and smooth. If $f: M \to N$ is a smooth map, $f_*: T(M) \to T(N)$ is the induced map on the tangent bundles. For every smooth map $f: M \to M$ we define the displacement function $\delta_f: M \to R$ ($=$ real numbers) by $\delta_f(x) = \rho(x, f(x))$.

(1.1.1) Definition. We say the map $f: M \to M$ has *small displacement* if for each $x \in M$ there is a unique minimizing geodesic from x to $f(x)$. Equivalently, f has small displacement if it takes each point into the complement of its cut locus. If $f: M \to M$ is a diffeomorphism of small displacement we define its *displacement vector field* V by: if $x \in M$ then V_x is the tangent at x to the minimizing geodesic from x to $f(x)$, with $|V_x| = g(V_x, V_x)^{1/2} = \rho(x, f(x))$.

(1.1.2) Lemma. *Let $f: M \to M$ be a diffeomorphism of small displacement. Then*:

(i) *the function $\delta_f^2: M \to R$ is smooth on M,*

(ii) *$\delta_f: M \to R$ is smooth outside the fixed point set of f,*

(iii) *the displacement vector field V is a smooth vector field on M.*

Proof. Fix $x \in M$, and let $U = M -$ (cut locus of x). U is an open cell in M, and there is a neighborhood $U'_x \subset T_x(m)$ such that $\exp: U'_x \to U$ is a diffeomorphism onto U. There is a neighborhood $W_1 \subset U$ containing x such that $f(W_1) \subset U$; and for each $y \in W_1$ there is an open set $N_y \subset T_y(M)$ such that $\exp: N_y \to U$ is a diffeomorphism into U. We assume $N_x = U'_x$, and we may choose the sets N_y so that $W = \bigcup_{W_1} N_y$ is open in $T(M)$. Then the map $h: W \to U \times U$ sending $Y \in N_y$ to $(y, \exp Y)$ is a diffeomorphism into $U \times U$. Since $N_x = U'_x$, we have $\{x\} \times U \subset h(W)$. The map $U \times U \to R$ given by $(y, z) \to \rho(y, z)$ coincides with $\| Y \|$ if $z = \exp Y$ and $Y \in N_y$. Now $\| Y \|^2$ is differentiable on W, so $\rho^2(y, z)$ is differentiable on $h(W)$.

Now by the assumption on f, $f(x) \in U$ so $(x, f(x)) \in h(W)$. Since the above argument holds for any $x \in M$, we see that δ_f^2 is differentiable everywhere on M because it is the composition of differentiable functions. This proves (i), and (ii) follows trivially since δ_f vanishes exactly on the fixed point set of f.

Let $Z \subset U$ be an open set with $f(x) \in Z$, and let $W_0 = f^{-1}(Z) \cap U$. Then the map $f_0: W_0 \to U \times U$ defined by $f_0(y) = (y, f(y))$ is differentiable, and the displacement vector field V restricted to W_0 is the image of the map $h^{-1}f_0: W_0 \to T(m)$ which is C^∞, since f_0 is C^∞ and h is a diffeomorphism. Since the choice of x is arbitrary, V is C^∞ on all of M.

(1.1.3) Remark. The displacement function δ_f may fail to be differentiable at a fixed point of f as in the following situation: Let $M = R^n$, g be the ordinary metric, and f be the symmetry about the origin 0 sending $R^n \ni x \to -x$. Then $\delta_f(x) = 2\sqrt{x_1^2 + \cdots + x_n^2}$, where $x = (x_1, \cdots, x_n)$, and this is not differentiable at $x = 0$.

(1.1.4) Definition. (i) For any map $f: M \to M$ we let Fix (f) denote the set of fixed points of f.

(ii) If f has small displacement and is a diffeomorphism, we let Crit (f) denote the set of critical points of δ_f^2 in M.

(1.1.5) Remark. Crit $(f) =$ Fix $(f) \cup$ (critical points of δ_f in $M -$ Fix (f)), since for every $X \in T_x(M)$, $X\delta_f^2 = 2\delta_f(x)X\delta_f$ whenever δ_f is differentiable.

(1.2) Suppose now that $f: M \to M$ is an isometry of small displacement. We wish to differentiate δ_f. To do this fix $x \in M -$ Fix (f), and let $X \in T_x(M)$ be any non-zero vector, and $b(s)$ a smooth curve through x with tangent X at $x = b(0)$. Then $X\delta_f = \left.\dfrac{d}{ds}\right|_{s=0} \rho(b(s), f(b(s)))$. Let $a = \rho(x, f(x))$. By assumption on x, $a > 0$. The displacement vector field V is C^∞, so we have a C^∞ map $Q: [0, a] \times [0, \infty) \to M$ given by $Q(s, t) = \exp_{b(s)}\left(t\dfrac{V}{a}\right)$. Here we may take t and s in slightly larger open intervals to avoid one-sided derivatives. For fixed $s = s_0$ the curve $Q(s_0, t)$ is the unique minimizing geodesic from $b(s_0)$ to $f(b(s_0))$, and is parametrized proportional to arc-length.

Let $T = Q_*\partial/\partial t$ and $X = Q_*\partial/\partial s$; these are C^∞ vector fields on the image of Q, and have the two properties: $[T, X] = 0$ and $\nabla_T T = 0$. The first follows from $[T, X] = [Q_*\partial/\partial s, Q_*\partial/\partial t] = Q_*[\partial/\partial t, \partial/\partial s] = 0$, and the second holds because T is the tangent field to a family of geodesics. Moreover, if $b(s)$ is a geodesic then $\nabla_X X = 0$ when $t = 0$ or a since $f(b(s))$ is also a geodesic. Evidently $g(T, T)$ is independent of t, and we let $C(s) = \sqrt{g(T, T)}$.

(1.2.1) Theorem. *Let $f: M \to M$ be an isometry of small displacement and $x \in M$. Then $x \in$ Crit (f) if and only if f preserves the minimizing geodesic from x to $f(x)$.*

Proof. Let c be the minimizing geodesic from x to $f(x)$ and assume $x \notin$ Fix (f). Then

$$\rho(b(s), f(b(s))) = \int_0^a \sqrt{g(T, T)}\,(s, t)\, dt\,,$$

so

$$\begin{aligned} X_{b(s)}\delta_f &= \frac{d}{ds}\rho(b(s), f(b(s))) = \int_0^a \partial/\partial s \sqrt{g(T, T)}\, dt \\ &= \frac{1}{C(s)}\int_0^a g(\nabla_X T, T)\, dt = \frac{1}{C(s)}\int_0^a g(\nabla_T X, T)\, dt \end{aligned}$$

$$= \frac{1}{C(s)} \int_0^a \partial/\partial t\, g(X, T)\, dt$$

$$= \frac{1}{C(s)} (g(X, T)(s, a) - g(X, T)(s, 0))\,.$$

Here we have used $\nabla_X T - \nabla_T X = [X, T] = 0$, and $\nabla_T T = 0$. Thus $X_x \delta_f = g(X, T)(0, a) - g(X, T)(0, 0)$, since $C(0) = 1$. If c is normal to $b(s)$ at x, then $g(X, T)(0, 0) = 0$, which shows that $x \in \operatorname{Crit}(f) - \operatorname{Fix}(f)$ implies $g(X, T)(0, a) = 0$. Now by definition of X, $X_{f(b(0))} = f_* X_{b(s)}$, so $x \in \operatorname{Crit}(f) - \operatorname{Fix}(f)$ implies that f preserves the normal space to c; this is equivalent to f preserving c.

Now suppose f preserves the geodesic c. If $X \in T_x(M)$ is tangent to c, then $X\delta_f = 0$ because δ_f is measured along c for all points on c. Thus, if X is any vector in $T_x(M)$, then $X\delta_f = X_0 \delta_f$ where X_0 is the component of X normal to c. But then $X_0 \delta_f = g(X_0, T)(0, a) = 0$, since if f preserves c it must also preserve the normal space to c. This shows that $X\delta_f = 0$ for all $X \in T_x(M)$, so $x \in \operatorname{Crit}(f)$. The theorem holds vacuously at every fixed point of f.

(1.2.2) Remark. By "f preserves the geodesic" we mean that f restricted to the geodesic is a simple translation along the geodesic. This excludes a reflection about some isolated fixed point.

