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THE TOPOLOGY OF TAUT RIEMANNIAN MANIFOLDS WITH POSITIVE RISEC PINCHING

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0. Introduction

Suppose N is a smooth compact Riemannian manifold of dimension n. Geometers have long known that innocuous-appearing restrictions on the set $K(N)$ of sectional curvatures of N can lead to strong implications for the topology of N. For example, if $n = 2$ and $K(N) > 0$, then the classical Gauss-Bonnet theorem [9] leads to the conclusion that N is homeomorphic to the 2-sphere S^2 or the real projective plane P^2. Bishop and Goldberg [2] show how to derive further information from the generalized Gauss-Bonnet formula in higher dimensions. Again, if n is even, N is orientable, and $K(N) > 0$, then the fundamental group $\pi_1(N)$ is trivial [14]. A most remarkable series of advances have recently culminated in the Sphere Theorem [5]: if $K(N) \subset (k, 1]$ where $k > 1/4$, and $\pi_1(N)$ is trivial, then N is homeomorphic to the n-sphere S^n.

Berger and Bott [1] have studied the sums of the Betti numbers (over an arbitrary coefficient field) of the loop space of N, assuming that the sectional curvatures are k-pinched. They show that the sum of the Betti numbers of dimension at most λ is majorized by an exponential-polynomial expression in λ. Under certain conditions causing the vanishing of lower-order Betti numbers of N, it is possible to derive estimates for sums of the Betti numbers of N itself.

We consider here the possibility that the sectional curvatures of N can take on negative values, but with the proviso that the Ricci curvatures are always positive. This leads us to replace the concept of pinching by two new numerical characters of the metric, namely, the *tautness* and *risec pinching*; these numbers are defined precisely in § 1. It is then possible to alter the method of [1] to get new estimates for sums of Betti numbers, given below in Theorem 5.3. We apply this estimate to examine γ-holomorphically pinched Kähler manifolds M in the case where $c_d < \gamma < 2/3$ (c_d is some constant depending on the complex dimension d of M). In this range for γ, M will have positive Ricci curvatures and may have negative sectional curvatures. Known information about γ-holomorphically pinched Kähler manifolds is scarce and typically restricted to the case when γ is closer to 1 (e.g. $\gamma > 4/5$ in [3]).

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We also consider a new situation in which a pair (M, N) is given, where N is a closed embedded totally-geodesic hypersurface of M, both manifolds being orientable. We study the space $\Omega(*, N)$ of paths in M from a fixed point to N, obtaining in Theorem 5.2 a majorization of the sum up to dimension λ of the Betti numbers of $\Omega(*, N)$ in terms of tautness and risec pinching. In case that M is a homology sphere, we derive in Corollary 6.3 a majorization for the Euler characteristic $\chi(N)$. Currently, it is not known whether the exotic sphere-structures admit Riemannian metrics of strictly positive curvature.

1. Notation and preliminary matters

All manifolds to be considered will be smooth and connected, and all Riemannian metrics will be smooth and complete. Let M be a compact manifold of dimension m carrying a Riemannian metric $\langle \cdot, \cdot \rangle$ with associated norm $\|\cdot\|$. Let N be a compact orientable embedded hypersurface in M with normal bundle $\perp(N)$ which carries the induced Riemannian and norm structures, again denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$. Let exp be the exponential map of the Levi-Civita connection ∇ of $\langle \cdot, \cdot \rangle$ on M, and denote the restriction of exp to $\perp(N)$ by $\exp^\perp$. Let

$$\perp_s(N) = \{y \in \perp(N) \mid \|y\| < s\} .$$

An element $y \in \perp(N)$ is called *focal* for N if it is a critical point of $\exp^\perp$: there is an element z in the tangent space $\perp(N)_y$ such that $d\exp^\perp(z) = 0$. The point $p = \exp^\perp y$ is also called *focal* for N, and there are two *focal loci*

$$\mathrm{Foc}^\perp(N) \quad \text{and} \quad \mathrm{Foc}^M(N)$$

consisting of those points in $\perp(N)$ and M, respectively, which are focal for N.

There is an alternate way to describe the focal points, which we will find indispensible. Any point $p \in \mathrm{Foc}^M(N)$ can be obtained as follows. Let

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

be the curvature transformation of ∇, and σ an N-geodesic from N to p, that is, σ: $[0, s] \to M$ be a geodesic parametrized by arc-length with $\sigma_*(0) \in \perp(N)$ and $\sigma(s) = p$. A *Jacobi field* along σ is a vector field Y along σ and orthogonal to σ satisfying the *Jacobi differential equation*

$$Y'' + R(\sigma_*, Y)\sigma_* = 0 ,$$

the primes denoting two successive covariant differentiations along σ. Let $S_{\sigma_*(0)}$ represent the second fundamental form of N in the direction $\sigma_*(0)$, operating as a symmetric linear endomorphism on the tangent space $N_{\sigma_*(0)}$. Then $p \in M$ is focal for N if and only if there is an N-geodesic σ: $[0, s] \to M$ from N to p

and a Jacobi field Y along σ satisfying the initial condition

$$S_{\sigma_*(0)}Y(0) - Y'(0) = 0$$

and the terminal condition $Y(s) = 0$.

If M is an orientable manifold, it bears a distinguished exterior m-form v called the *Riemannian volume element* (with respect to a chosen fixed orientation). In fact v may be uniquely determined as follows: if $y_1, \cdots, y_m \in M_p$ are orthonormal and form a positively-oriented frame, then $v(y_1, \cdots, y_m) = 1$. Trivially,

$$|v(y_1, \cdots, y_m)| \leq \|y_1\| \cdots \|y_m\|$$

for any m-tuple $(y_1, \cdots, y_m)$. We set $w = (\exp^\perp)^* v$, the pullback of v to $\perp(N)$ by the map $\exp^\perp$.

