

CLOSED CONFORMAL VECTOR FIELDS

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1. Introduction

Let M be a connected m -dimensional Riemannian manifold with metric g . A vector field X on M is conformal if and only if

$$(1.1) \quad L_X g = - (2/m) \delta \xi \cdot g,$$

where L_X denotes the Lie derivation with respect to X , $\xi = g(X, \cdot)$ is the covariant form of X with respect to g , and $\delta \xi$ is the corresponding codifferential form of ξ . Let d be the exterior differential operator, and Q the Ricci operator which is defined on the 1-form ξ by $Q\xi = R_1(X, \cdot)$, R_1 being the Ricci tensor.

If M is a connected 2-dimensional Riemannian manifold with constant scalar curvature $S > 0$, and M admits a conformal non-Killing vector field, then M is globally isometric with a sphere (cf. [1]).

Next let M be a connected, compact Riemannian manifold of dimension $m > 2$ with constant scalar curvature $S > 0$. Then M is globally isometric with a sphere, if M admits a conformal non-Killing vector field X and any one of the following conditions is satisfied:

- (a) M is an Einstein space [6], [8],
- (b) the Ricci tensor is parallel [5],
- (c) trace Q^2 is constant [4],
- (d) $Q\delta\xi = k\delta\xi$ for some constant k [9],
- (e) $Q\delta C_0^*(M) \subset d\delta C_0^*(M)$, where $C_0^*(M)$ denotes the space of covariant forms of all conformal vector fields on M [1],
- (f) ξ is an exact form [4].

(a) and (e) have been proved independently. Conditions (d) and (e) are related to (f). If ξ is exact, then ξ vanishes at some point of M . The first theorem of this note is a generalization of (f).

Theorem 1. *Let M be a connected compact Riemannian manifold with the constant scalar curvature $S > 0$. Then M is globally isometric with a sphere if M admits a closed conformal vector field X which satisfies one of the conditions:*

- (i) the harmonic part h of ξ vanishes at some point of M ,
- (ii) X vanishes at some point of M ,

- (iii) $R_1(X, X) = (S/m)g(X, X)$ holds at some point of M ,
- (iv) $R_1(X, X)$ is non-negative.

Corollary 1. *Let M be connected, compact and orientable Riemannian manifold with positive constant scalar curvature. Assume that the Euler number is not equal to zero (m : even) and that M admits a closed conformal vector field. Then M is globally isometric with a sphere.*

Remark. In Theorem 1, any closed conformal vector field satisfies (i), if M has one of the properties:

- (v) Q is non-singular at each point of M ,
- (vi) for any function f , we have $C_0(M, g) \neq I_0(M, e^f g)$, where $C_0(M, \cdot)$ or $I_0(M, \cdot)$ denotes the identity component of the group of conformal transformations or isometries, respectively.

In §4 we consider sufficient conditions for M to admit a closed or exact conformal vector field. We denote by $A^r(M)$ the space of all r -forms on M and by $H^1(M)$ the space of all harmonic 1-forms on M .

Theorem 2. *Suppose that a compact orientable Riemannian manifold M admits a conformal non-Killing vector field. Then M admits closed conformal vector field if any one of the following conditions is satisfied:*

- (a) $Q\delta A^2(M) \subset \delta A^2(M)$,
- (b) for any $w \in A^1(M)$ such that $dw = 0$, we have $dQw = 0$,
- (c) $dQdA^0(M) = (0)$ and $dQH^1(M) = (0)$,
- (d) $\delta Q\delta A^2(M) = (0)$ and $dQH^1(M) = (0)$,
- (e) $dQd\delta C_0^*(M) = (0)$, $dQH^1(M) = (0)$ and S is constant.

Theorem 3. *Suppose that a compact orientable Riemannian manifold M admits a conformal non-Killing vector field. Then M admits a conformal vector field whose covariant form is exact if any one of the following conditions is satisfied:*

- (f) $QdA^0(M) \subset dA^0(M)$,
- (g) for any $w \in A^1(M)$ such that $\delta w = 0$, we have $dQw = 0$,
- (h) $dQdA^0(M) = (0)$ and $dQH^1(M) = (0)$,
- (i) $\delta Q\delta A^2(M) = (0)$ and $\delta QH^1(M) = (0)$,
- (j) $dQd\delta C_0^*(M) = (0)$, $\delta QH^1(M) = (0)$ and S is constant.