(1.3) We now compute the second derivative of δ_f. Let $x \in M - \operatorname{Fix}(f)$, $b(s)$ be a geodesic with $b(0) = x$, and X be defined as before. In particular, $X_{b(s)}$ is the tangent to $b(s)$ and $X_{f(b(s))}$ is the tangent to $f(b(s))$. Then

$$X^2_{b(s)} \delta_f = \frac{d^2}{ds^2} d(b(s), f(b(s))) = \int_0^a \frac{\partial^2}{\partial s^2} \sqrt{g(T, T)}\, dt$$

$$= \int_0^a \frac{g(T, T)(g(\nabla_X \nabla_X T, T) + g(\nabla_X T, \nabla_X T)) - g(\nabla_X T, T)^2}{g(T, T)^{3/2}}\, dt\,.$$

Now $[X, T] = 0$ implies that $\nabla_X T = \nabla_T X$, and

$$\nabla_X \nabla_T X = \nabla_T \nabla_X X + R(X, T)X\,,$$

so

$$g(\nabla_X \nabla_X T, T) = \frac{\partial}{\partial t} g(\nabla_X X, T) + g(R(X, T)X, T)\,.$$

Moreover, $g(R(X, T)X, T) = -K(X, T)(g(T, T)g(X, X) - g(X, T)^2)$, so,

$$X^2_{b(s)} \delta_f = \frac{1}{C(s)^3} \int_0^a \Big\{ g(T, T) \Big(\frac{\partial}{\partial t} g(\nabla_X X, T) - K(X, T)$$

$$\times (g(T, T) g(X, X) - g(X, T)^2) + g(\nabla_X T, \nabla_X T) \Big) - g(\nabla_X T, T)^2 \Big\}\, dt$$

$$= \frac{1}{C(s)} \int_0^a \frac{\partial}{\partial t} g(\nabla_X X, T)\, dt$$
$$+ \frac{1}{C(s)^3} \int_0^a \{g(T,T)g(\nabla_X T, \nabla_X T) - g(\nabla_X T, T)^2\}\, dt$$
$$- K(X,T)(g(T,T)g(X,X) - g(X,T)^2)\} .$$

Since $b(s)$ is a geodesic, $\nabla_X X = 0$ at both $t = 0$ and $t = a$. Therefore,

$$\int_0^a \partial/\partial t\, g(\nabla_X X, T)\, dt = g(\nabla_X X, T)(s, a) - g(\nabla_X X, T)(s, 0) = 0 ,$$

and

$$X^2_{b(s)} \delta_f = \frac{1}{C(s)^3} \int_0^a \{g(T,T)g(\nabla_X T, \nabla_X T) - g(\nabla_X T, T)^2 - K(X,T)(g(T,T)g(X,X) - g(X,T)^2)\}\, dt . \tag{1.3.1}$$

Here $K(X, T)$ is the curvature of the 2-plane spanned by X and T. This equation is valid even when $x \notin \mathrm{Crit}\,(f)$.

Note. A subset S of a Riemannian manifold M is said to be *locally convex*, if for every pair of points $x, y \in S$, which are sufficiently close, the minimizing geodesic from x to y lies is S.

(1.3.2) Lemma. *Let M be a complete connected Riemannian manifold, and $S \subset M$ a closed, connected and locally convex subset. Then S is a totally geodesic submanifold of M with possibly non-empty boundary. (Here we do not assume the boundary is smooth or of codimension one.)*

Proof. Let $x \in S$, and N_x be a convex normal neighborhood of x in M. For the moment we restrict to N_x. Suppose $y \in S \cap N_x$ and $y \neq x$. Then the geodesic segment γ from x to y lies in S. Choose any interior point z_0 of γ and a ball $B_{r_1}(z_0)$ with radius $r_1 = \min\{d(x, z_0), d(y, z_0)\}$ and center z_0. Suppose $B_{r_1}(z_0) \cap S \not\subset \gamma$ and $z_2 \in B_{r_1}(z_0) \cap S - \gamma$. We construct a cone Δ_2 over $B_{r_1}(z_0) \cap \gamma$ with vertex z_2 and generators the geodesics from z_1 to the points of $B_{r_1}(z_0) \cap \gamma$. By the assumption on z_2, Δ_2 is a two-dimensional cell with boundary. Again choose an interior point z_2 of Δ_2 and let $r_2 = \inf\{\mathrm{d}(z_2, w) \mid w \in \partial\Delta_2\}$. Suppose $Z_3 \in B_{r_2}(z_2) \cap S - \Delta_2$, and construct the cone Δ_3 over $\Delta_2 \cap B_{r_2}(z_2)$ with vertex z_3 and geodesic generators. By choosing a possibly smaller r_2 we can make sure that the generators of Δ_3 are always transverse to $\Delta_2 \cap B_{r_2}(z_2)$. Then the cone Δ_3 is a three-dimensional cell with boundary. We continue in this manner, and must eventually stop since $\dim M < \infty$. Say the last cone constructed is Δ_k. Choose an interior point $z_k \in \Delta_k$; it is clear from the convexity of S that there is a geodesic segment from z_k to each point of Δ_k which lies in S (in fact, it lies in $\Delta_k \subset S$ since otherwise we could have constructed Δ_{k+1}). This means the interior of Δ_k is a convex neighborhood in S and therefore a totally geodesic submanifold (with boundary) of M. Such a submanifold has the property that each of its points w is contained in a convex normal neighborhood, which in this case is the image by the exponential map of a ball in some linear k-dimensional subspace of $T_w(M)$. We make the above construction for all choices of points in the interior of S, and compare nearby normal neighborhoods in S. We claim they must all have dimension k. This is seen by choosing a point in one of the neighborhoods U_1 which does not lie in the other neighborhood U_2 (we assume $\dim U_2 = k$). This point is then the vertex of a cone over a k-dimensional normal coordinate ball in N_2, and by maximality of the dimension of U_2 we must get back a k-dimensional cell. This implies $\dim U_1 = k$. Then by connectedness of S, we see that every ball in S has dimension k. The intersection of convex balls is a convex neighborhood, so the interior of S is in fact a k dimensional totally geodesic submanifold.

(1.3.3) Remark. It may happen that $\partial \operatorname{Crit}(f) \neq \phi$, as seen by the following example: We consider the Euclidean plane R^2 with the usual coordinates (x, y). Let $\varepsilon > 0$. Then there is a C^∞ function $\varphi(y)$ on R^1 with the property that $\varphi(y) = 1/y^2$ if $y > \varepsilon$, $\varphi(y) \equiv 1$ if $y < 0$, and $\varphi(y) > 0$ everywhere. Then let $ds^2 = \varphi(y)(dx^2 + dy^2)$ be the Riemannian metric. On the set $\{(x, y) \mid y > \varepsilon\}$, ds^2 is the metric of the hyperbolic plane (Poincaré upper half-space); and on $\{(x, y) \mid y < 0\}$, ds^2 is the usual Euclidean metric. If we let $a > 0$ be a small number then the map $f: R^2 \to R^2$ given by $f(x, y) = (x + a, y)$ is an isometry of the set R^2 considered as a Riemannian manifold with metric ds^2. f has small displacement and $\{(x, y) \mid y > 0\} \subset \operatorname{Crit}(f)$ since f has constant displacement on this set. However, $\{(x, y) \mid y > \varepsilon\} \cap \operatorname{Crit}(f) = \emptyset$ since the displacement in $\{(x, y)y > \varepsilon\}$ is decreasing in y. Therefore $\partial \operatorname{Crit}(f) \neq \emptyset$.

(1.3.4) Theorem. *Let M have curvature $K \leq 0$, and assume $f: M \to M$ is an isometry of small displacement. Then*

(i) Crit (f) *is a totally geodesic submanifold possibly with boundary,*

(ii) δ_f *takes its absolute minimum on* Crit (f).

(iii) *If* Fix $(f) = \emptyset$, *then* Crit (f) *is connected; if* Fix $(f) \neq \emptyset$, *then* Crit (f) = Fix (f).

(iv) *If M is simply connected, then* Fix (f) *is connected.*

(v) *If $K < 0$ and* Fix $(f) = \emptyset$, *then* Crit (f) *is either empty or consists of a single geodesic.*

Proof. Under the curvature assumption, we have $X^2_{b(s)}\delta_f \geq 0$ for every geodesic $b(s)$ by the following:

$$g(T, T)g(\nabla_X T, \nabla_X T) \rightarrow g(\nabla_X T, T)^2 \geq 0 ,$$

$$g(T, T)g(X, X) - g(X, T)^2 \geq 0$$

by the Cauchy-Schwarz inequality. Thus the right side of equation (1.3.1) is non-negative, and hence $X^2_{b(s)}\delta_f \geq 0$ whenever $b(s) \notin \operatorname{Fix}(f)$. Suppose now that

$x \in \text{Crit}\,(f) - \text{Fix}\,(f)$, and let $b(s)$ be any geodesic with $b(0) = x$. Suppose $b(s_0)$ is the first point on b, which lies in Fix (f). Then $X_{b(s)}\delta_f$ is non-decreasing along $b(s)$, so that in fact δ_f is non-decreasing along $b(s)$ because $X_{b(0)}\delta_f = 0$. But this is impossible since $\delta_f(x) > 0$ and $\delta_f(bs_0)) = 0$. Thus either Crit (f) $-$ Fix $(f) = \emptyset$ or Fix $(f) = \emptyset$. This proves the second part of (iii). Let Fix (f) $= \emptyset$. Then the above argument shows that if $b(s_0) \in \text{Crit}\,(f)$, then $X_{b(s)}\delta_f = 0$ for all $s \in [0, s_0]$. This means that δ_f is constant on Crit (f). The condition $X^2_{b(s)}\delta_f \geq 0$ shows that each point of Crit (f) is a relative minimum of δ_f, so that in fact it must be an absolute minimum. If Fix $(f) \neq \emptyset$, then Crit $(f) =$ Fix (f), so that again δ_f takes its absolute minimum on Crit (f), and hence (ii) is proved.

Now if Fix $(f) \neq \emptyset$, then we take $x, y \in \text{Fix}\,(f)$, which lie in the same component of Fix (f) and are sufficiently close so there is a unique minimizing geodesic c between them. Thus $f(c)$ is a geodesic of the same length between them, so in fact $c = f(c)$. Moreover, $c \subset \text{Fix}\,(f)$, and Fix (f) is totally geodesic. If Fix $(f) = \emptyset$, choose $x, y \in \text{Crit}\,(f)$, and let $b(s)$ be any geodesic between them with $b(0) = x$ and $b(s_0) = y$. Now δ_f is constant along $b(s)$, and δ_f takes its absolute minimum at x and y, so all points on $b(s)$ between x and y lie in Crit (f). This proves (i) by Lemma 1.3.2, and also proves the first part of (iii). (iv) follows froms the fact that in a simply connected manifold with curvature $K < 0$ there are no cut points, so every pair of points is connected by a unique minimizing geodesic, and the above argument for Fix $(f) \neq \emptyset$ applies.