The *sectional curvature* K is defined by the equation

$$K(x, y)\|x \wedge y\|^2 = \langle R(x, y)x, y\rangle$$

for each linearly independent pair $x, y \in M_p$. If x and y are linearly dependent, define $K(x, y) = 0$. The value of $K(x, y)$ depends only upon the two-dimensional subspace of M_p determined by x and y. The symbol $K(M)$ will denote the set of values taken by K in $\boldsymbol{R}$ as its argument runs through all two-planes tangent to M. Suppose that $-a^2 \leq K(M) \leq b^2$. Then the ratio $\rho^2 = a^2/b^2$ will be called the *tautness* of the metric, and M will be said to be *ρ^2-taut*.

For each $x \in M_p$ with $\|x\| = 1$, there is a linear endomorphism L_x of M_p defined by $L_x(y) = R(x, y)x$. The *Ricci curvature in the direction* x is the number $Ri(x) = (\text{trace } L_x)/(m-1)$. It may be simply expressed in terms of sectional curvatures. Extend x to an orthonormal basis $(x, x_2, x_3, \cdots, x_m)$ of M_p. Then

$$Ri(x) = \frac{1}{m-1} \sum_{j=2}^{m} K(x, x_j) .$$

The symbol $Ri(M)$ will denote the set of values taken by Ri in $\boldsymbol{R}$ as its argument runs through the set of unit vectors tangent to M. It is clear that $A \leq K(M) \leq B$ implies that $A \leq Ri(M) \leq B$. We shall be interested in the case that $-a^2 \leq K(M) \leq b^2$ and $Ri(M) \geq \delta^2 > 0$. In this case, we define the *risec pinching* of M to be the number $k^2 = \delta^2/b^2$.

2. Morse Theory for totally-geodesic hypersurfaces

We now specialize to the case where N is a compact *totally-geodesic* hypersurface of M. (For typographical convenience, let $n =$ dimension N. It is to be understood always that $n = m - 1$.) The second fundamental forms of N vanish identically. Let σ: $[0, s] \to M$ be an N-geodesic with $\sigma(s) = p \in M$, and $\mathscr{L}$ the linear space of continuous piecewise-smooth vector fields along σ and orthogonal to it and satisfying the end conditions $Y'(0) = 0$ and $Y(s) = 0$. The *index form* I on σ is the quadratic form whose value on $Y \in \mathscr{L}$ is

$$I(Y) = \int_0^s \{\|Y'\|^2 - \langle R(\sigma_*, Y)\sigma_*, Y\rangle\}|_u \, du .$$

The *index* of σ is the dimension of a maximal subspace of $\mathscr{L}$ on which I is negative definite.

Notice that if y moves in $\perp(N)$ without encountering the focal locus $\mathrm{Foc}^{\perp}(N)$, the index of the geodesic exp $([0, y])$ does not change.

Let $i^{\perp}(p; \lambda)$ be the number of N-geodesics terminating at p and having index λ. Denote by $\Omega(p, N)$ the space of paths in M originating at p and terminating on N, topologizing $\Omega(p, N)$ by the Frechet topology (which is equivalent to the compact-open topology). Let $b_\mu(\Omega(p, N))$ be the μth Betti number of $\Omega(p, N)$ over a fixed (commutative) field. It will follow from the results of § 3 that each $b_\mu(\Omega(p, N))$ is finite. Then we have the following *Morse inequalities* [11] valid if $p \notin \mathrm{Foc}^M(N)$:

$$i^{\perp}(p; \lambda) \geq b_\lambda(\Omega(p, N)) \tag{2.1}$$

for every integer $\lambda \geq 0$. Set

$$c_\lambda = \sum_{\mu=0}^{\lambda} b_\mu(\Omega(p, N)) , \tag{2.2}$$

$$g^{\perp}(p; \lambda) = \sum_{\mu=0}^{\lambda} i^{\perp}(p; \lambda) . \tag{2.3}$$

Then $g^{\perp}(p; \lambda)$ will be the number of N-geodesics of index $\leq \lambda$ terminating at p. From the Morse inequalities (2.1), we have immediately

$$g^{\perp}(p; \lambda) \geq c_\lambda$$

for each integer $\lambda \geq 0$, so long as $p \notin \mathrm{Foc}^M(N)$.

Proposition 2.1. *Let M be a Riemannian manifold of dimension $m = n + 1$ such that $Ri(M) \geq \delta^2 > 0$, N a compact totally-geodesic embedded hypersurface of M, and σ: $[0, s] \to M$ an N-geodesic of M of length s terminating at $p \in M$, where $p \notin \mathrm{Foc}^M(N)$. If*

$$s \geq \frac{\pi}{2\delta} + h\frac{\pi}{\delta} ,$$

then the index of $\sigma \geq h + 1$.

Proof. It is enough to construct a subspace $\mathscr{K}$ of $\mathscr{L}$ of dimension $h + 1$ on which I is negative semi-definite. The restriction that $p \notin \mathrm{Foc}^M(N)$ will then imply that I is negative definite on $\mathscr{K}$.

