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HOMOGENEOUS EINSTEIN SPACES OF DIMENSION FOUR

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Introduction

Given a manifold M with a Riemannian metric g_{ij}, Riemannian curvature tensor R_{ijkl}, and Ricci tensor $R_{ij} = \sum_k R_{kikj}$, then M is defined to be an Einstein space if $R_{ij} = \lambda g_{ij}$ for some scalar function λ on M. In fact, $\lambda = R/n$, where $R = \sum_i R_{ii}$ is the scalar curvature of M and $n = \dim M$. It is a classical theorem that, for $n \geq 3$, λ is a constant if M is connected. A general problem of Riemannian geometry is the determination of all Einstein spaces.

It is also of interest to consider pseudo-Riemannian metrics g_{ij}. Indeed, in the case of $\dim M = 4$ and for a Lorentzian metric, (signature $+ + + -$ say), the equations $R_{ij} = \lambda g_{ij}$ are specializations of Einstein field equations of General Relativity: $R_{ij} - \frac{1}{2} g_{ij} = T_{ij}$, where T_{ij} is some tensor field with variously specified properties.

From the physicist's point of view M is often not specified. Indeed M is often, on first consideration, taken to be an open subset of Euclidean space and the Einstein equations are solved locally. However, our point of view differs from theirs in that we desire to know whether a given manifold can be an Einstein space. This question has interest only for $\dim M \geq 4$, since all two dimensional Riemannian manifolds are Einstein spaces, and any three-dimensional Einstein space necessarily has constant curvature.

A fairly comprehensive list of known Einstein spaces is given by M. Berger [1, pp. 41, 42] and J. A. Wolf [9].

A natural starting point for the determination of Einstein spaces is with four-dimensional homogeneous Riemannian spaces. It is the purpose of this paper to determine all such spaces which are Einstein spaces. A homogeneous space M can be represented as a quotient G/H, where G is a transitive group of isometries on M and H is the isotropy group at some point $p_0 \in M$. In Chapter II it will be seen that when G is large enough, for example when $\dim H > 1$, then the solution of the problem is fairly easy. However, when G

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is small, for example when G acts simply transitively on M, the problem seems to be much more difficult. For this reason we were led into some general investigations of group manifolds, that is, a manifold with a group of isometries acting simply transitively on it. The results of this investigation are contained in Chapter I. Chapter II deals with the situation when G is large enough, as mentioned above, and finally Chapter III contains the determination of all four-dimensional group manifolds which are Einstein spaces.

All four-dimensional, simply connected, homogenous Einstein spaces were found and they are listed in the following table. All of these spaces are Riemannian symmetric spaces. An interesting problem therefore arises as to whether a direct proof of this fact can be found. However, in higher dimensions a homogenous Einstein space is not necessarily symmetric. Indeed, there exist bounded homogeneous, but non-symmetric, complex domains of complex dimensions four and five [7]. Bounded homogeneous complex domains with the Bergmann metric are Einstein spaces (cf. [2, p. 300]). J. A. Wolf [9] has recently classified all non-symmetric isotropy irreducible homogeneous Riemannian spaces, which are Einsteinian, and his list contains a space of dimension 7.

Four-dimensional, simply connected, homogeneous Einstein spaces

S = Ricci tensor field, g = metric tensor field on M, $S = \lambda g$, $\lambda \in \boldsymbol{R}$.

λ	M
$\lambda > 0$	$C(+, 4)$, $P(2, C)$, $C(+, 2) \times C(+, 2)$ (equal curvature on each factor),
$\lambda = 0$	$C(0, 4)$,
$\lambda < 0$	$C(-, 4)$, $H(2, C)$, $C(-, 2) \times C(-, 2)$ (equal curvature on each factor),

where $C\begin{pmatrix} + & \\ 0, & n \\ - & \end{pmatrix}$ = space of constant $\begin{pmatrix} + \\ 0 \\ - \end{pmatrix}$ curvature and dimension n, $P(2, C)$ = complex projective space of two complex dimensions, $H(2, C)$ = hermitian hyperbolic space of two complex dimensions.

The last four spaces listed in the table are realizable as group manifolds. Matrix representations of all the distinct Lie algebras $\mathfrak{g}$ giving rise to them are listed below. They are all solvable. $\mathfrak{g}$ and the metric on $\mathfrak{g}$ are defined by taking $X_1, \cdots, X_4$ to be an orthonormal basis over the reals.

$$C(0, 4): \quad X_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_2 = \left[\begin{array}{ccc|c} & & & 1 \\ & \bigcirc & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right],$$

$$X_3 = \left[\begin{array}{ccc|c} & & & 0 \\ & \bigcirc & & 1 \\ & & & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right], \qquad X_4 = \left[\begin{array}{ccc|c} & & & 0 \\ & \bigcirc & & 0 \\ & & & 1 \\ \hline 0 & 0 & 0 & 0 \end{array}\right].$$

$$C(-,4):\quad X_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 \\ 0 & -t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \qquad X_2 = \left[\begin{array}{ccc|cc} & & & -1 & 1 \\ & \bigcirc & & 0 & 0 \\ & & & 0 & 0 \\ \hline 1 & 0 & 0 & & \\ 1 & 0 & 0 & \multicolumn{2}{c}{\bigcirc} \end{array}\right],$$

$$X_3 = \left[\begin{array}{ccc|cc} & & & 0 & 0 \\ & \bigcirc & & -1 & 1 \\ & & & 0 & 0 \\ \hline 0 & 1 & 0 & & \\ 0 & 1 & 0 & \multicolumn{2}{c}{\bigcirc} \end{array}\right], \qquad X_4 = \left[\begin{array}{ccc|cc} & & & 0 & 0 \\ & \bigcirc & & 0 & 0 \\ & & & -1 & 1 \\ \hline 0 & 0 & 1 & & \\ 0 & 0 & 1 & \multicolumn{2}{c}{\bigcirc} \end{array}\right],$$

where $0 \leq t < \infty$. Distinct t give non-isomorphic groups.

$$H(2, C):\quad X_1 = \begin{bmatrix} -i\dfrac{2t}{3} & 0 & 0 \\ 0 & \dfrac{it}{3} & 1 \\ 0 & 1 & \dfrac{it}{3} \end{bmatrix}, \qquad X_2 = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

$$X_3 = \begin{bmatrix} 0 & i & -i \\ i & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \qquad X_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & i & -i \\ 0 & i & -i \end{bmatrix},$$

where $0 \leq t < \infty$. Distinct t give non-isomorphic groups.

$$C(-,2)\times C(-,2):\quad X_1 = \left[\begin{array}{cc|c} 1 & 0 & \\ 0 & 0 & \bigcirc \\ \hline \multicolumn{2}{c|}{\bigcirc} & \bigcirc \end{array}\right], \qquad X_2 = \left[\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \bigcirc \\ \hline \multicolumn{2}{c|}{\bigcirc} & \bigcirc \end{array}\right],$$

$$X_3 = \left[\begin{array}{c|cc} \bigcirc & \multicolumn{2}{c}{\bigcirc} \\ \hline & 1 & 0 \\ \bigcirc & 0 & 0 \end{array}\right], \qquad X_3 = \left[\begin{array}{c|cc} \bigcirc & \multicolumn{2}{c}{\bigcirc} \\ \hline & 0 & 1 \\ \bigcirc & 0 & 0 \end{array}\right].$$

Chapter I

Throughout this chapter G will denote a connected Lie group and $\mathfrak{g}$ will denote its Lie algebra. We regard $\mathfrak{g}$ as the tangent space at e of G, but as a

Lie algebra, $\mathfrak{g}$ is identified with the Lie algebra of all left-invariant vector fields on G. On G we consider a left-invariant Riemannian metric, which is determined uniquely by a positive definite inner product $\langle , \rangle$ on $\mathfrak{g}$. Namely, let $\mathfrak{g} \in G$, and $X, Y \in T_g(G)$, and define $\langle X, Y\rangle_g = \langle l_{g^{-1}*}X, l_{g^{-1}*}Y\rangle$, where l_g (or r_g) denotes left (right) multiplication by g in G. The curvature tensor R of a left-invariant metric is itself a left-invariant tensor field on G, and is therefore uniquely determined by its values on $\mathfrak{g}$. Thus we regard the curvature tensor R as a quardrilinear form on $\mathfrak{g}$.