Now assume that $K < 0$ everywhere on M, and $x \in \text{Crit}\,(f) - \text{Fix}\,(f)$. If $b(s)$ is a geodesic transverse to the minimizing geodesic c from x to $f(x)$, then we have $g(T, T)g(X, X) - g(X, T)^2 > 0$ at $s = 0$ and $t = 0$ or a, since

$$\begin{aligned} &g(T, T)g(X, X) - g(X, T)^2 \\ &\qquad = g(T, T)g(X, X)(1 - \cos^2 (\text{angle between } X \text{ and } T)) \,. \end{aligned}$$

Thus $X^2_{b(s)}\delta_f > 0$ at $s = 0$ so that $X_{b(s)}\delta_f > 0$ for s near 0. This means δ_f is strictly increasing along $b(s)$, so $b(s)$ cannot lie in Crit (f). Since c is evidently in Crit (f) the conclusion follows.

(1.3.5) Corollary. *If M is simply connected and $K \leq 0$, then the results of the above theorem hold for any isometry.*

(1.3.6) Remark. If M is an analytic manifold and has curvature $K \leq 0$, then Crit (f) is a real analytic submanifold, which is totally geodesic and has no boundary. The fact that Crit (f) has no boundary follows since if an interval of a geodesic γ lies in Crit (f), then the whole geodesic γ must lie in Crit (f) because δ^2_f is then an analytic function γ whose derivative is zero in an interval and hence zero everywhere. If there were a boundary point x, there would have to be a geodesic starting inside Crit (f) and leaving through x, contradicting the fact that γ must lie in Crit (f). Note that if M is analytic, then every isometry $f\colon M \to M$ is analytic and the displacement function δ^2_f for isometries of small displacement is also analytic.

(1.3.7) Theorem. *Let M be any complete connected Riemannian manifold, and $f: M \to M$ an isometry of small displacement. If $h: M \to M$ is any isometry, then* $\mathrm{Crit}\,(h \cdot f \cdot h^{-1}) = h(\mathrm{Crit}\,(f))$. *That is,* $h(x) \in \mathrm{Crit}\,(f)$ *if and only if* $x \in \mathrm{Crit}(h^{-1} \circ f \circ h)$.

Proof. $h(x) \in \mathrm{Crit}\,(f)$ if and only if f preserves the minimizing geodesic c from $h(x)$ to $fh(x)$. Let $c(0) = h(x)$, $c(a) = fh(x)$ with $a = d(h(x), fh(x))$. Then f preserves c if and only if $f^2h(x) = c(2a)$. Now $f^2h(x) = c(2a)$ when $h^{-1}f^2h(x) = h^{-1}c(2a)$. The geodesic $h^{-1}c$ is the minimizing geodesic from x to $h^{-1}fh(x)$, so $x \in \mathrm{Crit}\,(h^{-1}fh)$ if and only if $(h^{-1}fh)^2x = h^{-1}c(2a)$. But $(h^{-1}fh)^2 = h^{-1}f^2h$, so the result follows.

(1.3.8.) Theorem. *Suppose M has curvature $K \leq 0$ and f is an isometry of small displacement. Let $x \in$* $\mathrm{Crit}\,(f) - \mathrm{Fix}\,(f)$, *$b(s)$ be any geodesic in* $\mathrm{Crit}\,(f)$, *which is transverse to the displacement vector field at x ($b(0) = x$), V be the displacement vector field of f along $b(s)$, and $a = \delta_f(x)$. Then the surface Q defined by $Q(s, t) = \exp_{b(s)}(t\,V/a)$ has curvatuae $K \equiv 0$, and the vector fields $T = Q_*\partial/\partial t$ and $X = Q_*\partial/\partial s$ are parallel on Q.*

Proof. We know $X^2_{b(s)}\delta_f = 0$ since δ_f is constant along $b(s)$, so

$$\int_0^a \{(g(T, T)g(\nabla_X T, \nabla_X T) - g(\nabla_X T, T)^2)$$

$$- K(X, T)(g(T, T)g(X, X) - g(X, T)^2)\}\,dt = 0\,.$$

Since $b(s)$ is transverse to the geodesic c between x and $f(x)$, $g(T, T)g(X, X) - g(X, T)^2 > 0$, so we must have $K(X, T) = 0$ for all s and t. Furthermore, the curves $Q(s, t)$ for either s or t constant are then a Euclidean coordinate system in Q, so their tangents form parallel vector fields.

(1.3.9) Theorem. *Let M be a Riemannian manifold, X a Killing vector field on M, and g_t its 1-parameter group of isometries, and assume g_t has small displacement for $t \in [0, 1]$. Then* $\mathrm{Crit}\,(|X|^2) = \bigcap_{n=1}^{\infty} \mathrm{Crit}\,(g_{1/n!})$ *and* $\mathrm{Crit}\,(g_{1/(n+1)!}) \subset \mathrm{Crit}\,(g_{1/n!})$ *for all $n = 1, 2, \cdots$.*

Proof. It is clear that $\mathrm{Crit}\,(|X|^2) \subset \mathrm{Crit}\,(g_t)$ for all $t \in (0, 1]$ since $\mathrm{Crit}\,(|X|^2) = \{x \in M \mid g_t x \text{ is a geodesic}\}$. Suppose $x \in \bigcap_{n=1}^{\infty} \mathrm{Crit}\,(g_{1/n!})$. Since $\mathrm{Crit}\,(g_{1/n}) \subset \mathrm{Crit}\,(g_1)$ for all n, the geodesic preserved by $g_{1/n}$ is the same as that for g_1, and therefore the orbit $g_t x$ crosses the geodesic c from x to gx at the points $g_{m/n!}x$ for $1 \leq m \leq n!$. The set of points $\{g_{m/n!}x \mid 1 \leq m \leq n!, \text{ all } n\}$ is dense on c, so in fact $g_t x = c$. The fact that $\mathrm{Crit}\,(g_{1/(n+1)!}) \subset \mathrm{Crit}\,(g_{1/n!})$ is obvious from Theorem 1.2.1.

(1.3.10) Corollary. *Let M be analytic, and suppose its curvature K is non-positive. Let X be a Killing vector field, and g_t its 1-parameter group, and assume g_t has small displacement for all $t \in [0, 1]$. Then there is a $t_0 \in (0, 1]$ such that* $\mathrm{Crit}\,(|X|^2) = \mathrm{Crit}\,(g_{t_0})$.

Proof. If $X = 0$ at some point, then q_t has a fixed point and the corollary follows from Theorem 1.3.3 (iii). Suppose $X \neq 0$ everywhere. Then each Crit (g_t) is a connected submanifold of M without boundary (Remark 1.3.5). Let $k_n = \dim \operatorname{Crit}(g_{1/n!})$. Since the critical sets Crit $(g_{1/n!})$ are nested and converge to Crit $(|X|^2)$, we must have $k_n \to \dim \operatorname{Crit}(|X|^2)$, which means that for some n, $k_n = \dim \operatorname{Crit}(|X|^2)$. Then Crit $(|X|^2)$ is a connected submanifold of the connected manifold Crit $(g_{1/n!})$, and they must be equal since they have the same dimension.

Chapter II. Homogeneous and symmetric spaces

We now assume that $M = G/K$ is a reductive homogeneous space which is connected and has normal metric in which it is complete. We fix a direct sum decomposition

$$\boldsymbol{G} = \boldsymbol{K} + \boldsymbol{m}$$

of the Lie algebra $\boldsymbol{G}$ of G, where $\boldsymbol{K} =$ Lie algebra of K, and $\boldsymbol{m}$ is a complementary subspace with the property that ad $(K)\boldsymbol{m} \subset \boldsymbol{m}$. We consider only those isometries of M coming from elements of G.

(2.1) Let $g \in G$ be an isometry of small displacement, and let $x \in M$. We assume x is identified with the identity coset of its isotropy group K. Then there is a unique shortest $T \in \boldsymbol{m}$ such that $gx = (\exp T)x$, and $(\exp tT)x$, $0 \leq t \leq 1$, is the minimizing geodesic from x to gx. Thus $(\exp -T)gx = x$ so that $k = (\exp -T)g \in K$, and we have a unique decomposition $g = (\exp T)k$.

(2.1.1) Theorem. *$x(=K)$ is in* Crit (g) *if and only if* ad $(k) = T$, *where $g = (\exp T)k$ in the above decomposition.*

Proof. By Theorem 1.2.1, $x \in \operatorname{Crit}(g)$ if and only if g preserves the geodesic $(\exp tT)x$. This is true exactly when $g(\exp tT)x = (\exp(1 + t)T)x$. Now $g(\exp T)x = (\exp(1 + t)T)x$ if and only if $(\exp -tT)k(\exp tT)x = x$; that is, when $(\exp -tT)k(\exp tT) \in K$ for all t. This curve has tangent $dL_k(T) - dR_k(T) = T - \operatorname{ad}(k)T$ at $t = 0$, where L_k (resp. R_k) is the left (resp. right) translation by k. Since ad $(K)\boldsymbol{m} \in \boldsymbol{m}$, we have ad $(k)T \in \boldsymbol{m}$ so the tangent lies in $\boldsymbol{m}$. Since it must also lie in $\boldsymbol{K}$, it must vanish; that is, ad $(k)T = T$.

Conversely, if ad $(k)T = T$ then

$$\begin{aligned} g(\exp tT)x &= (\exp T)k(\exp tT)k^{-1}x = (\exp T)(\exp t \operatorname{ad}(k)T)x \\ &= (\exp T)(\exp tT)x = (\exp(1 + t)T)x\,. \end{aligned}$$

So g preserves the geodesic $(\exp tT)x$ from x to gx.