By a well-known theorem of Myers [12], every segment of σ of length $\geq \pi/\delta$ contains a pair of mutually conjugate points. Also, by an obvious modification of the proof of Myers' theorem, the segment $\sigma((0, \pi/2\delta])$ contains a point focal to N. Therefore, we may find Jacobi fields $Z_0, Z_1, \cdots, Z_n$ in $\mathscr{L}$ with $Z_0(t) = 0$ for $t \notin [0, \pi/2\delta]$ and $Z_i(t) = 0$ for $t \notin [\pi/2\delta + (i-1)\pi/\delta, \pi/2\delta + i\pi/\delta]$ and $1 \leq i \leq h$, and we will have $I(Z_j) < 0$ for $0 \leq j \leq h$. The desired subspace $\mathscr{K}$ is generated by $Z_0, Z_1, \cdots, Z_h$.

3. Two integrals

From this point on, we assume that M and N are oriented. The Riemannian volume element v of M induces a measure on M. Since M is compact its total measure will be a finite number to be denoted by $|M|$. Moreover, the induced Riemannian structure on the hypersurface N leads to a measure on N. Since N is compact, the measure of N will be a finite number to be denoted by $|N|$. The symbol $|\cdot|$ applied to various sets will always denote the appropriate measure. Each ray in $\perp(N)$ (properly, in a fiber of $\perp(N)$) meets the focal locus $\mathrm{Foc}^{\perp}(N)$ in isolated points only, so that $|\mathrm{Foc}^{\perp}(N)| = 0$ in $\perp(N)$. Since $\exp^{\perp}$ is smooth and $\dim \perp(N) = \dim M$, the image $\mathrm{Foc}^{M}(N) = \exp^{\perp}[\mathrm{Foc}^{\perp}(N)]$ again has zero measure as a subset of M.

We now give two propositions whose proofs are *mutatis mutandis* the same as the proofs of Propositions (4.1) and (4.2) of [1]. Therefore the proofs are omitted.

Suppose $N_s = \exp^{\perp}[\perp_s(N)]$, $Q_s^{\perp} = \perp_s(N) - (\perp_s(N) \cap \mathrm{Foc}^{\perp}(N))$, and $Q_s^M = \exp Q_s^{\perp}$. Finally, let $\exp_s^{\perp}$ denote the restriction of $\exp^{\perp}$ to $Q^{\perp}$.

Proposition 3.1. *The set $Q_s^{\perp}$ (resp. Q_s^M) is open in $\perp_s(N)$ (resp. in M), and $|Q_s^M| = |N_s|$.*

Proposition 3.2. *The mapping $\exp_s^{\perp} \to Q_s^M$ is a covering having everywhere a finite number of sheets.*

Now we can define the first of the two integrals. Since Q_s^M is open in M, it is measurable, as is each of its connected components. On each of its components U_β (where $Q_s^M = \cup_\beta U_\beta$ is the decomposition of Q_s^M into connected components) the number of sheets of the covering $\exp_s^{\perp}\colon Q_s^{\perp} \to Q_s^M$ is finite and constant. Denote the number of sheets above p by $f^{\perp}(p; s)$. It is important to notice that $f^{\perp}(p; s)$ is just the number of N-geodesics of length $\leq s$ terminating at p. Therefore the integral

$$I^{\perp}(s) = \int_{Q_s^M} f^{\perp}(p; s) \cdot v$$

exists. Moreover, $Q_s^{\perp}$ is open in $\perp(N)$, so measurable, and the integral $\int_{Q_s^{\perp}} |w|$ exists (where $w = (\exp^{\perp})^* v$).

Proposition 3.3. $I^{\perp}(s) = \int_{Q_s^{\perp}} |w|$.

Proof. Each connected component U_β of Q_s^M is measurable and the number of sheets $q(\beta)$ above U_β is finite and equal to $f^{\perp}(p; s)$ for any $p \in U_\beta$. Therefore

$$\int_{U_\beta} f^{\perp}(p; s) \cdot v = q(\beta) |U_\beta| ,$$

and

$$\int_{Q_s^M} f^{\perp}(p; s) \cdot v = \sum_\beta q(\beta) |U_\beta| .$$

On the other hand, $\{Q_s^{\perp} \cap (\exp_s^{\perp})^{-1}(U_\beta)\}$ defines a partition of $Q_s^{\perp}$ by open, therefore measurable, sets. Further, $\exp_s^{\perp}$ restricted to a connected component U_β and to $(\exp_s^{\perp})^{-1}(U_\beta)$ is a covering with $q(\beta)$ sheets. Since $w = (\exp^{\perp})^* v$, we have

$$\int_{(\exp_s^{\perp})^{-1}(U_\beta)} |w| = q(\beta) \int_{U_\beta} v = q(\beta) |U_\beta| ,$$

whence

$$\int_{Q_s^{\perp}} |w| = \sum \int_{(\exp_s^{\perp})^{-1}(U_\beta)} |w| = \sum q(\beta) |U_\beta|$$

$$= \int_{Q_s^M} f^{\perp}(p; s) \cdot v = I^{\perp}(s) . \qquad \text{q.e.d.}$$

Now define the second of our integrals by

$$J^{\perp}(\lambda) = \int_M g^{\perp}(p, \lambda) \cdot v .$$

Proposition 3.4. *Let M be an oriented Riemannian manifold of dimension $m = n + 1$ such that $Ri(M) \geq \delta^2 > 0$, and N a compact totally-geodesic oriented embedded hypersurface of M. For each integer $\lambda \geq 0$, let c_λ be defined by* (2.2), *and suppose*

$$s = \frac{\pi}{2\delta} + \frac{\lambda\pi}{\delta} .$$

Then

$$c_\lambda \leq |M|^{-1} \int_{\perp_s(N)} |w| .$$

Proof. By Proposition 2.1, each N-geodesic of length $\geq s = \pi/2\delta + \lambda\pi/\delta$ has index $\geq \lambda + 1 > \lambda$, whence any N-geodesic of index $\leq \lambda$ has length $\leq s$. Therefore, the definition (2.3) of $g^{\perp}(p; \lambda)$ leads immediately to

$$g^{\perp}(p; \lambda) \leq f^{\perp}(p; s) \qquad \text{for all } p \in Q_s^M . \tag{3.1}$$