The objectives of this chapter are to determine a formula for the curvature tensor in terms of the inner product and Lie bracket on $\mathfrak{g}$, to apply this formula in such a way as to classify G according to the nature of its Ricci tensor, and finally to determine what conditions a four-dimensional G must satisfy in order to be an Einstein space.

Before deriving a formula for R, it is necessary to establish some notation and conventions. The Riemannian connection on G is defined by the formula

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle &= X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle \\ &\quad + \langle [X, Y], Z\rangle + \langle Z, X], Y\rangle + \langle X, [Z, Y]\rangle, \end{aligned} \tag{I.1}$$

where X, Y, Z are any vector fields on G (cf. [4]). Define the operator L_X by $L_X(Y) = [X, Y]$, and let $A_X = \nabla_X - L_X$. It is well known that A_X is a (1, 1) tensor field on G, which is skew-symmetric with respect to the metric if and only if X is a killing vector field. Also it is known that $A_X Z = -\nabla_Z X$ for any vector field Z on G. (Cf. [4, pp. 235, 237]).

Let $A \in \mathfrak{g}$, X and Y be, respectively, the left-invariant and right-invariant vector fields generated by A, and $a_t = \exp tA$, $t \in R$. Then the 1-parameter groups of diffeomorphisms generated by X and Y are $\{r_{a_t}\}$ and $\{l_{a_t}\}$, respectively. Thus Y is a killing vector field, since l_{a_t} is an isometry of G for each t. Finally, observe that if Z is any right-invariant vector field on G then $[Z, X] = 0$. In fact, for $g \in G$,

$$\begin{aligned} [Z, X]_g &= \lim_{t\to 0} \frac{1}{t}\{(r_{a_t*}Z)_g - Z_g\} = \lim_{t\to 0} \frac{1}{t}\{r_{a_t*}Z_{ga_{-t}} - Z_g\} \\ &= \lim_{t\to 0} \frac{1}{t}\{r_{a_t*}r_{ga_{-t}*}Z_e - Z_g\} = \lim_{t\to 0} \frac{1}{t}(Z_g - Z_g) = 0\,. \end{aligned}$$

Theorem 1. *Let A and B be orthogonal unit vectors in $\mathfrak{g}$, and $A_1, \cdots, A_n$ an orthonormal basis of $\mathfrak{g}$ with $A_1 = A$, $A_2 = B$. Then*

$$\begin{aligned} R(A, B, A, B) &= -\langle [A, [A, B]], B\rangle \\ &\quad - \langle [A, B], [A, B]\rangle - \sum_{i=1}^{n} \langle A, [A, A_i]\rangle \langle B, [B, A_i]\rangle \\ &\quad + \frac{1}{4}\sum_i \{\langle A, [B, A_i]\rangle + \langle B, [A, A_i]\rangle + \langle A_i, [A, B]\rangle\}^2 . \end{aligned} \tag{I.2}$$

Proof. Let $X = X_1, \cdots, X_n$ be the left-invariant vector fields generated by $A_1, \cdots, A_n$, respectively, and Y the right-invariant vector field generated by B. A well-known formula for R is

$$R(X, Y, X, Y) = \langle \nabla_x \nabla_y Y - \nabla_y \nabla_x Y, X\rangle ,$$

since $[X, Y] = 0$. But now Y is a Killing field and $\nabla_y X - \nabla_x Y = [Y, X] = 0$. Thus

$$\langle \nabla_y \nabla_x Y, X\rangle = -\langle \nabla_x Y, \nabla_y X\rangle + Y\langle \nabla_x Y, X\rangle = -\langle \nabla_x Y, \nabla_x Y\rangle ,$$

since

$$\langle \nabla_x Y, X\rangle = -\langle A_y X, X\rangle = \langle X, A_y X\rangle = -\langle X, \nabla_x Y\rangle .$$

This gives the following formula for R:

$$R(X, Y, X, Y) = \langle \nabla_x \nabla_y Y, X\rangle + \langle \nabla_x Y, \nabla_x Y\rangle. \tag{1.3}$$

In the following, g will denote a point in G, and e the identity element of G. The first step is to determine the vector $\nabla_x Y$ in terms of the orthonormal frame field $X_1, \cdots, X_n$. Making use of (I.1) we have

$$\begin{aligned}\langle \nabla_x Y, X_i\rangle &= \frac{1}{2}\{X\langle Y, X_i\rangle - X_i\langle Y, X\rangle + \langle Y, [X_i, X]\rangle + Y\langle X_i, X\rangle \\ &\quad + \langle X, [X_i, Y]\rangle + \langle X_i, [X, Y]\rangle\} \\ &= \frac{1}{2}\{X\langle Y, X_i\rangle - X_i\langle Y, X\rangle + \langle Y, [X_i, X]\rangle\} ,\end{aligned}$$

since $\langle X_i, X\rangle$ is a constant on G and $[X_i, Y] = 0$, $i = 1, \cdots, n$. Let $a_t^i = \exp tA_i$, $t \in R$, $i = 1, \cdots, n$. Then

$$\begin{aligned}X\langle Y, X_i\rangle(g) &= X\langle ad(g^{-1})B, A_i\rangle = \left\langle \frac{d}{dt}\bigg|_{t=0} ad(a_{-t})ad(g^{-1})B, A_i\right\rangle \\ &= -\langle [A, ad(g^{-1})B], A_i\rangle .\end{aligned}$$

Thus $X\langle Y, X_i\rangle(e) = -\langle [A, B], A_i\rangle$.

Likewise, $X_i\langle Y, X\rangle(e) = -\langle [A_i, B], A\rangle$. So

$$\langle \nabla_x Y. X_i\rangle(e) = \frac{1}{2}\{-\langle [A, B], A_i\rangle + \langle A_i, B], A\rangle + \langle B, [A_i, A]\rangle\} , \tag{I.4}$$

$$\begin{aligned}&\langle \nabla_x Y, \nabla_x Y\rangle(e) \\ &\quad = \frac{1}{4}\sum_{i=1}^{n}\{\langle A, [B, A_i]\rangle + \langle B, [A, A_i]\rangle + \langle A_i, [A, B]\rangle\}^2 .\end{aligned} \tag{I.5}$$

We next determine $\nabla_y Y$ in terms of $X_1, \cdots, X_n$. By (I.1) and $[X_i, Y] = 0$, we have $\langle \nabla_y Y, X_i \rangle = Y\langle Y, X_i\rangle - \frac{1}{2} X_i \langle Y, Y\rangle$. But

$$Y\langle Y, X_i\rangle(g) = Y\langle ad(g^{-1})B, A_i\rangle = \left\langle \frac{d}{dt}\Big|_{t=0} ad(g^{-1})ad(a^2_{-t})B, A_i \right\rangle$$
$$= \langle - ad(g^{-1})[B, B], A_i\rangle = 0 ,$$
$$X_i\langle Y, Y\rangle(g) = \frac{d}{dt}\Big|_{t=0} \langle ad(a^i_{-t})ad(g^{-1})B, ad(a^i_{-t})ad(g^{-1})B\rangle$$
$$= - 2\langle A_i, ad(g^{-1})B], ad(g^{-1}) B\rangle .$$

Thus $\nabla_y Y = \sum_{i=1}^{n} \langle [A_i, ad(g^{-1})B], ad(g^{-1})B\rangle X_i$.