(2.1.2) Corollary.

$$\operatorname{Crit}(g) = \{hx \mid \operatorname{ad}(k_h)T_h = T_h,\ h^{-1}gh = (\exp T_h)k_h,\ h \in G\}\,,$$

where $h^{-1}gh = (\exp T_h)k_h$ is the unique decomposition of Theorem 2.1.1.

Proof. Clearly $h^{-1}gh$ has small displacement if g does, so the proof follows from Theorems 1.3.7 and 2.1.1.

(2.2) Let $M = G/K$ be a compact connected Riemannian homogeneous space with normal metric. Assume G is compact and semi-simple, so that the Killing form B is negative definite on $\boldsymbol{G}$ and is invariant under the adjoint action of G. Let $\boldsymbol{G} = \boldsymbol{K} + \boldsymbol{m}$ as usual, with $\boldsymbol{K}$ and $\boldsymbol{m}$ orthogonal by $-B$.

(2.2.1) Lemma. *There is a number* $r > 0$ *such that if* $g \in \exp B$, $B_r = \{Y \in \boldsymbol{G} \mid (-B(Y, Y))^{1/2} < r\}$, *then* $g = (\exp T)(\exp S)$ *for unique shortest* $T \in \boldsymbol{m}$, $S \in \boldsymbol{K}$; *and* $(\exp S)(\exp T) = (\exp T)(\exp S)$ *if and only if* $[T, S] = 0$.

Proof. Define a map $\boldsymbol{K} \times \boldsymbol{m} \xrightarrow{\varphi} G$ by $\varphi(S, T) = (\exp T)(\exp S)$. φ is clearly regular at $(0, 0)$ and is differentiable everywhere. Then by the inverse function theorem there is a neighborhood of $(0, 0)$ in $\boldsymbol{K} \times \boldsymbol{m}$ on which φ is a diffeomorphism. Let $r_0 > 0$ be maximal for the property that $\exp\colon \boldsymbol{G} \to G$ is a diffeomorphism on $B_{r_0} = \{Y \in \boldsymbol{G} \mid -B(Y, Y) < r^2\}$. Let $V_1 = \boldsymbol{K} \cap B_{r_0}$, $V_2 = \boldsymbol{m} \cap B_{r_0}$, and $V \subset V_1 \times V_2$ be the maximal neighborhood of the form $V = \varphi^{-1}(\exp(B_r))$ on which φ is a diffeomorphism. It is clear that $r > 0$.

Suppose now that $g \in \exp B_r$; then g is written uniquely as $g = (\exp T)(\exp S)$ for $T \in \boldsymbol{m}$, $S \in \boldsymbol{K}$. Assume $(\exp T)(\exp S) = (\exp S)(\exp T)$, which means $\exp \mathrm{ad}(\exp S)T = \exp T$. But since B_r is ad(G)-invariant, we have $\mathrm{ad}(\exp S)T$, $T \in B_r$ so that $\mathrm{ad}(\exp S)T = T$ as exp is a diffeomorphism on $B_r \subset B_{r_0}$. Similarly, $ad(\exp T)S = S$, which means $(\exp S)(\exp tT) = (\exp tT)(\exp S)$ for all t, Applying the above argument to tT and S, for any $t \in [0, 1]$, we get $(\exp tS)(\exp tT) = (\exp tT)(\exp tS)$, which is equivalent to $[T, S] = 0$. It is obvious that $[T, S] = 0$ implies $(\exp T)(\exp S) = (\exp S)(\exp T)$.

(2.2.2) Theorem. *Let* $M = G/K$ *be a compact homogeneous space with normal metric, and assume* G *is compact semisimple. Let* $X \in B_r$, $g = \exp X$ *be the associated isometry, and* $x = K$. *Then* $hx \in \mathrm{Crit}(g)$ *for* $h \in G$ *if and only if* $h^{-1}gh = (\exp T)(\exp S)$ *with* $[T, S] = 0$, *where* $S = (\mathrm{ad}(h^{-1})X)_{\boldsymbol{K}}$, $T = (\mathrm{ad}(h^{-1})X)_{\boldsymbol{m}}$, *and* $g = \exp X$.

Proof. We know that $hx \in \mathrm{Crit}(g)$ if and only if $h^{-1}gh = (\exp T)k$ for $T \in \boldsymbol{m}$, $k \in K$ where $\mathrm{ad}(k)T = T$. Here there is no question of uniqueness of T since B_r is ad(G)-invariant and φ is a diffeomorphism on $\exp B_r$. Thus $h^{-1}gh \in \exp B_r$; if $\mathrm{ad}(k)T = T$ then $(\exp T)k = k(\exp T)$, and since $k = \exp S$ Lemma 2.2.1 shows that $[S, T] = 0$. In this case $(\exp T)(\exp S) = \exp(S + T) = h^{-1}(\exp X)h$, so that $S = (\mathrm{ad}(h^{-1})X_{\boldsymbol{K}}$ and $T = (\mathrm{ad}(h^{-1})X_{\boldsymbol{m}}$. Conversely, if $[S, T] = 0$ then obviously $\mathrm{ad}(k) = T$.

(2.2.3) Corollary. *Let* $M = G/K$ *be a connected symmetric space of compact type, with* σ *the symmetry in both* G *and* $\boldsymbol{G}$, *and let* $x = K$. *If* $g \in \exp B_r$ *as in Theorem* 2.2.2, *then* $\mathrm{Crit}(g) = \{h^{-1}x \mid h \in G$ *and* $[\mathrm{ad}(h)X, \sigma\,\mathrm{ad}(h)X] = 0\}$; $g = \exp X$.

Proof. For any $Y \in \boldsymbol{G}$, $[Y, \sigma Y] = [Y_{\boldsymbol{K}} + Y_m, Y_{\boldsymbol{K}} - Y_m] = 2[Y_{\boldsymbol{K}}, Y_m]$, so the result follows from Theorem 2.2.2.

(2.2.4) Corollary. *Under the assumptions in Theorem* 2.2.2, $\mathrm{Crit}(|X|^2) = \mathrm{Crit}(g_t)$ *where* $g_t = \exp tX$ *and* $t \in (0, 1]$.

Proof. Clearly $\mathrm{Crit}(|X|^2) \subset \mathrm{Crit}(g_t)$ for each $t \in (0, 1]$. Conversely, if for any $t \in (0, 1]$, $[(tX)_K, (tX)_m] = 0$, then this is true for all $t \in (0, 1]$. This does not depend on the choice of decomposition $G = K + m$; therefore, $x \in \mathrm{Crit}(g_{t_1})$ if and only if $x \in \mathrm{Crit}(g_{t_2})$. Since $\bigcap_{0<t\leq 1} \mathrm{Crit}(g_t) = \mathrm{Crit}(|X|^2)$ the result follows.

(2.3) In this section we assume $M = G/K$ is a connected Riemannian symmetric space of compact type, and $g \in \exp B_r$ an isometry having $x = K$ in $\mathrm{Crit}(g)$ with $g = \exp X$.

(2.3.1) Lemma. *If* $X \in G$ *is such that* $[X, \sigma X] = 0$, *then there is a Cartan subalgebra* $\mathfrak{h}$ *of* G *such that* $X \in \mathfrak{h}$ *and* $\sigma\mathfrak{h} = \mathfrak{h}$.

Proof. Let $X_m = \frac{1}{2}(X - \sigma X)$ and $X_K = \frac{1}{2}(X + \sigma X)$ so that $X_m \in m, X_K \in K$. Let $Z_G(X_K) =$ centralizer of X_K in G. σ is the identity on K, so if $Y \in Z_G(X_K)$ then $[\sigma Y, X_K] = [\sigma Y, \sigma X_K] = \sigma[Y, X_K] = 0$. Therefore $\sigma Z_G(X_K) = Z_G(X_K)$, and $Z_G(X_K) = Z_K(X_K) + Z_m(X_K)$. Since $[X, \sigma X] = 0, [X_m, X_K] = 0$, so $X_m \in Z_m(X_K)$. Choose $A \subset Z_m(X_K)$ a maximal abelian subspace containing X_m, and let $B \subset$ (centralizer of A in $Z_K(X_K)$) be a maximal abelian subspace necessarily containing X_K. It is clear that A and B are non-empty since $X_m \in A$ and $X_K \in B$. The subspace $A + B$ of G is an abelian subalgebra which is invariant under σ. Suppose $Y \in G$ commutes with every element of $A + B$. If we let $Y = Y_K + Y_m$ with $Y_K \in K, Y_m \in m$, then $[Y, A] = 0 = [Y, B]$ implies $[Y_K, A] = 0 = [Y_K, B]$, and $[Y_m, A] = 0 = [Y_m, B]$. Since A is maximal abelian in $Z_m(X_K)$, $Y_m \in A$. Y_K centralizes A and also B, so by maximality of B, $Y_K \in B$, and $Y = Y_K + Y_m \in A + B$. Thus $A + B$ is a maximal abelian subalgebra of G, and is a Cartan subalgebra, since G is compact.

(2.3.2) Theorem. *Let* $M = G/K$ *be a connected symmetric space of compact type, and* $g \in \exp B_r$ *an isometry. If* $x \in \mathrm{Crit}(g)$, *then the component of* $\mathrm{Crit}(g)$ *containing* x *is* $Z_G^0(g) \cdot x$. *Here* $Z_G^0(g)$ *is the identity component of the centralizer* $Z_G(g)$ *of* g *in* G.

Proof. By Corollary 2.2.3, $h^{-1}x \in \mathrm{Crit}(g)$ if and only if $[\mathrm{ad}(h)X, \sigma\,\mathrm{ad}(h)X] = 0$. It suffices to consider only those $h \in \exp m$ since M is complete.