We have already remarked that $g^{\perp}(p; \lambda)$ is constant on each connected component U_β of Q_s^M, so the integral

$$J^{\perp}(\lambda) = \int_{Q_s^M} g^{\perp}(p; \lambda)\cdot v$$

exists. As a direct consequence of the Morse inequalities, $J^{\perp}(\lambda) \geq c_\lambda |Q_s^M|$. From Proposition 3.1, $J^{\perp}(\lambda) \geq c_\lambda |N_s|$.

Let $CL^{\perp}(N)$ be the *cut-locus* of N in $\perp(N)$, that is, $CL^{\perp}(N)$ be the set of points $y \in \perp(N)$ such that the N-geodesic $\sigma(t) = \exp^{\perp}(ty)$ does not absolutely minimize distance to N for $t > 1$, but does absolutely minimize for $0 < t < 1$. Let $CL^M(N) = \exp^{\perp}[CL^{\perp}(N)]$. Since $CL^{\perp}(N)$ intersects each ray of $\perp(N)$ at most once, $|CL^{\perp}(N)| = 0$ in $\perp(N)$. Since $\exp^{\perp}$ is smooth and $\dim \perp(N) = \dim M$, $|CL^M(N)| = 0$. But each point of M is joined to N by a minimal N-geodesic, which must have index 0. Therefore,

$$|M| \geq |Q_s^M| \geq |M - CL^M(N)| = |M| ,$$

so we have

$$J^{\perp}(\lambda) \geq c_\lambda |M| .$$

But (3.1) implies that $I^{\perp}(s) \geq J^{\perp}(\lambda)$, and Proposition 3.3 shows that

$$I^{\perp}(s) = \int_{Q_s^{\perp}} |w| .$$

Further, $Q_s^{\perp} \subset \perp_s(N)$ so that

$$c_\lambda |M| \leq \int_{\perp_s(N)} |w| .$$

4. Estimation of Jacobi fields

We now suppose that $-a^2 \leq K(M) \leq b^2$, and let $\rho^2 = a^2/b^2$ be the tautness of M (we do not assume that ρ^2 has the best possible value for the given metrization of M). Let $D = \pi/2b$. The proof of the first of the following two theorems may be found in [5] and the proof of the second is similar.

Rauch comparison theorem. *Let Y be a Jacobi field with $Y(0) = 0$. Then for $0 \leq t \leq 2D$,*

$$\frac{\sin bt}{b}\|Y'(0)\| \leq \|Y(t)\| \leq \frac{\sinh at}{a}\|Y'(0)\| . \tag{4.1}$$

Berger comparison theorem. *Let Y be a Jacobi field with $Y'(0) = 0$. Then for $0 \leq t \leq D$,*

$$\cos bt \cdot \|Y(0)\| \leq \|Y(t)\| \leq \cosh at \cdot \|Y(0)\| . \tag{4.2}$$

Lemma 4.1. *For any Jacobi field Y, we have*

$$\|Y(D)\| \leq \cosh aD \cdot \|Y(0)\| + \frac{\sinh aD}{a}\|Y'(0)\| , \tag{4.3}$$

$$\|Y(D)\| \geq \frac{1}{b}\|Y'(0)\| - \cosh aD \cdot \|Y(0)\| , \tag{4.4}$$

and, for any $t \in [0, D]$,

$$\|Y(t)\| \leq \cosh aD \cdot \|Y(0)\| + \frac{\sinh aD}{a}\|Y'(0)\| . \tag{4.5}$$

Proof. Given Y, define Jacobi fields Y_1 and Y_2 by $Y_1(0) = 0$, $Y_1'(0) = Y'(0)$ and $Y_2(0) = Y(0)$, $Y_2'(0) = 0$. Then $Y = Y_1 + Y_2$. For $0 \leq t \leq D$, (4.1) yields

$$\|Y_1(t)\| \leq \frac{\sinh at \cdot \|Y'(0)\|}{a} ,$$

and (4.2) yields

$$\|Y_2(t)\| \leq \cosh at \cdot \|Y(0)\| .$$

Since $\|Y(t)\| \leq \|Y_1(t)\| + \|Y_2(t)\|$, and $\cosh at$ and $\sinh at/a$ are monotone in t, (4.5) and (4.3) follow from putting $t = D$ in (4.5).

From (4.1),

$$\|Y_1(D)\| \geq \frac{\sin bD \cdot \|Y'(0)\|}{b} = \frac{\|Y'(0)\|}{b} .$$

From (4.2),

$$\|Y_2(D)\| \leq \cosh aD \cdot \|Y(0)\| .$$

Since $\|Y(t)\| \geq \|Y_1(t)\| - \|Y_2(t)\|$,

$$\|Y(D)\| \geq \frac{1}{b}\|Y'(0)\| - \cosh aD \cdot \|Y(0)\| ,$$

which is (4.4).