Finally, we compute $\nabla_x \nabla_y Y$ from this expression. Since

$$\nabla_x \nabla_y Y$$
$$= \sum_i \{X\langle [A_i, ad(g^{-1})B], ad(g^{-1})B\rangle X_i + \langle [A_i, ad(g^{-1})B], ad(g^{-1})B\rangle \nabla_x X_i\},$$

we have

$$\langle \nabla_x \nabla_y Y, X\rangle$$
$$= X\langle [A, ad(g^{-1})B], ad(g^{-1})B\rangle + \sum_i \langle [A_i, ad(g^{-1})B], ad(g^{-1})B\rangle \langle \nabla_x X_i, X\rangle .$$

Now

$$X\langle [A, ad(g^{-1})B], ad(g^{-1})B\rangle(e)$$
$$= \frac{d}{dt}\Big|_{t=0} \langle [A, ad(a_{-t})ad(g^{-1})B], ad(a_{-t})ad(g^{-1})B\rangle(e)$$
$$= - \langle [A, [A, B]], B\rangle - \langle [A, B], [A, B]\rangle ,$$

and $\langle \nabla_x X_i, X\rangle(e) = \langle [A, A_i], A\rangle$, from (I.1). Putting these together we obtain

$$\langle \nabla_x \nabla_y Y, X\rangle(e) = - \langle [A, [A, B]], B\rangle - \langle [A, B], [A, B]\rangle - \sum_i \langle A, [A, A_i]\rangle \langle B, [B, A_i]\rangle . \tag{I.6}$$

The formula for $R(A, B, A, B)$ now follows immediately from (I.3), (I.5) and (I.6).

As an immediate application of the above formula for R we have

Theorem 2. *Denote the Ricci tensor of G by S and regard S as a symmetric bilinear form on $\mathfrak{g}$. If $X \in \mathfrak{g}$ is a central element,* i.e., $[X, Y] = 0$ *for every $Y \in \mathfrak{g}$, then $S(X, X) \geq 0$, and $S(X, X) = 0$ if and only if $\langle X, \mathfrak{g}'\rangle = 0$, where $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is the derived algebra of $\mathfrak{g}$.*

Proof. Let $X_1, \cdots, X_n$ be an orthonormal basis of $\mathfrak{g}$ with $X_1 = X$. Then $S(X, X) = \sum_{i=1}^{n} R(X, X_i, X, X_i)$. Now use formula (I.2) and the assumption that $[X, Y] = 0$ for every $Y \in \mathfrak{g}$ to get

$$R(X, X_i, X, X_i) = \frac{1}{4} \sum_j \langle X, [X_j, X_i]\rangle^2 .$$

Hence $S(X, X) = \frac{1}{4} \sum_{i,j} \langle X, [X_i, X_j]\rangle^2$. The conclusion is now obvious if one observes that the vectors $\{[X_i, X_j]\}_{i,j}$ span $\mathfrak{g}'$.

Theorem 3. *Let $\mathfrak{z}(\mathfrak{g})$ denote the center of $\mathfrak{g}$.*

1) *If $\mathfrak{z}(\mathfrak{g}) \neq 0$ and $S = \lambda \langle , \rangle$, then $\lambda = 0$ and $\mathfrak{z}(\mathfrak{g}) \perp \mathfrak{g}'$.*
2) *If S is positive definite, then G is compact semisimple.*
3) *If S is negative definite, then $\mathfrak{z}(\mathfrak{g}) = 0$.*

Remark. In Chapter III it will be seen that, if $\dim \mathfrak{g} = 4$ and $S = 0$, then $\mathfrak{z}(\mathfrak{g}) \neq 0$. The author does not know whether the converse of 1) is true for higher dimensions.

Proof. Suppose $\mathfrak{z}(\mathfrak{g}) \neq 0$ and $S = \lambda \langle , \rangle$. Then Theorem 2 implies that $\lambda \geq 0$, with $\lambda = 0$ if and only if $\mathfrak{z}(\mathfrak{g}) \perp \mathfrak{g}'$. If $\lambda > 0$, then S is positive definite, and thus Myers' theorem says that G and every covering manifold of G is compact. In particular the simply connected covering group $\tilde{G}$ of G is compact. Thus $\mathfrak{g}$ is a compact semisimple Lie algebra. (cf. [2, p. 122]). This proves 1) and 2).

If S is negative definite then Theorem 2 implies that $\mathfrak{z}(\mathfrak{g}) = 0$.

The following corollary will be needed in the proof of Theorem 5.

Corollary 1. *Suppose* $\dim G = 4$ *and* $\mathfrak{z}(\mathfrak{g}) \neq 0$. *If G is an Einstein space, then it must be locally flat.*

Proof. Since $\mathfrak{z}(\mathfrak{g}) \neq 0$ we have $S = 0$ by Theorem 3. It suffices to prove this corollary for simply connected G.

Let $G = G_0 \times G_1 \times \cdots \times G_k$ be the de Rham decomposition of G, where G_0 is Euclidean and $G_1, \cdots, G_k$ are the irreducible non-Euclidean factors. Then $G_1, \cdots, G_k$ are subgroups of G, although the product is only that of Riemannian spaces. (cf. [6, p. 51]).

Claim. $\dim G_0 \geq 1$. In fact, let $B \in \mathfrak{z}(\mathfrak{g})$ be a unit vector, and Y the left-invariant vector field on G generated by B. Then Y is also right-invariant, since B is central. Let X be any left-invariant vector field on G, and $X_1, \cdots, X_4$ a left-invariant orthonormal frame field on G with $X_1 = X$, $X_2 = Y$. Then from Formula (I.4), we have $\langle \nabla_X Y, X_i\rangle(e) = \langle B, [X_i(e), X(e)]\rangle = 0$, since $\mathfrak{z}(\mathfrak{g}) \perp \mathfrak{g}'$ by Theorem 3. Hence Y is a parallel vector field and consequently $Y(e) \in T_e(G_0)$, i.e., $\dim G_0 \geq 1$.

Thus $\dim G_i \leq 3$, $i = 1, \cdots, k$; and consequently G_i is flat since its Ricci tensor is zero. Hence $G = G_0$.

Remark. In Chapter III it will be shown that a four-dimensional G with zero Ricci tensor must have $\mathfrak{z}(\mathfrak{g}) \neq 0$. Thus the above corollary can be easily extended to five-dimensional G.

Following the lead of J. A. Wolf we can sharpen the results of his theorem in [8].