Let $\mathfrak{h}_i$ be the distinct Cartan subalgebras of G which contain X, and choose regular elements $X_i \in \mathfrak{h}_i$ which lie in B_r. This is possible since tX_i is regular when X_i is regular and $t \neq 0$. Now for any $h \in G$, $\mathrm{ad}(h)\mathfrak{h}_i$ are the distinct Cartan subalgebras containing $\mathrm{ad}(h)X$, so if $h^{-1}x \in \mathrm{Crit}(g)$ then by Lemma 2.3.1 there is an index i such that $\sigma\,\mathrm{ad}(h)\mathfrak{h}_i = \mathrm{ad}(h)\mathfrak{h}_i$. In particular, this means that $[\mathrm{ad}(h)X_i, \sigma\,\mathrm{ad}(h)X_i] = 0$, so $h^{-1}x \in \mathrm{Crit}(\exp X_i)$. Conversely, if $h^{-1}x \in \mathrm{Crit}(\exp X_i)$ then $[\mathrm{ad}(h)X_i, \sigma\,\mathrm{ad}(h)X_i] = 0$. Now $\mathrm{ad}(h)X_i$ and $\sigma\,\mathrm{ad}(h)X_i$ are regular elements which commute, so we must have $\sigma\,\mathrm{ad}(h)\mathfrak{h}_i = \mathrm{ad}(h)\mathfrak{h}_i$. This means $[\mathrm{ad}(h)X, \sigma\,\mathrm{ad}(h)X] = 0$, so $h^{-1}x \in \mathrm{Crit}(g)$. Thus we have $\mathrm{Crit}(g) = \bigcup_i \mathrm{Crit}(\exp X_i)$.

We assume now that $g = \exp X$ with X a regular element of G, and $x = K$ is in $\operatorname{Crit}(g)$.

If $\mathfrak{h}$ is the Cartan algebra of G containing X, then the assumption that x is a critical point implies $[X, \sigma X] = 0$ which means $\sigma\mathfrak{h} = \mathfrak{h}$. Suppose $h \in \exp m$ is such that $h^{-1}x \in \operatorname{Crit}(g)$. Then $[\operatorname{ad}(h)X, \sigma\operatorname{ad}(h)X] = 0$, or equivalently, $\operatorname{ad}(h)\mathfrak{h} = \sigma\operatorname{ad}(h)\mathfrak{h}$. But this just means $\operatorname{ad}(h^2)\mathfrak{h} = \sigma\mathfrak{h} = \mathfrak{h}$, so $h^2 \in$ normalizer of $\mathfrak{h}$ in G. If h is sufficiently close to the identity e, then this condition implies $h \in T$, where T is the identity component of the normalizer. T is the maximal torus of G corresponding to $\mathfrak{h}$, and equals $Z_G^0(g) =$ identity component of the centralizer of g in G. If y is in the same component of $\operatorname{Crit}(g)$ as x, then we cover a curve c in $\operatorname{Crit}(g)$ from x to y by neighborhoods U_j where $U_j = Vx_j$ for a neighborhood $V \subset T$ of e, and a finite number of points $x_i \in c$ such that $x_0 = x$, $x_n = y$, and $x_j \in Vx_{j-1}$ for all $1 \leq j \leq n$. Since c is compact this is possible for some n. We choose n large enough and V so small that the transvection $h \in (\exp m) \cap V$ always satisfies the property that if $h^2 \in$ normalizer of T then $h \in T$. Note that the set m of transvections may change with j, but this does not affect the above construction. Then $x_j = g_j x_{j-1}$ for $g_j \in V$, so $y = g_n g_{n-1} \cdots g_1 x$, which means $y \in Tx$. This shows that the component of $\operatorname{Crit}(g)$ which contains x is contained in Tx. The other inclusion is obvious since T is in the centralizer of g. Thus $\operatorname{Crit}(g) = \bigcup_m Tx_m$ for a set $\{x_m\}$ of representative elements of the components of $\operatorname{Crit}(g)$.

If g is not regular then $Z_G^0(g) = \bigcup_i T_i$, where the T_i are the distinct maximal tori containing g. Therefore

$$\begin{aligned}\operatorname{Crit}(g) &= \bigcup_i \operatorname{Crit}(\exp X_i) = \bigcup_i \bigcup_{m_i} T_i x_{m_i} \\ &\subseteq \left(\bigcup_i T_i\right) \cdot \left(\bigcup_m x_m\right) = \bigcup_m Z_G^0(g) \cdot x_m ,\end{aligned}$$

where $\{x_m\}$ is a set of representatives of the components of $\operatorname{Crit}(g)$. Since $\bigcup_m Z_G^0(g)x_m \subset \operatorname{Crit}(g)$, the result follows.

(2.3.3) Remark. In the case where X is a regular element we see from the proof of the above theorem that in fact the orbit of x by the normalizer of $\mathfrak{h}$ in G is contained in $\operatorname{Crit}(\exp X)$. It would be interesting to know if this is an equality.

(2.3.4) Corollary. *If $g = \exp X$ for a regular element X of G, then* $\operatorname{Crit}(g)$ *is a flat totally geodesic submanifold of M.*

Proof. Since the components of $\operatorname{Crit}(g)$ are orbits by an abelian subgroup, they must be flat, and are totally geodesic because this subgroup is invariant by the symmetry σ of G corresponding to the geodesic symmetry at each point of $\operatorname{Crit}(g)$.

(2.3.5) Example. In the proof Corollary 2.3.4 we use regularity of X to get that $Z_G^0(\exp X)$ is invariant by σ. This in turn requires $Z_G(X)$ to be σ-invariant. The assumption of regularity cannot be dropped, as seen from the following example of J.A. Wolf: Let $M = SU(6)/SO(6)$, and let $e_1, \cdots, e_6$ be a basis of $Su(6)$. Let X_K and X_m have eigenvalues $\sqrt{-1}, -\sqrt{-1}, 2\sqrt{-1}, -2\sqrt{-1}, 10^{20}\sqrt{-1}, -10^{20}\sqrt{-1}$ and $\sqrt{-1}, \sqrt{-1}, 2\sqrt{-1}, 2\sqrt{-1}, -3\sqrt{-1}, -3\sqrt{-1}$, respectively, corresponding to the vectors $e_1, \cdots, e_6$. Then $[X_K, X_m] = 0$. $X_K + X_m$ has eigenvalues 0, $2\sqrt{-1}$, $4\sqrt{-1}$, $-3\sqrt{-1} + 10^{20}\sqrt{-1}$, $-3\sqrt{-1} - 10^{20}\sqrt{-1}$ corresponding to the eigenspaces spanned by $\{e_2, e_4\}$, $\{e_1\}, \{e_3\}, \{e_5\}, \{e_6\}$ respectively, and $X_K - X_m$ has eigenvalues $0, -2\sqrt{-1}$, $-4\sqrt{-1}$, $3\sqrt{-1} + 10^{20}\sqrt{-1}$, $3\sqrt{-1} - 10^{20}\sqrt{-1}$, corresponding to the eigenspaces spanned by $\{e_1, e_3\}, \{e_2\}, \{e_4\}, \{e_5\}, \{e_6\}$ respectively. The centralizers of $X_K + X_m$ and $X_K - X_m$ consist of matrices which are scalar multiples of the identity in each of their eigenspaces; but as the eigenspaces do not correspond, the centralizers are not equal. If $X = X_K + X_m$ then $\sigma X = X_K - X_m$; and clearly $Z_G(\sigma X) = \sigma Z_G(X)$, so we have $\sigma Z_G(X) \neq Z_G(X)$.

(2.4) In this section we consider symmetric spaces of noncompact type.

(2.4.1) Theorem. *Let $M = G/K$ be a connected Riemannian symmetric space of non-compact type, and assume $g \in G$ is any isometry. If $x \in \mathrm{Crit}\,(g)$, then* $\mathrm{Crit}\,(g) = Z_G^0(g)\cdot x$.

Proof. Since M is simply connected with curvature $K \leq 0$ there are no cut points so every isometry is of small displacement, and every pair of points is joined by a unique minimizing geodesic.

Suppose $y \neq x$ is another critical point, and let $(\exp sS)x, S \in m$, be the geodesic from x to y. We assume S is transverse to the geodesic c from x to gx. Construct the surface Q as in Chapter I, and let $T \in m$ be the tangent to c. Then by Theorem 1.3.6 we have that Q is flat and the vector fields S and T are parallel on Q, where flatness implies $[S, T] = 0$. Now in a symmetric space $dL_{\exp sS}(T)$ is parallel along $(\exp sS)x$, and since T itself is parallel, $T = dL_{\exp sS}(T)$. Therefore the translation $L_{\exp sS}$ takes the geodesic $(\exp tT)x$ to the geodesic from $(\exp sS)x$ to $g(\exp sS)x$ for each s. Thus $(\exp sS)gx = g(\exp sS)x$, or $g^{-1}(\exp\text{-}sS)g(\exp sS)x = x$ which means $g^{-1}(\exp\text{-}sS)g(\exp sS) \in K$. Now $g = (\exp T)k$ with $T \in m, k \in K$, and $[S, T] = 0$. Therefore we get $k^{-1}(\exp\text{-}T)\cdot(\exp\text{-}sS)(\exp T)k(\exp sS) \in K$, which implies $(\exp\text{-}sS)k(\exp sS) \in K$ for all s. Then $dR_k(S) - dL_k(S) \in K$, or $S - \mathrm{ad}(k)S \in K$.

Since $S \in m$ and $k \in K, \mathrm{ad}(k)S \in m$, so $\mathrm{ad}(k)S = S$. Thus $g(\exp sS)g^{-1} = (\exp T)k(\exp sS)k^{-1}(\exp\text{-}T) = (\exp T)(\exp s\,\mathrm{ad}(k)S)(\exp\text{-}T) = \exp s\,\mathrm{ad}(\exp T)S = \exp sS$ for every s. Thus $\mathrm{Crit}\,(g) \subset Z_G^0(g)x$. The other inclusion is obvious, so $\mathrm{Crit}\,(g) = Z_G^0(g)x$.