Lemma 4.2. *Suppose* $Y_1(0) = 0$. *Then*

$$\|Y_1(D)\| \le \frac{b}{a} \cosh aD \sinh aD \cdot \|Y_1'(0)\| \ .$$

Suppose $Y_2'(0) = 0$. *Then*

$$\|Y_2'(D)\| \le b(1 + \cosh^2 aD)\|Y_2(0)\| \ .$$

Proof. Define Y by $Y(x) = Y_1(D - x)$ for all x. From (4.4),

$$\|Y_1(0)\| = \|Y(D)\| \ge \frac{\|Y'(0)\|}{b} - \cosh aD \cdot \|Y(0)\| \ .$$

But $\|Y_1(0)\| = 0$ by hypothesis, so

$$\|Y_1'(D)\| \le b \cosh aD \cdot \|Y_1(D)\| \ .$$

From (4.1),

$$\|Y_1(D)\| \le \frac{\sinh aD \cdot \|Y_1'(0)\|}{a} \ ,$$

so

$$\|Y_1'(D)\| \le \frac{b}{a} \cosh aD \sinh aD \cdot \|Y_1'(0)\| \ ,$$

which is the first inequality.

Next, define Y by $Y(x) = Y_2(D - x)$ for all x. From (4.4),

$$\begin{aligned} \|Y_2(0)\| = \|Y(D)\| &\ge \frac{1}{b} \|Y'(0)\| - \cosh aD \cdot \|Y(0)\| \\ &= \frac{1}{b} \|Y_2'(D)\| - \cosh aD \cdot \|Y_2(D)\| \ . \end{aligned}$$

Therefore,

$$\|Y_2'(D)\| \le b\|Y_2(0)\| + b \cosh aD \cdot \|Y_2(D)\| \ .$$

From (4.2), $\|Y_2(D)\| \le \cosh aD \cdot \|Y_2(0)\|$, whence

$$\|Y_2(D)\| \le b(1 + \cosh^2 aD)\|Y_2(0)\| \ ,$$

which is the second inequality.

Let us note that for any Jacobi field Y,

$$\|Y'(D)\| \le \|Y_1'(D)\| + \|Y_2'(D)\| \ . \tag{4.6}$$

Using Lemma 4.2,

$$\|Y'(D)\| \leq b(1 + \cosh^2 aD)\|Y(0)\| + \frac{b}{a}\cosh aD \sinh aD \cdot \|Y'(0)\| . \tag{4.7}$$

Lemma 4.3. *Let a_{11}, a_{12}, a_{21}, a_{22} be non-negative real numbers, and x_k, y_k be defined recursively by*

$$x_{k+1} = a_{11}x_k + a_{12}y_k, \qquad y_{k+1} = a_{21}x_k + a_{22}y_k , \tag{4.8}$$

with initial values x_0, y_0 both non-negative real numbers. Suppose that $A = \max(a_{11} + a_{12}, a_{21} + a_{22})$. Then

$$\max(x_k, y_k) \leq A^k \max(x_0, y_0) . \tag{4.9}$$

Proof. Introduce a norm in $\boldsymbol{R}^2$ by $\|(x, y)\| = \max(|x|, |y|)$. Equations (4.8) come from a linear endomorphism $T\colon \boldsymbol{R}^2 \to \boldsymbol{R}^2$ which is easily seen to have operator norm $\|T\| \leq A$. The inequality (4.9) is just a translation of the inequality

$$\|T^k(x, y)\| \leq A^k\|(x_0, y_0)\| .$$

Lemma 4.4. *Let Y be a Jacobi field such that $Y'(0) = 0$. Then, for any $t \geq 0$,*

$$\|Y(t)\| \leq \|Y(0)\|\left(\cosh\frac{\pi}{2}\rho + \frac{1}{\rho}\sinh\frac{\pi}{2}\rho\right) \cdot\left(1 + \cosh^2\frac{\pi}{2}\rho + \frac{1}{\rho}\cosh\frac{\pi}{2}\rho \sinh\frac{\pi}{2}\rho\right)^{[t/D]},$$

where $[w]$ stands for the greatest integer $\leq w$.

Proof. For any integer $k \geq 0$, put $f_k = \|Y(kD)\|$ and $g_k = \|Y'(kD)\|/b$. Apply (4.3) to the Jacobi field Z defined by $Z(x) = Y(x + kD)$ for all x. Then

$$\|Z(D)\| \leq \cosh aD \cdot \|Z(0)\| + \frac{\sinh aD}{a}\|Z'(0)\| .$$

But $Z(D) = Y((k + 1)D)$, $Z(0) = Y(kD)$, and $Z'(0) = Y'(kD)$, so

$$f_{k+1} \leq \cosh aD \cdot f_k + \frac{b}{a}\sinh aD \cdot g_k . \tag{4.10}$$

Next, apply (4.7) to Z, using $Z'(D) = Y'((k+1)D)$:

$$\|Z'(D)\| \leq b(1 + \cosh^2 aD)\|Z(0)\| + \frac{b}{a}\cosh aD \sinh aD \cdot \|Z'(0)\| ,$$

so

$$g_{k+1} \leq (1 + \cosh^2 aD)f_k + \frac{b}{a}\cosh aD \sinh aD \cdot g_k . \tag{4.11}$$

Now we can apply Lemma 4.3 with

$$A = 1 + \cosh^2 aD + \frac{b}{a}\cosh aD \sinh aD$$

and $f_0 = \|Y(0)\|$, $g_0 = 0$ to find

$$f_k, g_k \leq \|Y(0)\| \left(1 + \cosh^2 aD + \frac{b}{a}\cosh aD \sinh aD\right)^k . \tag{4.12}$$