Theorem 4. *Suppose G is nilpotent and non-abelian. Then there exist unit vectors $X \in \mathfrak{g}$ and $Y \in \mathfrak{g}$ such that $S(X, X) > 0$ and $S(Y, Y) < 0$.*

Proof. Let $\mathfrak{g} = D^0\mathfrak{g} \supset D^1\mathfrak{g} \supset \cdots \supset D^{k-1}\mathfrak{g} \supset D^k\mathfrak{g} = 0$ be the central series of $\mathfrak{g}$, where $D^i\mathfrak{g} = [\mathfrak{g}, D^{i-1}\mathfrak{g}]$. $\mathfrak{g}$ being non-abelian implies that $k \geq 2$, $D^{k-1}\mathfrak{g} \neq 0$. Now $0 = D^k(\mathfrak{g}) = [\mathfrak{g}, D^{k-1}\mathfrak{g}]$ means that $D^{k-1}\mathfrak{g} \subset \mathfrak{z}(\mathfrak{g})$. Thus there exists a unit vector $X \in \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}'$. By Theorem 2 we have that $S(X, X) > 0$.

For each $i = 1, \cdots, k$ let $\mathfrak{a}^i \subset D^{i-1}\mathfrak{g}$ be the orthogonal complement of $D^i\mathfrak{g}$ in $D^{i-1}\mathfrak{g}$. Thus $D^{i-1}\mathfrak{g} = \mathfrak{a}^i + D^i\mathfrak{g}$, orthonogonal vector space direct sum. Now $\mathfrak{g} = \mathfrak{a}^1 + \mathfrak{a}^2 + \cdots + \mathfrak{a}^k$, orthogonal direct sum as vector spaces. Since $\mathfrak{g}$ is non-abelian there is a smallest integer t, $1 \leq t < k$, such that $\mathfrak{a}^t$ contains a non-central unit vector, Y say. Let $\mathfrak{a} = \sum_{1 \leq i < t} \mathfrak{a}^i$. Then $\mathfrak{a}$ is central, and $\mathfrak{g} = \mathfrak{a} + \mathfrak{a}^t + D^t\mathfrak{g}$. Choose an orthonormal basis $X_1, \cdots, X_n$ of $\mathfrak{g}$ so that the first dim $\mathfrak{a}^1$ vectors are in $\mathfrak{a}^1$, the second dim $\mathfrak{a}^2$ vectors are in $\mathfrak{a}^2$, etc. Then there are integers $q < p$ such that $X_1, \cdots, X_q \in \mathfrak{a}$, $X_{q+1}, \cdots, X_p \in \mathfrak{a}^t$, and $X_{p+1}, \cdots, X_n \in D^t\mathfrak{g}$. We may assume that $X_{q+1} = Y$.

$(*)$ Observe that if $i \leq j$ then $X_i \perp [X, X_j]$ for any $X \in \mathfrak{g}$.

Claim. $S(Y, Y) < 0$.

For $1 \leq i \leq q$:

$R(X_i, Y, X_i, Y) = 0$ by (I.2), $(*)$ and the fact that X_i is central for $1 \leq i \leq q$.

For $q + 1 \leq i \leq p$:

$R(X_i, Y, X_i, Y) = -\dfrac{3}{4}\|[X_i, Y]\|^2$ by (1.2), $(*)$ and that $\langle X_i, [Y, X_j]\rangle = 0$ for $1 \leq j \leq n$.

For $p + 1 \leq i \leq n$:

$$\begin{aligned} R(X_i, Y, X_i, Y) &= -\|[X_i, Y]\|^2 + \frac{1}{4}\sum_{1 \leq j \leq n}(\langle X_i, [Y, X_j]\rangle + \langle X_j, [X_i, Y]\rangle)^2 \\ &= -\|[X_i, Y]\|^2 + \frac{1}{4}\sum_{q+1 \leq j \leq i}\langle X_i, [Y, X_j]\rangle^2 \\ &\quad + \frac{1}{4}\sum_{i < j \leq n}\langle X_j, [X_i, Y]\rangle^2 \end{aligned}$$

by (I.4) and repeated applications of $(*)$. Thus

$$
\begin{aligned}
&S(Y, Y)\\
&\quad = \sum_{1\le i\le n} R(X_i, Y, X_i, Y)\\
&\quad = -\frac{3}{4}\sum_{q+1\le i\le p} \|[X_i, Y]\|^2 - \sum_{p+1\le i\le n} \|[X_i, Y]\|^2\\
&\qquad + \frac{1}{4}\sum_{p+1\le i\le n}\sum_{q+1\le j\le i} \langle X_i, [Y, X_j]\rangle^2 + \frac{1}{4}\sum_{p+1\le i\le n}\sum_{i<j\le n} \langle X_j, [X_i, Y]\rangle^2\\
&\quad = -\frac{3}{4}\sum_{q+1\le i\le p} \|[X_i, Y]\|^2 - \sum_{p+1\le i\le n} \|[X_i, Y]\|^2\\
&\qquad + \frac{1}{4}\sum_{q+1\le j\le p}\sum_{p+1\le i\le n} \langle X_i, [Y, X_j]\rangle^2 + \frac{1}{4}\sum_{p+1\le j\le n}\sum_{j\le i\le n} \langle X_i, [Y, X_j]\rangle^2\\
&\qquad + \frac{1}{4}\sum_{p+1\le i\le n}\sum_{1\le j\le n} \langle X_j, [X_i, Y]\rangle^2, \quad \text{by } (*)\\
&\quad = -\frac{3}{4}\sum_{q+1\le i\le p} \|[X_i, Y]\|^2 - \sum_{p+1\le i\le n} \|[X_i, Y]\|^2\\
&\qquad + \frac{1}{4}\sum_{q+1\le j\le p}\sum_{1\le i\le n} \langle X_i, [Y, X_j]\rangle^2 + \frac{1}{4}\sum_{p+1\le j\le n}\sum_{1\le i\le n} \langle X_i, [Y, X_j]\rangle^2\\
&\qquad + \frac{1}{4}\sum_{p+1\le i\le n} \|[X_i, Y]\|^2, \quad \text{by } (*)\\
&\quad = -\frac{3}{4}\sum_{q+1\le i\le n} \|[X_i, Y]\|^2 + \frac{1}{4}\sum_{q+1\le j\le p} \|[X_j, Y]\|^2\\
&\qquad + \frac{1}{4}\sum_{p+1\le j\le n} \|[Y, X_j]\|^2 = -\frac{1}{2}\sum_{q+1\le i\le n} \|[X_i, Y]\|^2 < 0,
\end{aligned}
$$

since Y is non-central, and so $[X_i, Y] \neq 0$ for some i.

Corollary 2. *A nilpotent G can be an Einstein space if and only if it is abelian.*

Theorem 5. *Suppose that G is non-abelian, four-dimensional and Einsteinian. Then G must be solvable, but non-nilpotent.*

Proof. G cannot be nilpotent by Corollary 2.

Let r be the largest solvable ideal of $\mathfrak{g}$. Then $\mathfrak{g}/r$ is semi-simple and therefore must have dimension three or zero, because there are no two or four-dimensional semi-simple Lie algebras. Thus $\dim r$ is one or four, i.e., either $\dim r = 1$ or G is solvable.