(2.5) In [7] and [8] R. Hermann discussed the critical points of the squared length function f_x of a Killing vector field X. We shall reformulate a part of Theorem 1 in [7], and then a comparison with our results show that in the case of a symmetric space, the critical manifold of f_X coincides with that of $g_t =$

$\exp tX$ for any small t. That $\mathrm{Crit}(\exp t_1X) = \mathrm{Crit}(\exp t_2X)$ for any small t_1, t_2 is obvious from Theorem 2.2.2 for the case of compact spaces.

We again fix a decomposition $\boldsymbol{G} = \boldsymbol{K} + \boldsymbol{m}$ of the Lie algebra $\boldsymbol{G}$ of G, where $M = G/K$ is the symmetric space and $\boldsymbol{K}$(resp. $\boldsymbol{m}$) the $+1$(resp. -1) eigenspaces of the symmetry σ. For each $g \in G$, we let $P_g\colon \boldsymbol{G} \to \mathrm{ad}(g)\boldsymbol{m}$ be the projection. Notice that P_g depends only on gK.

It is easy to see that $\mathrm{ad}(g) \circ P_e = P_g \circ \mathrm{ad}(g)$, so $P_g = \mathrm{ad}(g) \circ P_e \circ \mathrm{ad}(g^{-1})$. Now for every $X \in \boldsymbol{G}$ we have a Killing vector field, which we also denote by X, on M coming from the 1-parameter group $\exp tX$ of isometries of M. Identify the tangent space of M at gK with $\mathrm{ad}(g)\boldsymbol{m}$ for each $g \in G$; then we may view P_gX as a vector field on M. In fact, P_gX is the Killing vector field of the 1-parameter group $\exp tX$. Let $\langle\, ,\rangle$ be the invariant metric on M, so that

$$\begin{aligned} f_X(gX) &= \langle P_gX, P_gX\rangle = \langle \mathrm{ad}(g) \circ P_e \circ \mathrm{ad}(g^{-1})X, \mathrm{ad}(g)P_e\mathrm{ad}(g^{-1})X\rangle \\ &= \langle P_e \circ \mathrm{ad}(g^{-1})X, P_e \circ \mathrm{ad}(g^{-1})X\rangle\,. \end{aligned}$$

f_X is evidently differentiable on M. We will use the abbreviation $f_X(gK) = f_X(g)$. Then f_X has a critical point at gK exactly when

$$\frac{d}{dt}\bigg|_{t=0} f_X((\exp tH)g) = 0$$

for all $H \in \boldsymbol{m}$. Now

$$\begin{aligned} f_X((\exp tH)g) &= \langle P_e \circ \mathrm{ad}((\exp tH)g)X, P_e \circ \mathrm{ad}((\exp tH)g)X\rangle \\ &= \langle (e^{t\,\mathrm{ad}\,H} \circ \mathrm{ad}(g)X)_{\boldsymbol{m}}, (e^{t\,\mathrm{ad}\,H} \circ \mathrm{ad}(g)X)_{\boldsymbol{m}}\rangle\,. \end{aligned}$$

Here $e^{t\,\mathrm{ad}\,H} = \cosh(t\,\mathrm{ad}\,H) + \sinh(t\,\mathrm{ad}\,H)$, cosh and sinh denoting the usual power series.

Since $H \in \boldsymbol{m}$, and M is a symmetric space, we have $\cosh(t\,\mathrm{ad}\,H)\boldsymbol{m} \subset \boldsymbol{m}$, $\cosh(t\,\mathrm{ad}\,H)\boldsymbol{K} \subset \boldsymbol{K}$, $\sinh(t\,\mathrm{ad}\,H)\boldsymbol{m} \subset \boldsymbol{K}$, and $\sinh(t\,\mathrm{ad}\,H)\boldsymbol{K} \subset \boldsymbol{m}$, so

$$\begin{aligned} &(e^{t\,\mathrm{ad}\,H} \circ \mathrm{ad}(g)X)_{\boldsymbol{m}} \\ &\quad = \{[\cosh(t\,\mathrm{ad}\,H) + \sinh(t\,\mathrm{ad}\,H)][(\mathrm{ad}(g)X)_{\boldsymbol{m}} + (\mathrm{ad}(g)X)_{\boldsymbol{K}}]\}_{\boldsymbol{m}} \\ &\quad = \cosh(t\,\mathrm{ad}\,H)(\mathrm{ad}(g)X)_{\boldsymbol{m}} + \sinh(t\,\mathrm{ad}\,H)(\mathrm{ad}(g)X)_{\boldsymbol{K}}\,. \end{aligned}$$

Thus

$$\frac{d}{dt}\bigg|_{t=0} (e^{t\,\mathrm{ad}\,H}\,\mathrm{ad}(g)X)_{\boldsymbol{m}} = \mathrm{ad}\,H(\mathrm{ad}(g)X)_{\boldsymbol{K}}\,.$$

Now

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} f\, f_X((\exp tH)g) &\\
&= 2\Big\langle \frac{d}{dt}\Big|_{t=0} (e^{t \operatorname{ad} H} \operatorname{ad}(g)X)_m, (\operatorname{ad}(g)X)_m \Big\rangle \\
&= 2\langle \operatorname{ad} H(\operatorname{ad}(g)X)_K, (\operatorname{ad}(g)X)_m \rangle \\
&= 2\langle H, [(\operatorname{ad}(g)X)_K, (\operatorname{ad}(g)X)_m] \rangle .
\end{aligned}$$

Now $\langle\, ,\rangle$ is non-degenerate so the above vanishes for all $H \in m$ if and only if $[(\operatorname{ad}(g)X)_K, (\operatorname{ad}(g)X)_m] = 0$. Since we are in a symmetric space this is equivalent to $[\operatorname{ad}(g)X, \sigma \operatorname{ad}(g)X] = 0$.

We assume that M is complete and connected so that every point $g\cdot K$ in M can be represented by a transvection; that is, an element $g \in G$ such that $\sigma g = g^{-1}$. Every $g \in G$ can be expressed as a product $g = pk$ with $k \in K$ and $\sigma p = p^{-1}$, so that $\operatorname{ad}(g)m = \operatorname{ad}(p)m$, which shows that P_g depends only on the transvective component p of g, so we may assume g is a transvection. Then

$$\begin{aligned}
[\operatorname{ad}(g)X, \sigma \operatorname{ad}(g)X] &= [\operatorname{ad}(g)X, \operatorname{ad}(g^{-1})\sigma X] \\
&= \operatorname{ad}(g^{-1})[\operatorname{ad}(g^2)X, \sigma X] .
\end{aligned}$$

Thus $g^{-1}K$ is a critical point of f_X if and only if $[\operatorname{ad}(g^2)X, \sigma X] = 0$. This is the first part of Theorem 1 in [7].

Now let M be connected, symmetric and of non-compact type, and consider the critical set $\operatorname{Crit}(f_X)$ of f_X.

(2.5.1) Theorem.

$$\operatorname{Crit}(f_X) = \operatorname{Crit}(\exp X) = Z_G^0(\exp X)\cdot x$$

for any $x \in \operatorname{Crit}(\exp X)$ *if* X *is sufficiently small.*

Proof. By the remarks of (2.5), we have that for $h \in \exp m$, hK is a critical point of f_X if and only if $[\operatorname{ad}(h^{-2})X, \sigma X] = 0$. We will find the tangent space of the critical set of f_X at $x = K$ assuming x is a critical point of f_X. Suppose $H(t)$ is a C^∞-curve in m with $H(0) = 0$ such that $\exp H(t)x \in \operatorname{Crit}(f_X)$ for small t. Then $[\operatorname{ad}(\exp\text{-}2H(t))X, \sigma X] = 0$ for all t near zero. Assume that

$$\frac{d}{dt}\Big|_{t=0} H(t) = V .$$

Now $\operatorname{ad} \sigma X(e^{-2 \operatorname{ad} H(t)}X) = 0$ so

$$\operatorname{ad} \sigma X(X - 2 \operatorname{ad} H(t)(X) + 4(\operatorname{ad} H(t))^2 X - \cdots) = 0 .$$

Differentiating at $t = 0$, this shows that $2 \operatorname{ad} \sigma X \operatorname{ad} V(X) = 0$, that is,

$$\begin{aligned}0 = [S - T, [V, S + T]] &= [S, [V, S]] - [T, [V, S]] \\ &\quad + [S, [V, T]] - [T, [V, T]] \\ &= (\operatorname{ad} T)^2 V - (\operatorname{ad} S)^2 V + (\operatorname{ad} T \operatorname{ad} S - \operatorname{ad} S \operatorname{ad} T) V .\end{aligned}$$