For any t, $t = [t/D]D + x$, $0 \leq x < D$. Apply (4.5) to Z defined by $Z(v) = Y(v + [t/D]D)$ for all v. Then $Z(x) = Y(t)$, so

$$\|Y(t)\| = \|Z(x)\| \leq \cosh aD \cdot \|Z(0)\| + \frac{\sinh aD}{a}\|Z'(0)\| .$$

Therefore, we find

$$\|Y(t)\| \leq \cosh aD f_{[t/D]} + \frac{\sinh aD}{a} g_{[t/D]} .$$

Using (4.12),

$$\|Y(t)\| \leq \|Y(0)\| \left(\cosh aD + \frac{\sinh aD}{a}\right) \cdot \left(1 + \cosh^2 aD + \frac{b}{a}\sinh aD \cosh aD\right)^{[t/D]} .$$

The substitutions $\rho = a/b$ and $D = \pi/2b$ yield the claimed inequality.

5. Majorization of Betti numbers of $\Omega(p, N)$

In this section, we will assume that M is a compact oriented Riemannian manifold of dimension $m = n + 1$ with $-a^2 \leq K(M) \leq b^2$ and $Ri(M) \geq \delta^2 > 0$. We will let the tautness of M be denoted by $\rho^2 = a^2/b^2$ and the risec pinching of M by $k^2 = \delta^2/b^2$. As in the previous section $D = \pi/2b$. For brevity, we will set

$$A_\rho = \cosh \frac{\pi}{2}\rho + \frac{1}{\rho} \sinh \frac{\pi}{2}\rho ,$$

$$B_\rho = 1 + \cosh^2 \frac{\pi}{2}\rho + \frac{1}{\rho} \cosh \frac{\pi}{2}\rho \sinh \frac{\pi}{2}\rho .$$

Proposition 5.1. *Let N be a compact orientable smoothly-embedded totally-geodesic hypersurface of M. Then, for every $s \geq 0$,*

$$\int_{\perp_s(N)} |w| \leq \frac{2|N|DA_\rho^n B_\rho^{ns/D}}{n \log B_\rho} .$$

Proof. Let x be any vector of $\perp(N)$ and complete x to a positively-oriented orthonormal frame $\{x/\|x\|, y_1, \cdots, y_n\}$ in the tangent space $\perp(N)_x$ (which has a Euclidean metric). By the definition of w,

$$w(x/\|x\|, y_1, \cdots, y_n) = v(\|x\|^{-1} d\exp^\perp(x), d\exp^\perp(y_1), \cdots, d\exp^\perp(y_n)) .$$

But

$$\|d\exp^\perp(y_i)\| = \|Y_i(\|x\|)\| ,$$

where Y_i is the Jacobi field along $\sigma(t) = \exp^\perp(tx/\|x\|)$ such that $Y_i(0) = y_i$ and $y_i'(0) = 0$. From Lemma 4.4, we can obtain an estimate for $\|Y_i(\|x\|)\|$ and so give an estimate for $|w_x|$:

$$|w_x| \leq A_\rho^n B_\rho^{n[\|x\|/D]} .$$

Let $\bar{w}_x$ be the Euclidean volume in fibers of $\perp(N)$. Then

$$\int_{\perp_s(N)} |w| \leq \int_{\perp_s(N)} A_\rho^n B_\rho^{n[\|x\|/D]} \cdot |\bar{w}_x| .$$

Now let $\bar{v}$ be the volume element induced on N by the embedding. Since N is totally-geodesic we may write

$$\int_{\perp_s(N)} |w| \leq \int_{[-s,s]\times N} A_\rho^n B_\rho^{n[\|x\|/D]} \cdot |dt \wedge \bar{v}| .$$

By Fubini's theorem and the inequality $[u] \leq u$,

$$\int_{\perp_s(N)} |w| \leq 2|N|A_\rho^n \int_0^s B_\rho^{nt/D} dt = \frac{2|N|DA_\rho^n B_\rho^{ns/D}}{n \log B_\rho} .$$

q.e.d.

It is convenient to use the notation

$$\Omega_n = \int_0^\pi \sin^n \theta \, d\theta \ .$$

It is easy to get an asymptotic estimate for Ω_n by expressing it in terms of the Euler Γ-function:

$$\Omega_n = \sqrt{\pi} \frac{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)} \sim \sqrt{\frac{2\pi}{n}} \quad \text{as} \quad n \to \infty \ .$$

Theorem 5.2. *Let N be a compact orientable smoothly-embedded totally-geodesic hypersurface of M, and c_λ be defined by* (2.2). *Then*

$$c_\lambda \le \frac{\pi A_\rho^n B_\rho^{(2\lambda+1)n/k}}{n\Omega_n \log B_\rho} \ . \tag{5.1}$$

Proof. Apply Proposition 3.4 with $\delta = kb$. We will have

$$c_\lambda \le |M|^{-1} \int_{\perp_s(N)} |w|$$

with $s = \pi/2\delta + \lambda\pi/\delta$. Then $ns/D = (2\lambda + 1)n/k$ and this is the smallest exponent we may use in applying Proposition 5.1, obtaining

$$c_\lambda \le \frac{2|N|DA_\rho^n B_\rho^{(2\lambda+1)n/k}}{n|M| \log B_\rho} \ . \tag{5.2}$$

From the calculations on p. 261 of [8], we have $2|N|D/|M| \le \pi/\Omega_n$, which and (5.2) hence imply (5.1). q.e.d.