Suppose $\dim r = 1$. We shall prove that in this case G cannot be an Einstein space. $\mathfrak{g}$ is the semi-direct sum $r + \mathscr{S}$, where $\mathscr{S}$ is a three-dimensional simple subalgebra of $\mathfrak{g}$. Explicitly, there is a Lie algebra homomorphism $\eta\colon \mathscr{S} \to \operatorname{End}(r)$, given by $\eta(X)(Y) = [X, Y]$ for $X \in \mathscr{S}$, and $Y \in r$. Then $\dim \operatorname{Ker} \eta \geq 2$. Since $\mathscr{S}$ has no non-trivial ideals, $\operatorname{Ker} \eta = \mathscr{S}$, i.e., η is trivial. This just means that $\mathfrak{g}$ is the direct sum $r \oplus \mathscr{S}$, and $\mathscr{S}$ is an ideal of $\mathfrak{g}$. But then it is clear that $r = \mathfrak{z}(\mathfrak{g})$ and, consequently, $\mathfrak{z}(\mathfrak{g}) \neq 0$. Hence G must be locally flat by Corollary 1.

Since $r = \mathfrak{z}(\mathfrak{g})$, the invariant vector fields on G generated by elements of r are parallel, as was seen in the proof of Corollary 1. Thus r is invariant under the linear holonomy representation on $\mathfrak{g}$. But then so is $\mathscr{S}$, since $\mathscr{S} \perp r$ by Theorem 3. Hence it follows that the simply connected covering group $\tilde{G}$ of G is a direct product—as groups and as Riemannian spaces—$\tilde{G} = \boldsymbol{R} \times S$, where S is the analytic subgroup of $\tilde{G}$ corresponding to $\mathscr{S}$. Now $\tilde{G}$ being simply connected and flat implies that S is simply connected and flat also. But this is impossible since a semi-simple connected Lie group cannot act transitively on $\boldsymbol{R}^3$ as a group of Euclidean motions. (cf. [5, Chapter X]).

Theorem 4 reduces the problem of finding all four-dimensional G with a left-invariant Einstein metric to the case where G is solvable but non-nilpotent. It will be seen in Chapter III that G must have a discrete center in order to be an Einstein space. This result is indeed stronger than Theorem 4 since there are four-dimensional solvable non-nilpotent Lie groups with non-discrete center. For example

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & c & d \\ 0 & 0 & 1 \end{pmatrix} \middle| \begin{array}{c} c > 0 \\ a, b, c, d \in R \end{array} \right\}.$$

Chapter II

Throughout this chapter M denotes a simply connected, homogeneous, Riemannian manifold and G is a connected group of isometries acting transitively and effectively on M. Our problem is to determine all such four-dimensional M which are Einstein spaces. S. Ishihara [3] used a general method of E. Cartan's in order to determine the topological structure of four-dimensional M. Ishihara's methods solve most of our problem except for the case when G acts simply transitively on M, that is, the case when M is a group manifold with a left-invariant metric.

This chapter is divided into three sections. §1 contains a determination of the subalgebras of $so(4)$, §2 is a summary of Cartan's method as applied by Ishihara, and §3 contains the application of these and some other methods to our problem.

1. Fix a point $p_0 \in M$, and let H be the isotropy subgroup of G at p_0, and $SO(M)$ be the principal bundle of oriented orthonormal frames on M with structure group $SO(n)$, $n = \dim M$. Fix a frame $u_0 \in SO(M)$ over $p_0 \in M$. Define $F: G \to SO(M)$ by $F(q) = q_* u_0$, and the homomorphism $f: H \to SO(n)$ by $F(h) = u_0 f(h)$ for $h \in H$. F defines a bundle map of $G(G/H, H)$ into $SO(M)(M, SO(n))$:

$$\begin{tikzcd} G \arrow[r, "F"] \arrow[d, "H"'] & SO(M) \arrow[d, "\pi"] & SO(n) \\ G/H \arrow[r, "1"'] & M & \end{tikzcd}$$

Consider the map $f_*: \mathfrak{h} \to so(n)$, where $\mathfrak{h}$ is the Lie algebra of H. An evident way to divide this investigation of M into more tenable pieces is to determine all the subalgebras of $so(n)$ and then to consider separately each case as $f_*(\mathfrak{h})$ ranges over this collection of subalgebras. For this reason we determine the subalgebras of $so(4)$.

Let θ be the canonical 1-form on $SO(4)$, $\theta = (\theta_{ij})$, $\theta_{ij} + \theta_{ji} = 0$, $0 \leq i$, $j \leq 3$. The Maurer-Cartan equations for $SO(4)$ are then

$$d\theta_{ij} = \sum_k \theta_{ik} \wedge \theta_{kj} \ .$$

Define the following forms:

$$\varphi_1 = \theta_{01} - \theta_{23}, \qquad \varphi_2 = \theta_{02} - \theta_{31}, \qquad \varphi_3 = \theta_{03} - \theta_{12} \ ,$$
$$\psi_1 = -\theta_{01} - \theta_{23}, \qquad \psi_2 = -\theta_{03} - \theta_{31}, \qquad \psi_3 = -\theta_{03} - \theta_{12} \ .$$

Without difficulty one can obtain the following proposition.

Proposition. *Let $\mathfrak{h}$ be a subalgebra of the Lie algebra $so(4)$. Then under an adjoint transformation induced on $so(4)$ by an element of $O(4)$, $\mathfrak{h}$ is equivalent to one of the subalgebras defined by the following equations:*

I. $\mathfrak{h} = so(4)$.
II. $\varphi_1 = \varphi_3 = 0$.
III. $\varphi_1 - \psi_1 = 0$, $\varphi_2 - \psi_2 = 0$, $\varphi_3 - \psi_3 = 0$.
IV. $\varphi_1 = \varphi_2 = \varphi_3 = 0$.
V. $\varphi_2 = \varphi_3 = \psi_2 = \psi_3 = 0$.
VI. $\varphi_2 = \varphi_3 = \psi_2 = \psi_3 = 0$, $m\varphi_1 = \psi_1$ $(m > 0)$.
VII. $\varphi_1 = \varphi_2 = \varphi_3 = \psi_2 = \psi_3 = 0$.
VIII. $\mathfrak{h} = 0$.

2. Let ω denote the Riemannian connection form on $SO(M)$, and θ the canonical 1-form on $SO(M)$. Note that θ no longer denotes the canonical form on $SO(n)$ as it did in § 1, but all other notation in this section is the same as that in § 1. Write $\omega = (\omega^i_j)$, $\omega^i_j + \omega^j_i = 0$, and $\theta = (\theta^i)$, $0 \leq i, j, \cdots, \leq n-1$. Then the structure equations are

$$d\theta^i = -\sum_k \omega^i_k \wedge \theta^k \ ,$$
$$d\omega^i_j = -\sum_k \omega^i_k \wedge \omega^k_j + \Omega^i_j \ ,$$

where Ω^i_j is the curvature form.

$G(G/H, H)$ is a subbundle of $SO(M)(M, SO(n))$, but the Riemannian connection cannot, in general, be reduced to G. Recall that $F(e) = u_0$, and thus $F_*\mathfrak{g} = T_{u_0}(F(G))$. The vertical subspace of $F_*\mathfrak{g}$ is $F_*\mathfrak{h}$, which means that $\mathfrak{h} = \{X \in \mathfrak{g} \mid F^*\theta_{u_0}(X) = 0\}$ and $F^*\omega$ restricted to $\mathfrak{h}$ is the identity homomorphism.