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2. Preliminaries

Let M be a compact orientable Riemannian manifold. By Yano's theorem [7] a vector field X with the covariant form ξ is a Killing vector field if and only if

$$(2.1) \quad \delta\xi = 0, \quad \langle \Delta\xi - 2Q\xi, \xi \rangle = 0,$$

where $\Delta = d\delta + \delta d$ is the Laplace-Beltrami operator, and $\langle \cdot, \cdot \rangle$ denotes

the global inner product. Subsequently Lichnerowicz [3] has shown that a necessary and sufficient condition for X to be a conformal vector field is

$$(2.2) \quad \Delta\xi + (1 - 2/m)d\delta\xi - 2Q\xi = 0,$$

or

$$(2.2)' \quad \langle \Delta\xi + (2 - 2/m)d\delta\xi - 2Q\xi, \xi \rangle = 0.$$

Now we assume that the scalar curvature S is constant. We can assume also that S is positive (cf. [3]). The relation $L_X S = 0$ for a conformal vector field X is equivalent to

$$(2.3) \quad \Delta\delta\xi = (m - 1)^{-1}S\delta\xi.$$

Let h be the harmonic part of the covariant form ξ of X . Then by the Hodge-de Rham decomposition theorem we have

$$(2.4) \quad \xi = d\delta\eta + \delta d\eta + h$$

for some 1-form η , and substitution of $\delta\xi = \delta d\delta\eta$ into (2.3) gives $\delta(d\Delta\delta\eta - (m - 1)^{-1}S d\delta\eta) = 0$. Therefore we get $\Delta d\delta\eta = (m - 1)^{-1}S d\delta\eta$. By $\Delta d\delta\eta = d\delta\xi$, we have

$$(2.5) \quad d\delta\eta = (m - 1)^{-1}S d\delta\xi.$$

3. Proof of Theorem 1

To prove Theorem 1 we can assume that $m > 2$ and that M is orientable (cf. [1]). Let X be a closed conformal vector field on M . Then by (2.2) we have

$$(3.1) \quad \Delta\xi = d\delta\xi = m(m - 1)^{-1}Q\xi.$$

Since $d\xi = 0$, by (2.4) and (2.5) the harmonic part h of ξ is

$$(3.2) \quad h = \xi - (m - 1)^{-1}S d\delta\xi.$$

Operating L_X to $\xi_i = g_{ij}X^j$ gives

$$(3.3) \quad d(g(X, X)) = L_X\xi = -2m^{-1}\delta\xi \cdot \xi.$$

Since ξ is closed we have $d\delta\xi \wedge \xi = 0$. Now let M_0 be the set of points where ξ vanishes, and let $M^* = M - M_0$. Then we get a C^∞ -function A^* on M^* such that

$$(3.4) \quad d\delta\xi = A^*\xi$$

on M^* . Similarly, we have a C^∞ -function B^* on M^* such that

$$(3.5) \quad dA^* = B^*\xi$$

on M^* . By (3.1) we have

$$(3.6) \quad Q\xi = (m-1)m^{-1}A^*\xi.$$

Lemma. *The function A^* on M^* is extendable to a continuous function A on M , and*

(A-1) *if $A = (m-1)^{-1}S$ at a point of M^* , then it holds on M ,*

(A-2) *if M^* has no point where $A = (m-1)^{-1}S$ holds, then $A < (m-1)^{-1}S$ holds on M , A takes negative values somewhere on M , and M_0 is empty.*

Proof. First we show that $(\delta\xi)_P \neq 0$ for any point P of M_0 when $M \neq M^*$. In fact, if $(\delta\xi)_P = 0$, then (1.1) shows that $(D_j\xi_i)_P = 0$ since ξ is closed, where D is the Riemannian connection by G . By (3.1) we have $(d\delta\xi)_P = 0$. Namely, we have $\xi_P = (D\xi)_P = (\delta\xi)_P = (d\delta\xi)_P = 0$. However, by the differential equations satisfied by conformal vector fields ξ must vanish on M . Therefore we have $(\delta\xi)_P \neq 0$ for any point P of M_0 . Then by (1.1) we have $(D_Y\xi)_P \neq 0$ for any non-zero tangent vector Y at P . This means that P is an isolated point. By (3.6) and the continuity of eigenvalues of Q we can extend A^* on M^* to a continuous function A on M . We operate δ to (3.4) and get

$$(3.7) \quad \delta\Delta\xi = -g(dA^*, \xi) + A^*\delta\xi$$

on M^* . By (2.3) and (3.5) we have

$$(3.8) \quad (m-1)^{-1}S\delta\xi = -B^*g(\xi, \xi) + A^*\delta\xi$$

on M^* . We solve (3.8) for B^* and substitute into (3.5). Then using (3.3) we have a differential equation

$$(3.9) \quad d(A^* - (m-1)^{-1}S) = -2^{-1}m(A^* - (m-1)^{-1}S)d(\log g(X, X))$$

on M^* . Thus, if $A = (m-1)^{-1}S$ at one point of M^* , then $A = (m-1)^{-1}S$ on M . It is known that $\langle Qh, h \rangle \leq 0$ holds for any harmonic 1-form, and so by (3.2) and (3.6) we have

$$\langle (1 - (m-1)S^{-1}A)^2 Q\xi, \xi \rangle = (m-1)m^{-1} \langle (1 - (m-1)S^{-1}A)^2 \xi, A\xi \rangle \leq 0.$$