We remark that our hypotheses lead to an estimate for the sum of Betti numbers of dimension at most λ of the loop space of M. Following Berger and Bott [1], we define

$$a_\lambda = \sum_{\mu=0}^{\lambda} b_\mu(\Omega(M)) \ ,$$

where $\Omega(M)$ is the loop space of M, and the Betti numbers are taken over a fixed, but arbitrary, field of coefficients. Our assumptions are that M is ρ^2-taut and k^2-risec pinched. Since $Ri(M)$ is strictly positive, the fundamental group of M has order $|\pi_1(M)| < \infty$ by a theorem of Myers [12]. Denote by ω_j the volume of the unit sphere S^j in R^{j+1}. Then we note the relation

$$\frac{\omega_{m-1}}{\omega_m} = \frac{m\Omega_m}{2\pi} \ .$$

The following estimate is derived in a fashion similar to the derivation of Theorem (6.2) of [1].

Theorem 5.3. *Under the hypothesis that M is complete, ρ^2-taut, k^2-risec pinched, and of dimension m,*

$$a_\lambda \leq \frac{m(\lambda+1)\Omega_m}{2k|\pi_1(M)|} A_\rho^{m-1} B_\rho^{2(m-1)(\lambda+1)/k} .$$

The preceding estimates of this section remain valid when $K(M)$ is to be non-negative. This is the same as setting $a = 0$, whence $\rho = 0$. We have

$$\lim_{\rho\to 0} A_\rho = 1 + \frac{\pi}{2} ,$$

and

$$\lim_{\rho\to 0} B_\rho = 2 + \frac{\pi}{2} .$$

The following estimates now follow.

Corollary 5.4. *Suppose that $K(M) \subset [0, b]$ and $Ri(M) \geq \delta^2 > 0$, so that the risec pinching of M is $k^2 = \delta^2/b^2$, and let N be a totally-geodesic hypersurface of M, as before. Then*

$$a_\lambda(M) \leq \frac{m(\lambda+1)\Omega_m}{2k|\pi_1(M)|}\left(1+\frac{\pi}{2}\right)^{m-1}\left(2+\frac{\pi}{2}\right)^{2(m-1)(\lambda+1)/k} ,$$

and

$$c_\lambda(N) \leq \frac{\pi\left(1+\frac{\pi}{2}\right)^{n}\left(2+\frac{\pi}{2}\right)^{(2\lambda+1)n/k}}{n\Omega_n \log\left(2+\frac{\pi}{2}\right)} .$$

We close this section by explaining why we have replaced the condition of pinched sectional curvatures used in [1] and in common use in current investigations into the relation between curvature and topology by the condition of taut sectional curvatures. It was, in fact, the sectional pinching condition, which we had in mind when first considering the case of totally-geodesic hypersurfaces and bounds for c_λ, which can be obtained under this assumption. But more can be said in this case. If $K(M)$ is strictly positive, and N is a totally-geodesic hypersurface of M separating $M - N$ into two components U_1 and U_2, then each pair $(U_i \cup N, N)$ is diffeomorphic with the disc-sphere pair (D^m, S^{m-1}) in all dimensions where the h-cobordism theorem is valid. The method of proof is implicit in [7] and has been developed independently and explicitly by Gromoll and Meyer [6]. Thus, the pair (M, N) is homeomorphic with the pair (S^m, S^{m-1}) for $m \geq 5$. So, in particular, for $n \geq 5$, $\chi(N) = 0$ or 2.

6. Application: hypersurfaces of spheres

We again assume that M is a compact orientable Riemannian mannifold of dimension $m = n + 1$, leaving the curvature unrestricted for the present. We consider a compact orientable manifold N of dimension n and seek topological consequences of the assumption that N can be embedded into M as a *totally-geodesic* hypersurface. The basic topological framework is a certain fibration whose structure we now describe.

We have already described in § 2 the space $\Omega(p, N)$ of paths in M starting at $p \notin \mathrm{Foc}^M(N)$ and terminating in N. There is a natural projection $\Omega(p, N) \to N$ taking each path to its terminus. Therefore, we get a Serre Fibration

$$\begin{array}{ccc} \Omega(p, *) & \longrightarrow & \Omega(p, N) \\ & & \downarrow \\ & & N \end{array} \tag{6.1}$$

where $\Omega(p, *)$ is the space of paths from p to a fixed point in M. The homotopy type of $\Omega(p, *)$ is independent of the choice of the fixed point and, in fact, is the same as that of $\Omega(M)$, the loop space of M.

We have the following lemma, in which the homology groups and Betti numbers are taken over a fixed field.

Lemma 6.1. *Let the homology of M be that of an m-sphere: $H_*(M) \approx H_*(S^m)$, and N be any compact hypersurface of M. Then*

$$b_\mu(\Omega(*, N)) = b_\mu(N) \,, \qquad \mu \leq n - 1 \,.$$

Proof. It is well-known (cf. [13]) that $b_\mu(\Omega(M)) = 0$ for $1 \leq \mu \leq n$. By the Hurewicz isomorphism, $\pi_\mu(\Omega(M)) = 0$ for $\mu \leq n - 1$. The homotopy sequence of (6.1) yields

$$\pi_\mu(\Omega(*, N)) \approx \pi_\mu(N) \,, \qquad \mu \leq n - 1 \,.$$

An argument based on the mapping cylinder of $\Omega(p, N) \to N$ and the relative Hurewicz theorem [10, prob. V.D.] show that

$$H_\mu(\Omega(*, N)) \approx H_\mu(N) \quad \text{for} \quad \mu \leq n - 1 \,,$$

whence the lemma.