$f_*\mathfrak{h}$ is a subalgebra of $so(n)$, and thus, is defined by a set $\{A_a\}$ of linear functionals on $so(n)$, i.e., $f_*\mathfrak{h} = \{X \in so(n) \mid A_a(X) = 0$, for all $a\}$. Write $A_a = (A_{aij})$, $A_{aij} + A_{aji} = 0$. For each a, the linear 1-form $\sum_{i<j} A_{aij}F^*\omega^i_j$ vanishes on $\mathfrak{h}$, i.e., it is a horizontal form, and therefore

$$\sum_{i<j} A_{aij}F^*\omega^i_j = \sum_i c_{ai}F^*\theta^i, \text{ for certain constants } c_{ai} ,$$

since the θ^i span the space of all horizontal forms.

We can obtain relations on the c_{ai} by differentiating these equations and evaluating each side on pairs of the form (X, Y), where $X \in \mathfrak{h}$, $Y \in \mathfrak{g}$. Let $F_*X = (\chi_{ij})$, $\chi_{ij} + \chi_{ji} = 0$, and $\sum_{i<j} \chi_{ij}A_{aij} = 0$, for all a. Then

$$\begin{aligned}
F^*d\omega^i_j(X, Y) &= -\sum_k \omega^i_k \wedge \omega^k_j(F_*X, F_*Y) + \Omega^i_j(F_*X, F_*Y) \\
&= -\frac{1}{2}\sum_k \omega^i_k(F_*X)\omega^k_j(F_*Y) - \omega^k_j(F_*X)\omega^i_k(F_*Y) \\
&= -\frac{1}{2}\sum_k (\chi_{ik}\omega^k_j + \chi_{kj}\omega^k_i)(F_*Y) , \\
F^*d\theta^i(X, Y) &= -\sum_k \omega^i_k \wedge \theta^k(F_*X, F_*Y) = -\frac{1}{2}\sum_k \omega^i_k(F_*X)\theta^k(F_*Y) \\
&= -\frac{1}{2}\sum_k \chi_{ik}\theta^k(F_*Y) .
\end{aligned}$$

We have used the fact that F_*X is vertical while Ω and θ are horizontal.

Hence we get the following relations on the c_{ai}:

$$\text{(II.1)} \qquad F^* \sum_{\substack{i<j \\ k}} A_{aij}(\chi_{ik}\omega^k_j + \chi_{kj}\omega^k_i) = F^* \sum_{k,i} c_{ai}\chi_{ik}\theta^k ,$$

where

$$\chi_{ik} + \chi_{ki} = 0 , \qquad \sum_{i<k} A_{aik}\chi_{ik} = 0$$

for each a.

3. In this section we determine all four-dimensional simply connected homogeneous Riemannian Einstein spaces $M = G/H$, except for the case when M is a group manifold. This remaining case will be dealt with in Chapter III. We consider eight cases, each case being numbered according to which type—I through VIII of the proposition in § 1—$f_*\mathfrak{h}$ belongs. In some of the cases the results are easily extended to higher dimensional M. In such cases the results are stated and proved in as much generality as possible. Case II is done in additional detail in order to illustrate the general method, which is essentially the same in each case, except for Case IV, which requires a few different methods.

For convenience we let $C(+,n)$, $C-,n)$ and $C(0,n)$ denote the simply connected Riemannian spaces of n dimensions with constant positive, negative and zero curvature, respectively.

Case I. $f_*\mathfrak{h} = so(4)$. In general, if $\dim M = n$, and $f_*\mathfrak{h} = so(n)$, then M is one of $C(+,n)$, $C(-,n)$ or $C(0,n)$.

Case II. $f_*\mathfrak{h}$ is defined by the equations $\varphi_1 = \varphi_3 = 0$. Thus

$$f_*\mathfrak{h} = \left\{ \begin{bmatrix} 0 & a & b & c \\ -a & 0 & c & d \\ -b & -c & 0 & a \\ -c & -d & -a & 0 \end{bmatrix} \middle| a, b, c, d \in R \right\} \cong u(2) .$$

In general, suppose that $\dim M = n = 2m$, and $f_*\mathfrak{h}$ is the standard imbedding of $u(m)$ in $so(n)$ given by

$$u(m) \ni h = h_1 + ih_2 \leftrightarrow \begin{bmatrix} h_1 & -h_2 \\ h_2 & h_1 \end{bmatrix} \in so(n) ,$$

where h_1 and h_2 are real $m \times m$ matrices.

We shall determine what M must be in this general case. To do this, we make use of the equations (II.1) in § 2, and shall omit the F^* in these equations, so in the following it is to be understood that ω and θ are restricted to the subbundle $G(G/H, H)$. Now there are $m(m-1)$ linear equations defining $f_*\mathfrak{h}$, namely,

$$\chi_{ii} = \chi_{i+m,j+m} , \tag{II.2}$$

$$\chi_{i,j+m} = \chi_{j,i+m} , \tag{II.3}$$

where $1 \le i, j, k, \cdots \le m$, (in the following we make the convention that $1 \le i, j, k, \cdots \le m$), and where $\chi = (\chi_{ab})$, $1 \le a, b \le n$, denotes any element in $so(n)$. Thus, for *each fixed* pair i, j, with $1 \le i < j \le m$, we get the equation, from (II.2) and (II.1),

$$\begin{aligned} &\sum_{k \ne j} \{\chi_{ik}(\omega_j^k - \omega_{j+m}^{k+m}) + \chi_{i,k+m}(\omega_j^{k+m} + \omega_{j+m}^k)\} \\ &\quad + \sum_{k \ne i} \{\chi_{kj}(\omega_i^k - \omega_{i+m}^{k+m}) + \chi_{k+m,i}(\omega_i^{k+m} + \omega_{i+m}^k)\} \\ &= \sum_{k<l} \chi_{kl}(c_k\theta^l - c_l\theta^k - c_{k+m}\theta^{l+m} - c_{l+m}\theta^{k+m}) \\ &\quad + \sum_{k \le l} \chi_{k,l+m}(c_k\theta^{l+m} - c_{l+m}\theta^k + c_l\theta^{k+m} - c_{l+m}\theta^l) , \end{aligned} \tag{II.4}$$

and the equation, from (II.3) and (II.1),

$$\begin{aligned}\text{(II.5)}\qquad &\sum_{k\neq j}\{\chi_{ik}(\omega^k_{j+m}+\omega^{k+m}_j)+\chi_{i,k+m}(\omega^{k+m}_{j+m}-\omega^k_j)\}\\ &\quad+\sum_{k\neq i}\{\chi_{kj}(\omega^k_{i+m}+\omega^{k+m}_i)+\chi_{k,j+m}(\omega^k_i-\omega^{k+m}_{i+m})\}\\ &=\sum_{k<l}\chi_{kl}(d_k\theta^l-d_l\theta^k+d_{k+m}\theta^{l+m}-d_{l+m}\theta^{k+m})\\ &\quad+\sum_{k\leq l}\chi_{k,l+m}(d_k\theta^{l+m}-d_{l+m}\theta^k+d_l\theta^{k+m}-d_{k+m}\theta^l)\ .\end{aligned}$$

In (II.4) and (II.5) the variables χ_{ab} which appear are all independent, that is, these equations are identities which must hold for any choice of the χ_{ab} which appear. Hence the coefficients of the χ_{ab} on each side of the equation must be the same. Equating the coefficients of χ_{ij} in (II.4) we get that $0=c_i\theta^j-c_j\theta^i+c_{i+m}\theta^{j+m}-c_{j+m}\theta^{i+m}$. Thus $c_i=c_j=c_{i+m}=c_{j+m}=0$, since the θ^k are linearly independent. Equating the coefficients of $\chi_{i,i+m}$ in (II.4) we then get

$$\text{(II.6)}\qquad \omega^{i+m}_j+\omega^i_{j+m}=0\ .$$

Similarly, we get, from (II.5),

$$\text{(II.7)}\qquad \omega^{i+m}_{j+m}-\omega^i_j=0\ .$$

Equations (II.6) and (II.7) hold for any i, j satisfying $1\leq i<j\leq m$. Thus these equations imply that the connection form ω, when restricted to $G(G/H,H)$, takes values in $f_*\mathfrak{h}$. This means that the connection can be restricted to $G(G/H,H)$.