Here $X = S + T$ with $S \in K$ and $T \in m$. Now $(\operatorname{ad} T)^2 V$ and $(\operatorname{ad} S)^2 V$ are in m, and $\operatorname{ad} T \operatorname{ad} S(V)$ and $\operatorname{ad} S \operatorname{ad} T(V)$ are in K, so in particular we get $(\operatorname{ad} T)^2 V - (\operatorname{ad} S)^2 V = 0$. The Killing form B is negative definite on K and positive definite on m, so we can define a new form B_σ on G by $B_\sigma(X, Y) = -B(X, \sigma Y)$. B_σ is positive definite on G, but no longer invariant under the adjoint action of G on G. Let $Y, Z \in G$. Then

$$\begin{aligned}B_\sigma(\operatorname{ad} S(Y), Z) &= -B(\operatorname{ad} S(Y), \sigma Z) = B(Y, \operatorname{ad} S(\sigma Z)) \\ &= B(Y, \sigma(\operatorname{ad} S(Z))) = -B_\sigma(Y, \operatorname{ad} S(Z)) ,\end{aligned}$$

so $\operatorname{ad} S$ is skew-symmetric with respect to B_σ. Similarly,

$$\begin{aligned}B_\sigma(\operatorname{ad} T(Y), Z) &= -B(\operatorname{ad} T(Y), \sigma Z) = B(Y, \operatorname{ad} T(\sigma Z)) \\ &= -B(Y, \sigma(\operatorname{ad} T(X))) = B_\sigma(Y, \operatorname{ad} T(Z)) ,\end{aligned}$$

so $\operatorname{ad} T$ is symmetric on G with respect to B_σ. Since $[X, \sigma X] = 0$, we have $[S, T] = 0$ so $\operatorname{ad} S$ and $\operatorname{ad} T$ commute on G. Now $\operatorname{ad} S$ has pure imaginary eigenvalues since it is skew, and $\operatorname{ad} T$ has real eigenvalues since it is symmetric. Therefore $(\operatorname{ad} S)^2$ is negative-semidefinite, and $(\operatorname{ad} T)^2$ is positive semi-definite. This means that if $(\operatorname{ad} S)^2 V = (\operatorname{ad} T)^2 V$, we must have $(\operatorname{ad} S)^2 V = 0 = (\operatorname{ad} T)^2 V$. $0 = B_\sigma((\operatorname{ad} S)^2 V, V) = (B_\sigma(\operatorname{ad} S(V), \operatorname{ad} S(V))$ and $0 = B_\sigma((\operatorname{ad} T)^2 V, V) = B_\sigma(\operatorname{ad} T(V), \operatorname{ad} T(V))$ so $\operatorname{ad} S(V) = 0 = \operatorname{ad} T(V)$ since B_σ is positive definite. Hence $[V, X] = 0$. Since $[V, X] = 0$ implies $(\exp tV)x \in \operatorname{Crit}(f_X)$ for all t, we see that the tangent space of $\operatorname{Crit}(f_X)$ at x is $Z_m(X) =$ centralizer of X in m. In Theorem 3.1 (*f*) of [8] it is shown that $\operatorname{Crit}(f_X)$ is connected and convex so that every point $y \in \operatorname{Crit}(f_X)$ lies on a geodesic in $\operatorname{Crit}(f_X)$ which passes through x; this geodesic has the form $(\exp tH)x$ for $H \in m$, and the above shows $H \in Z_G(X)$. Thus $\operatorname{Crit}(f_X) = Z_G^0(\exp X) \cdot x$. Now fix $x \in \operatorname{Crit}(f_X)$. If we let $x = K$, then we have $[X, \sigma X] = 0$ so $X = S + T$ with $S \in K, T \in m$ and $[S \cdot T] = 0$. Therefore $\operatorname{ad}(\exp S)T = T$ and $x \in \operatorname{Crit}(\exp tX)$ for all sufficiently small t. Conversely, suppose $x \in \operatorname{Crit}(\exp tX)$ for t small enough so that $\exp tX = (\exp T)(\exp S)$ for unique stortest $T \in m, s \in K$ and such that $(\exp T)(\exp S) = (\exp S)(\exp T)$ if and only if $[S, T] = 0$. It is possible to choose t so small by an argument used in the proof of Lemma 2.2.1. The choice of how small t has to be depends on x, and since M is non-compact there may be no value which works for all x. However, the above shows that this particular x is in $\operatorname{Crit}(f_X)$. But since $\operatorname{Crit}(f_X) = Z_G^0(\exp tX) \cdot x$ and $\operatorname{Crit}(\exp tX) = Z_G^0(\exp tX) \cdot x$, we have $\operatorname{Crit}(f_X) = \operatorname{Crit}(\exp tX)$. q.e.d.

We now compute the Hessian $\mathscr{H}_X$ of f_X at $g = e$. To do this, let $H_1, H_2 \in m$

and differentiate the expression $f_X((\exp sH_1)(\exp tH_2)e)$ at $t = s = 0$.

$$\begin{aligned}
&\frac{\partial^2}{\partial s \partial t}\Big|_{0,0} f_X((\exp s\, H_1)(\exp t\, H_2)) \\
&= \frac{\partial^2}{\partial s \partial t}\Big|_{0,0} \langle (e^{s \operatorname{ad} H_1} e^{t \operatorname{ad} H_2} X)_m, (e^{s \operatorname{ad} H_1} e^{t \operatorname{ad} H_2} X)_m \rangle \\
&= 2 \frac{\partial}{\partial s}\Big|_0 \left\langle \frac{\partial}{\partial t}\Big|_0 (e^{s \operatorname{ad} H_1} e^{t \operatorname{ad} H_2} X)_m, (e^{s \operatorname{ad} H_1} e^{t \operatorname{ad} H_2} X)_m \right\rangle \\
&= 2 \left\{ \left\langle \frac{\partial^2}{\partial s \partial t}\Big|_{0,0} (e^{s \operatorname{ad} H_1} e^{t \operatorname{ad} H_2} X)_m, X_m \right\rangle \right. \\
&\qquad \left. + \left\langle \frac{\partial}{\partial t}\Big|_{0,0} (e^{s \operatorname{ad} H_1} e^{t \operatorname{ad} H_2} X)_m, \frac{\partial}{\partial s}\Big|_{0,0} (e^{s \operatorname{ad} H_1} e^{t \operatorname{ad} H_2} X)_m \right\rangle \right\} .
\end{aligned}$$

Now $e^{t \operatorname{ad} H} = \cosh(t \operatorname{ad} H) + \sinh(t \operatorname{ad} H)$, and

$$\frac{d}{dt}\Big|_0 \cosh(t \operatorname{ad} H) = 0\,, \quad \frac{d}{dt}\Big|_0 \sinh(t \operatorname{ad} H) = \operatorname{ad} H\ ;$$

also,

$$\begin{aligned}
&(e^{s \operatorname{ad} H_1} e^{t \operatorname{ad} H_2} X)_m \\
&\quad = \{(\cosh(s \operatorname{ad} H_1) + \sinh(s \operatorname{ad} H_1))(\cosh(t \operatorname{ad} H_2) + \sinh(t \operatorname{ad} H_2))X\}_m \\
&\quad = \cosh(s \operatorname{ad} H_1) \cosh(t \operatorname{ad} H_2) X_m + \cosh(s \operatorname{ad} H_1) \sinh(t \operatorname{ad} H_2) X_K \\
&\qquad + \sinh(s \operatorname{ad} H_1) \cosh(t \operatorname{ad} H_2) X_K + \sinh(s \operatorname{ad} H_1) \sinh(t \operatorname{ad} H_2) X_m\,.
\end{aligned}$$

So,

$$\frac{\partial^2}{\partial s \partial t}\Big|_{0,0} (e^{s \operatorname{ad} H_1} e^{t \operatorname{ad} H_2} X)_m = \operatorname{ad} H_1 \operatorname{ad} H_2(X_m)\,,$$

and

$$\begin{aligned}
\frac{\partial}{\partial t}\Big|_{0,0} (e^{s \operatorname{ad} H_1} e^{t \operatorname{ad} H_2} X)_m &= \operatorname{ad} H_2(X_K)\,, \\
\frac{\partial}{\partial s}\Big|_{0,1} (e^{s \operatorname{ad} H_1} e^{t \operatorname{ad} H_2} X)_m &= \operatorname{ad} H_1(X_K)\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{\partial^2}{\partial s \partial t}\Big|_{0,0} f_X((\exp sH_1)(\exp tH_2)) \\
&\qquad = 2(\langle \operatorname{ad} H_1 \operatorname{ad} H_2(X_m), X_m \rangle + \langle \operatorname{ad} H_2(X_K), \operatorname{ad} H_1(X_K) \rangle)\,.
\end{aligned}$$

For any $Z \in G$, ad Z is skew-symmetric with respect to $\langle\, ,\, \rangle$ so the above becomes:

$$\begin{aligned} &2(\langle \operatorname{ad} X_K(H_2), \operatorname{ad} X_K(H_1)\rangle - \langle \operatorname{ad} X_m(H_2), \operatorname{ad} X_m(H_1)\rangle) \\ &\quad = 2(\langle (\operatorname{ad} X_m)^2 H_2, H_1\rangle - \langle (\operatorname{ad} X_K)^2 H_2, H_1\rangle) \\ &\quad = 2\langle ((\operatorname{ad} X_m)^2 - (\operatorname{ad} X_K)^2) H_2, H_1\rangle\,. \end{aligned}$$

It is clear that $(\operatorname{ad} X_m)^2 - (\operatorname{ad} X_K)^2$ is symmetric with respect to $\langle\, ,\, \rangle$. Thus we have

(2.5.2) Theorem.

$$\begin{aligned} \mathscr{H}_X(H_1, H_2) &= (\text{Hessian of } f_X)(H_1 H_2) \\ &= 2\langle ((\operatorname{ad} X_m)^2 - (\operatorname{ad} X_K)^2) H_1, H_2\rangle\,. \end{aligned}$$

(2.5.3) Corollary. *If M is of non-compact type, then* Crit(f_X) *is a non-degenerate critical manifold. If M is of compact type and X is a regular element, then* Crit(f_X) *is also non-degenerate.*

Proof. The nullity of $\mathscr{H}_X$ is the nullity of the form $(\operatorname{ad} X_m)^2 - (\operatorname{ad} X_K)^2$. The proof of Theorem 2.5.1 shows that this is just $Z_m(X) =$ (centralizer of X in m) $=$ (tangent space of Crit(f_X) at $x = K$), if M is non-compact.