Theorem 6.2. *Let M be a complete orientable Riemannian manifold of dimension $m = n + 1$, which is ρ^2-taut and k^2-risec pinched. Suppose the* *homology of M over a fixed field is that of an m-sphere*: $H_*(M) \approx H_*(S^m)$, *and let N be a compact orientable totally-geodesic hypersurface of M. Then*

$$\sum_{\mu=0}^{n-1} b_\mu(N) \leq \frac{\pi A_\rho^{n-1} B_\rho^{n(2n-1)/k}}{(n-1)\Omega_{n-1} \log B_\rho} . \tag{6.2}$$

Proof. Lemma 6.1 and Theorem 5.2.

Corollary 6.3. *Let C be the bound obtained in* (6.2), *and n be even. Then the Euler characteristic of N has the same bound*: $|\chi(N)| \leq C$.

It is interesting to compare the results of this and the preceding section with results obtained by Flaherty. We have considered a manifold-hypersurface pair (M, N) in which M is relatively loosely restricted while N is tightly restricted, being totally geodesic. We obtain information about the Betti numbers of $\Omega(*, N)$ and subsequently (in Theorem 6.2) about the Betti numbers of N itself in dimensions less than that of N. It would be pleasing to be able to conclude that some of those Betti numbers vanish, as we could if the bound in (6.2) were less than $n - 1$. But within the limits set for the parameters ρ and k, there seems to be no way to force this conclusion.

Flaherty [4] deals with assumptions which are in a sense dual to our assumptions. He allows N to bend in M, but not too severely, requiring that N be locally convex and that its principal curvatures lie in some fixed interval $[0, b]$. To balance the freedom of N, M is required to have Riemannian pinching at least $\delta^2 > 1/4$, so that it is a sphere [5]. If $b < \cot \pi/4\delta$, Flaherty concludes that

$$\sum_{\mu=1}^{n-1} b_\mu(N) = 1 ,$$

so that N is a homotopy sphere (a standard sphere if $\dim N \neq 3, 4$).

It is tempting to try to blend our assumptions on (M, N) with assumptions resembling those used by Flaherty, and it is easy to visualize the forms of theorems one might hope to derive from such assumptions. But there are difficulties of a technical nature between the hope and the realization. We shall develop this situation further in a subsequent paper.

7. Application: holomorphically-pinched Kähler manifolds

Let M be a compact Kähler manifold of complex dimension d, so $m = 2d$ is the real dimension. Suppose that M is γ-holomorphically pinched (cf. [2] for the relevant definitions.) We may normalize the metric and assume that $\gamma \leq H(x) \leq 1$ for all tangent vectors x. We will determine restrictions on γ so that M is taut and risec pinched. The basic inequalities we use are given on p. 519 of [2], where we note that the definition of Ricci curvature differs from ours by a constant divisor. There are two cases to be distinguished, according as $d \leq 5$ or not. We have

$$Ri(M) \geq \begin{cases} \dfrac{(3d+1)\gamma - (d-1)}{4(2d-1)}, & \text{if } d \leq 5, \\ \dfrac{2(2d-1)\gamma - (2d-3)}{2(2d-1)}, & \text{if } d \geq 6. \end{cases} \tag{6.1}$$

Therefore,

$$Ri(M) > 0 \quad \text{if} \quad \gamma > \begin{cases} \dfrac{d-1}{3d+1}, & \text{for } d \leq 5, \\ \dfrac{2d-3}{2(2d-1)}, & \text{for } d \geq 6. \end{cases}$$

We know from [2] that

$$\frac{3\gamma - 2}{4} \leq K(M) \leq 1 .$$

Being interested only in the case where negative sectional curvatures can occur, we may as well *assume* $\gamma < 2/3$. Since $K(M) \leq 1$, we find that M is k^2-risec pinched, where k^2 is given in each case by the right-hand side of the inequality (6.1):

$$k^2 = \begin{cases} \dfrac{(3d+1)\gamma - (d-1)}{4(2d-1)}, & \text{if } d \leq 5, \\ \dfrac{2(2d-1)\gamma - (2d-3)}{2(2d-1)}, & \text{if } d \geq 6. \end{cases} \tag{6.2}$$

Next we calculate the tautness, which is at most $(2 - 3\gamma)/4$, finding that M is ρ^2-taut, where

$$\rho^2 = \begin{cases} \dfrac{3d+5}{4(3d+1)}, & \text{if } d \leq 5, \\ \dfrac{2d+5}{8(2d-1)}, & \text{if } d \geq 6. \end{cases} \tag{6.3}$$

We are now in a position to apply Theorem 5.3 to estimate for the loop space $\Omega(M)$ the quantity

$$a_\lambda = \sum_{\mu=0}^{\lambda} b_\mu(\Omega(M)) .$$

Theorem 7.1. *Let M be a compact Kähler manifold of complex dimension* *d whose curvatures are γ-holomorphically pinched with*

$$\frac{2}{3} > \gamma > \begin{cases} \dfrac{d-1}{3d+1}, & \text{if } d \leq 5, \\ \dfrac{2d-3}{2(2d-1)}, & \text{if } d \geq 6. \end{cases}$$

Then

$$a_\lambda \leq \frac{d(\lambda+1)\Omega_{2d}}{k} A_\rho^{2d-1} B_\rho^{2(2d-1)(\lambda+1)/k},$$

where k and ρ are given by (6.2) *and* (6.3), *and A_ρ and B_ρ are defined at the beginning of* § 5.

The proof follows by substitution in Theorem 5.3 with the additional remark that $\pi_1(M)$ is trivial ([2, p. 528]).

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