Define

$$\begin{aligned}\pi^i&=\theta^i+i\theta^{i+m}\ ,\\ \pi^i_j&=\omega^i_j+i\omega^{i+m}_j,\ 1\leq i,j\leq m,\\ \Phi^i_j&=\Omega^i_j+i\Omega^{i+m}_j,\quad\text{on}\quad G(G/H,H)\ .\end{aligned}$$

Then

$$\begin{aligned}d\pi^i&=-\sum_j\pi^i_j\wedge\pi^j\ ,\\ d\pi^i_j&=-\sum_k\pi^i_k\wedge\pi^k_j+\Phi^i_j\ ,\\ \bar\pi^i_j&=-\pi^j_i,\ \bar\Phi^i_j=-\Phi^j_i\ .\end{aligned}$$

Thus $\pi^1,\cdots,\pi^m$ define a Hermitian structure on M, which is torsion free, so that M is a Kählerian space. Now $H\cong U(m)$ acts transitively on the unit sphere in $T_{p_0}(M)$. Hence M has constant holomorphic curvature, that is, M is either $P(m,C)=$ complex projective space, or $H(m,C)=$ hermitian hyperbolic space, or $C(0,n)$.

Case III. $f_*\mathfrak{h}$ is defined by the equations $\chi_{01} = \chi_{02} = \chi_{03} = 0$.

$$f_*\mathfrak{h} = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & -a & 0 & c \\ 0 & -b & -c & 0 \end{bmatrix} \middle| \, a, b, c \in R \right\} \cong so(3) \ .$$

In general, suppose $\dim M = n$, and $f_*\mathfrak{h} = so(n-1)$, which is contained in $so(n)$ as

$$\left[\begin{array}{c|c} 0 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & so(n-1) \\ 0 & \end{array} \right] .$$

Then the linear equations defining $f_*\mathfrak{h}$ are $\chi_{1j} = -\chi_{j1} = 0$, $j = 1, \cdots, n$, and as a consequence of equations (II.1) we get the following equations, when the ω^i_j and θ^k are restricted to G:

$$\omega^1_j = c\theta^j, \quad \text{where} \quad c = \text{constant}, \ 1 \leq j \leq n \ .$$

Notice then that $d\theta^1 = -\sum_j \omega^1_j \wedge \theta^j = -\sum_j c\theta^j \wedge \theta^j = 0$. If $c = 0$ then it is not difficult to see that M is $R^1 \times C(\pm, n-1)$ or $C(0, n)$. If $c \neq 0$, then K. Yano [10] shows that M is $C(-, n)$.

Case IV. $f_*\mathfrak{h}$ is defined by the equations

$$\chi_{01} - \chi_{23} = \chi_{02} - \chi_{31} = \chi_{03} - \chi_{12} = 0 \ ,$$

$$f_*\mathfrak{h} = \left\{ \begin{bmatrix} 0 & a & b & c \\ -a & 0 & c & -b \\ -b & -c & 0 & a \\ -c & b & -a & 0 \end{bmatrix} \middle| \, a, b, c \in R \right\} \cong su(2) \ .$$

In general, suppose $\dim M = n = 2m$, and $f_*\mathfrak{h} \cong su(m)$ which is contained in $so(n)$ by the standard imbedding described in Case II for $u(m)$.

Proposition. *If $m \neq 3$, $m \geq 2$, then M is a Kählerian space with constant holomorphic curvature equal to zero, that is, $M = C(0, n)$.*

Remark. J. A. Wolf has shown the author that in the case $m = 3$, M must be either $C(0, 6)$ or $G(2)/SU(3)$.

Proof. The linear equations defining $f_*\mathfrak{h}$ in $so(n)$ are:

$$\text{(II.8)} \qquad \begin{aligned} \chi_{ij} &= \chi_{i+m, j+m} \ , \\ \chi_{i, j+m} &= \chi_{j, i+m} \ , \\ \sum_{k=1}^{m} \chi_{k, k+m} &= 0, \quad \text{where} \quad 1 \leq i, j \leq m \ . \end{aligned}$$

Thus from (II.1) and these equations (II.8) we obtain $m(m-1)+1$ equations relating ω^α_β and θ^γ restricted to $G(G/H, H)$, $1 \leq \alpha, \beta, \gamma \leq n$. When $m = 2$ these three equations all reduce to identities $0 = 0$. When $m = 3$ these seven equations give no information. But when $m \geq 4$ we get

$$\omega^i_j = \omega^{i+m}_{j+m} ,$$

$$\omega^i_{j+m} = \omega^j_{i+m}, \quad \text{for} \quad 1 \leq i, j \leq m .$$

Thus, when restricted to $G(G/H, H)$, ω takes values in $u(m) \subset so(n)$. If we let $\pi^j = \theta^j + i\theta^{j+m}$, $1 \leq j \leq m$, then $\pi^1, \cdots, \pi^m$ define a hermitian inner product on M, and $G(G/H, H) \subset U(M)$, where $U(M)$ is the bundle of unitary frames on M.

Claim. *When restricted to $U(M)$, ω takes values in $u(m) \subset so(n)$.*

Proof. Let $u \in G(G/H, H) \subset U(M)$, and $X \in T_u(U(M))$. Then $X = X_1 + X_2$, where $X_1 \in T_u(G)$ and X_2 is vertical. Thus $\omega(X) = \omega(X_1) + \omega(X_2) \in u(m)$, since $\omega(X_1) \in u(m)$ as shown above, and $\omega(X_2) \in u(m) \subset so(n)$ because X_2 is a vertical vector tangent to $U(M)$ and ω is a connection form. Now, if u is any point of $U(M)$, then $u = va$ for some $v \in G(G/H, H)$ and some $a \in U(m) \subset SO(n)$, and any vector $X \in T_u(U(M))$ is given by $X = R_{a*}Y$ for some $Y \in T_v(U(M))$, where $R_a \colon U(M) \to U(M)$ denotes the right action of $U(m) \subset SO(n)$ on $U(M)$. Thus, using the Ad-invariance of ω, $\omega_u(X) = \omega_u R_{a*}Y = Ad(a^{-1})(\omega_v(Y)) \in u(m)$, where Ad denotes the adjoint representation of $U(m)$ in $u(m)$, and the claim is proved.

The fact that ω restricted to $U(M)$ takes values in $u(m) \subset so(n)$ implies that the connection can be restricted to $U(M)$. Hence M is a Kählerian space. Since the isotropy group $H \cong SU(m) \subset SO(n)$ acts transitively on the unit sphere in $T_{p_0}(M)$, it follows that M has constant holomorphic curvature.

In the case when $m = 2$, a different argument is needed in order to show that M is a Kählerian space of constant holomorphic curvature. The proof for $m = 2$ is valid for any even value of m, say $m = 2k$.