Now assume M is of compact type and $X \in B_r$, so that Crit$(f_X) =$ Crit$(\exp X)$. We assume X is a regular element of G, so $Z_G(X) =$ Cartan algebra containing X. Since we assume $x = K$ is in Crit(f_X), we have $[X, \sigma X] = 0$, so $Z_G(X) = Z_G(\sigma X)$. Now $H \in$ nullity of $\mathscr{H}_X$ if and only if $\operatorname{ad} X \operatorname{ad} \sigma X(H) = 0$; equivalently, $\operatorname{ad} \sigma X(H) \in Z_G(X) = Z_G(\sigma X)$, or $(\operatorname{ad}(\sigma X))^2 H = 0$. Similarly, $(\operatorname{ad} X)^2 H = 0$. Now

$$(\operatorname{ad} \sigma X)^2 H = (\operatorname{ad} X_K)^2 H + (\operatorname{ad} X_m)^2 H - 2 \operatorname{ad} X_K \operatorname{ad} X_m(H)\,,$$

and

$$(\operatorname{ad} X)^2 H = (\operatorname{ad} X_K)^2 H + (\operatorname{ad} X_m)^2 H + 2 \operatorname{ad} X_K \operatorname{ad} X_m(H)\,.$$

Therefore, $(\operatorname{ad} X_K)^2 H + (\operatorname{ad} X_m)^2 H = 0$. Since also $(\operatorname{ad} X_K)^2 H = (\operatorname{ad} X_m)^2 H$, we have $(\operatorname{ad} X_K)^2 H = 0$ and $(\operatorname{ad} X_m)^2 H = 0$, which implies that $\operatorname{ad} X_K(H) = 0$ and $\operatorname{ad} X_m(H) = 0$, so $H \in Z_G(X)$. Now every $H \in Z_G(X)$ is in the nullity of $\mathscr{H}_X$, so Crit(f_X), is non-degenerate.

(2.6) We now treat the Euclidean space $\boldsymbol{R}^n$. Let $E(n)$ be the Euclidean group of isometries of $\boldsymbol{R}^n$; then each $g \in E(n)$ is a pair $g = (A, v)$ for $A \in \mathrm{O}(n)$, $v \in \boldsymbol{R}^n$, and acts on $\boldsymbol{R}^n$ as follows: if $x \in \boldsymbol{R}^n$ then $gx = Ax + v$. $E(n)$ is a semi-direct product of $\mathrm{O}(n)$ with $\boldsymbol{R}^n$, so that $\boldsymbol{R}^n = E(n)/\mathrm{O}(n)$ is a Riemannian homogeneous space with normal metric. Furthermore, if $A \in \mathrm{O}(n)$, $v \in \boldsymbol{R}^n$ then $\operatorname{ad}(A)v = Av$. We now choose a particular isometry $g = (A, v)$ and find Crit(g). Note that since $\boldsymbol{R}^n$ has no cut points we can use Corollary 2.1.2 for any $g \in E(n)$. Assume that $\mathrm{Crit}(g) \neq \emptyset$, and choose $x \in \mathrm{Crit}(g)$ to be the origin of $\boldsymbol{R}^n$. Then we must have $v = \mathrm{ad}(A)v = Av$. Now let $h \in \boldsymbol{R}^n$ be any vector. Then

$$\begin{aligned} h\cdot g\cdot h^{-1}(y) &= h + g(y-h) \\ &= h + v + A(y-h) = h + v - Ah + Ay\,, \end{aligned}$$

so that $h\cdot g\cdot h^{-1} = (A, h + v - Ah)$. Now $-h \in \mathrm{Crit}(g)$ if and only if $h + v - Ah = A(h + v - Ah)$, that is, $h - 2Ah + A^2h = 0$, which means $(I - A)^2h = 0$.

Since $A \in \mathrm{O}(n)$, $A = \{A_1, \cdots, A_k, \underbrace{1, \cdots, 1}_{p}, -1, \cdots, -1\}$ with

$$A_i = \begin{pmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{pmatrix}, \quad \text{and} \quad \theta_i \neq n\pi\,.$$

Then

$$(A - I)^2 = \{(A_1 - I)^2, \cdots, (A_k - I)^2, 0, \cdots, 0, 4, \cdots, 4\}\,.$$

If $(A - I)^2h = 0$ for $h = (h_1, \cdots, h_n)$, we must have

$$(A_i - I)^2\begin{pmatrix} h_{2i-1} \\ h_{2i} \end{pmatrix} = 0 \qquad \text{for } i = 1, \cdots, k\,,$$

and $h_{2k+p+j} = 0$ for $j = 1, \cdots, n - 2k - p$. Now $\det(A_i - I)^2 = (\det(A_i - I))^2 = ((\cos\theta_i - 1)^2 + \sin^2\theta_i)^2$, and this is zero only when $\theta_i = n\pi$ which is impossible. Therefore, $h_j = 0$ for $j = 1, \cdots, 2k$, and we have $Ah = h$. Conversely, $Ah = h$ clearly implies $A(h + v - Ah) = h + v - Ah$, so $\mathrm{Crit}(g) = Z_{E(n)}(g)\cdot x$. Hence we have proved

(2.6.1) Theorem. *Let M be a Euclidean space and $g\colon M \to M$ any isometry. If $x \in \mathrm{Crit}(g)$, then $\mathrm{Crit}(g) = Z_{E(n)}(g)\cdot x$.*

(2.7) Suppose $M = M' \times M''$ is the Riemannian product of Rimannian manifolds M' and M'', and let g be the product metric on M. Suppose $f\colon M \to M$ is an isometry of small displacement satisfying $f = f' \times f''$, where $f'\colon M' \to M'$ and $f''\colon M'' \to M''$ are isometries, and let $b(s)$ be a curve through some point $x = (x', x'') \in M$. Then $\delta_f(b(s)) = \int_0^a \sqrt{g(T,T)}dt$, where T is as defined in (1.2).

$$\begin{aligned} \frac{d}{ds}\delta_f(b(s)) &= \int_0^a \frac{\partial}{\partial s}\sqrt{g(T',T') + g(T'',T'')}dt \\ &= \int_0^a \frac{g(\nabla_{X'}T',T') + g(\nabla_{X''}T'',T'')}{\sqrt{g(T,T)}}dt\,, \end{aligned}$$

where T', X' and T'', X'' are the components of $Q^*\partial/\partial t, Q_*\partial/\partial s$ in $T(M')$

and $T(M'')$ respectively. From this it is clear that the derivative vanishes at x for all values of $X' + X''$ exactly when $x' \in \mathrm{Crit}\,(f')$ and $x'' \in \mathrm{Crit}\,(f'')$.

(2.7.1) Theorem. *Let M be a simply connected Riemannian symmetric space with $M = M_0 \times M_1 \times \cdots \times M_p$, where M_0 is a Euclidean space and the M_i, $1 \leq i \leq k$, are irreducible. Suppose $g \in I^0(M) = I^0(M_0) \times I^0(M_1) \times \cdots \times I^0(M_k)$, and the components g_i of g acting on the M_i which are compact satisfy the hypotheses of Theorem* 2.3.2. *If $x \in$* Crit (g), *then the component of* Crit (g) *containing x is $Z^0_{I^0(M)}(g) \cdot x$.*

Proof. From the above remarks we see that $\mathrm{Crit}\,(g) = \mathrm{Crit}\,(g_0) \times \mathrm{Crit}\,(g_1) \times \cdots \times \mathrm{Crit}\,(g_k)$. Then the result follows from Theorems 2.3.2, 2.4.1, and 2.5.1.

(2.7.2) Lemma. *Let $\bar{M} \xrightarrow{\pi} M$ be a Riemannian covering of Riemannian manifolds with simply connected $\bar{M}$, and $f\colon M \to M$ an isometry of small displacement. Then there is a unique lift $\bar{f}\colon \bar{M} \to \bar{M}$ of f which is an isometry covering f, and such that $\rho(x, f(x)) = \bar{\rho}(\bar{x}, \bar{f}(\bar{x}))$ for all $\bar{x}$ such that $\pi(\bar{x}) = x$, $x \in M$. (Here $\bar{\rho}$ is the distance on $\bar{M}$.)*

Proof. For each $x \in M$ and each $\bar{x} \in \bar{M}$ where $\pi(\bar{x}) = x$, let c_x be the minimizing geodesic from x to $f(x)$, and $c_{\bar{x}}$ the lift of c_x to $\bar{M}$ starting at $\bar{x}$. Then define $\bar{f}(\bar{x}) =$ endpoint of $\bar{c}_{\bar{x}}$ over $f(x)$. f obviously covers f so it is an isometry of M. Moreover, $\bar{c}_x$ is a geodesic which minimizes the distance from $\bar{x}$ to $\bar{f}(\bar{x})$ and has the same length as c_x, so $\rho(x, f(x)) = \bar{\rho}(\bar{x}, \bar{f}(\bar{x}))$. q.e.d.

Now if Γ is the group of deck transformations of $\bar{M} \xrightarrow{\pi} M$, it is evident that Γ preserves $\mathrm{Crit}\,(\bar{f})$, so that $\mathrm{Crit}\,(f) = \mathrm{Crit}\,(\bar{f})/\Gamma$.

(2.7.3) Corollary. *Let M be a connected Riemannian symmetric space, and g an isometry whose lifting $\bar{g}$ satisfies the hypotheses of Theorem* 2.7.1. *If $x \in$* Crit (g), *then the comyonent of* Crit (g) *containing x is $Z^0_{\bar{G}}(g) \cdot \bar{x}/\Gamma$, where $\pi(\bar{x}) = x$, and $\tilde{G}$ is the isometry group of $\bar{M}$.*

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