This shows that if $A \neq (m-1)^{-1}S$ anywhere on M^* then A takes a negative value at some point on M^* , and hence $(m-1)^{-1}S > A$ on M^* . Then we can solve (3.9) for A as follows:

$$(3.10) \quad A = (m-1)^{-1}S - e^c(g(X, X))^{-m/2}$$

on M^* , where c is a constant. (3.10) means that A can not be continuous on M unless M_0 is empty.

Proof of Theorem 1. In each case we show that the harmonic part of ξ vanishes, and this is equivalent to the relation $A = (m - 1)^{-1}S$ by (3.2) and (3.4). (i) follows from the Lemma (A-1). (ii) follows from (A-2) and (A-1) of the Lemma, since M_0 is not empty. (iii) and (iv) follow also from the Lemma. q. e. d.

Condition (v) of the Remark in the introduction implies that the function A^* does not take value zero. By relations

$$0 = \langle \Delta h, \xi \rangle = \langle h, \Delta \xi \rangle = \langle (1 - (m - 1)S^{-1}A)A\xi, \xi \rangle ,$$

A can not be negative. Then we apply our Lemma.

Let $\xi \in C_0^*(M)$ be closed. If M_0 is non-empty, then ξ is exact. Assume condition (vi) and that M_0 is empty. Set $e^f = (g(X, X))^{-1}$, and define g' by $g' = e^f g$. Then we have $L_X g' = 0$. Next define ξ' by $\xi' = g'(X, \cdot) = (g(X, X))^{-1}\xi$. By (3.3) ξ' is also closed. Therefore ξ' must be a parallel vector field with respect to g' , and we have $C_0(M, g') = I_0(M, g')$ (cf. [2]). Since $C_0(M, g') = C_0(M, g)$, this contradicts the assumption.

4. Proofs of Theorems 2 and 3

It may be easily verified that (a), (b), (c) and (d) are equivalent, and (f), (j), (h) and (i) are equivalent. Assuming (c), we show that M admits a closed conformal vector field. Let (2.4) be the decomposition of ξ . Then by (2.2) we have

$$\begin{aligned} 0 &= \langle \Delta \xi + (1 - 2m^{-1})d\delta\xi - 2Q\xi, \delta d\eta \rangle \\ &= \langle \Delta d\delta\eta + (1 - 2m^{-1})d\delta d\delta\eta - 2Qd\delta\eta \\ &\quad - 2Qh + \Delta\delta d\eta - 2Q\delta d\eta, \delta d\eta \rangle . \end{aligned}$$

By (c) we have

$$(4.1) \quad \langle \Delta\delta d\eta - 2Q\delta d\eta, \delta d\eta \rangle = 0 .$$

By (2.1) $\delta d\eta$ defines a Killing vector field, and so $d\delta\eta + h$ is a closed conformal vector field. If S is constant, then $2Qd\delta\eta \in Qd\delta C_0^*(M)$ by (2.5). Thus (e) implies (4.1). This proves Theorem 2. To prove Theorem 3, assume (i). Then by (2.2) we have

$$\begin{aligned} &\langle \Delta d\delta\eta + (1 - 2m^{-1})d\delta d\delta\eta - 2Qd\delta\eta - 2Qh \\ &\quad + \Delta\delta d\eta - 2Q\delta d\eta, d\delta\eta \rangle = 0 . \end{aligned}$$

By (i) we get

$$\langle \Delta d\delta\eta + (1 - 2m^{-1})d\delta d\delta\eta - 2Qd\delta\eta, d\delta\eta \rangle = 0.$$

Hence $d\delta\eta$ defines a conformal vector field by (2.2)', which is exact. Case (j) is similar to (e).

Added in proof. Another proof for Theorem 1 appears in W. C. Weber & S. I. Goldberg, *Conformal deformations of Riemannian manifolds*, Queen's Papers in Pure and Applied Mathematics, No. 16, Queen's University, Kingston, Ontario, Canada, 1969.

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