Alternate proof for the case $m = 2k$. The linear isotropy representation of H is $SU(m) \subset SO(n)$, which implies that there exists an almost complex structure J_{p_0} on $T_{p_0}(M)$ which is invariant under the action of H. J_{p_0} can be extended to an almost complex structure J on M using G and the fact that J_{p_0} is invariant under H. Then the torsion tensor field N defined by J, i.e. by $N(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]$ for vector fields X, Y on M, must be zero since it is of type $(1, 2)$ and invariant under $SU(2k)$, namely, $-I \in SU(2k) \subset SO(4k)$, where I is the $4k \times 4k$ unit matrix, and thus

$$N(X, Y) = (-IN)(X, Y) = -N(-X, -Y) = -N(X, Y) ,$$

i.e., $N = 0$. Hence this almost complex structure on M is integrable.

Now ∇J, the covariant differential of J, is a tensor field of type $(1, 2)$ which is invariant under $SU(2k)$, and thus, as for N above, $\nabla J = 0$. Hence M is a Kählerian space.

M has constant holomorphic curvature since $H \cong SU(m)$ acts transitively on the unit sphere in $T_{p_0}(M)$.

We have now established that for $m \neq 3$, ≥ 2, M must be a Kählerian space of constant holomorphic curvature. We must now show that M is actually flat.

Suppose M has positive constant holomorphic curvature. Then $I_0(M)$, the connected component of the full group of isometries on M, is isomorphic to

$$\frac{\dfrac{SU(m+1)}{D}}{\dfrac{S(U(m)\times U(1))}{D}},$$

where D is the (discrete) center of $SU(m+1)$, and $S(U(m)\times U(1))$ is the subgroup of $U(m)\times U(1)$ consisting of elements with determinant equal to one. Now $H \cong SU(m)$ implies that $1 + \dim H = \dim S(U(m)\times U(1))$. Thus G must be a subgroup of $SU(m+1)/D$ of codimension one. But it can be easily shown that $SU(m+1)/D$ has no subgroups of codimension one if $m \geq 2$.

Suppose M has negative constant holomorphic curvature. Then

$$I_0(M) \cong \frac{\dfrac{SU(m,1)}{D}}{\dfrac{S(U(m)\times U(1))}{D}},$$

where $SU(m,1)$ is the subgroup of $SL(m+1,C)$ which leaves invariant the hermitian form $-z_1\bar{z}_1 - \cdots - z_m\bar{z}_m + z_{m+1}\bar{z}_{m+1}$, and D is the (discrete) center of $SU(m,1)$. Moreover, as above, G must be a subgroup of $SU(m,1)/D$ of codimension one. But, as for $SU(m+1)$, such subgroups of $SU(m,1)/D$ do not exist.

Hence M must be flat.

Case V. $f_*\mathfrak{h}$ is defined by the equations

$$X_{02} = X_{03} = X_{31} = X_{12} = 0,$$

$$f_*\mathfrak{h} = \left\{ \begin{bmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{bmatrix} \middle|\, a, b \in R \right\} \cong so(2) \oplus so(2)\,.$$

As a consequence of equations (II.1) we have $\omega_2^0 = \omega_3^0 = \omega_1^3 = \omega_2^1 = 0$, i.e., the connection can be restricted to $G(G/H, H)$. Hence $M = M_1 \times M_2$, where M_i is a two-dimensional space of constant curvature K_i, $i = 1, 2$.

Case VI. $f_*\mathfrak{h}$ is defined by the equations

$$X_{02} = X_{03} = X_{31} = X_{12} = 0, \qquad X_{23} = mX_{01} \qquad (m > 0) ,$$

$$f_*\mathfrak{h} = \left\{ \begin{bmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & ma \\ 0 & 0 & -ma & 0 \end{bmatrix} \middle| a \in R \right\} \cong so(2) .$$

As a consequence of equations (II.1) we have $\omega_2^0 = \omega_3^0 = \omega_1^3 = \omega_2^1 = 0$, $\omega_3^2 = m\omega_1^0$, i.e., again the connection can be restricted to $G(G/H, H)$. Thus $\Omega_j^i = 0$, except $\Omega_1^0 = d\omega_1^0$ and $\Omega_3^2 = md\omega_1^0$; and $d\omega_1^0 \wedge \theta^i = 0$, $0 \leq i \leq 3$, from the first Bianchi identity: $\sum_i \Omega_i^j \wedge \theta^i = 0$. Therefore $d\omega_1^0 = 0$, and $\Omega_j^i = 0$ for all i and j. Hence M is $C(0, 4)$.

Case VII. $f_*\mathfrak{h}$ is defined by the equations

$$\chi_{02} = \chi_{03} = \chi_{12} = \chi_{23} = \chi_{31} = 0 ,$$

$$f_*\mathfrak{h} = \left\{ \begin{bmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| a \in R \right\} \cong so(2) .$$

As a consequence of equations (II.1) we have

$$\begin{aligned} \omega_2^0 &= a\theta^0 + b\theta^1 , & \omega_3^0 &= c\theta^0 + e\theta^1 , \\ \omega_2^1 &= -b\theta^0 + a\theta^1 , & \omega_1^3 &= e\theta^0 - c\theta^1 , \\ \omega_3^2 &= r\theta^2 + t\theta^3 , \end{aligned}$$

where a, b, c, e, r, t are constants which must be determined. The first Bianchi identity gives the following relations on these constants:

$$\begin{aligned} 2ab + re &= 0 , & 2cb - rb &= 0 , \\ 2ae + te &= 0 , & 2ce - tb &= 0 , \\ ar + ct &= 0 , \end{aligned}$$

$$d\omega_1^0 = -2K\theta^0 \wedge \theta^1 - (br + et)\theta^2 \wedge \theta^3 , \tag{II.9}$$

where K is a constant. From the second Bianchi identity, $d\Omega_j^i = \sum_k \omega_k^i \wedge \Omega_j^k - \Omega_k^i \wedge \omega_j^k$, we get

$$\begin{aligned} 2aK + e(br + et) &= 0 , \\ 2cK - b(br + et) &= 0 . \end{aligned} \tag{II.10}$$

Consider each of four cases separately.

(i) $b \neq 0,\ e \neq 0$;

(ii) $b \neq 0,\ e = 0$;

(iii) $b = 0,\ e \neq 0$;

(iv) $b = 0,\ e = 0$.

(i) If $a \neq 0$, set $\mu = \omega_1^0 - b\theta^2 - e\theta^3$. Then $d\mu = 0$. Since $\mu \not\equiv 0$, the equation $\mu = 0$ therefore defines a four-dimensional ideal of $\mathfrak{g}$, which in turn defines a four-dimensional normal subgroup $\bar{H}$ of G. $\bar{H}$ also acts transitively on M, because $\bar{H} \cap H$ is discrete. Since $\dim \bar{H} = 4$ and M is simply connected, $\bar{H}$ acts simply transitively on M.

If $a = 0$, put $\mu = e\omega_1^0 - K\theta^3$. Then $d\mu = 0$ and $\mu \not\equiv 0$. As above the subgroup of G defined by $\mu = 0$ acts simply transitively on M.

(ii) If $c \neq 0$ put $\mu = \omega_1^0 - b\theta^2$, and if $c = 0$ put $\mu = b\omega_1^0 - K\theta^2$. Then, in either case, $d\mu = 0$, $\mu \not\equiv 0$, and $\mu = 0$ defines a subgroup of G, which acts simply transitively on M.

(iii) Same as (ii).

(iv) By virture of equation (II.9), $d\omega_1^0 = -2K\theta^0 \wedge \theta^1$ in this case. If $a \neq 0$ or $c \neq 0$, then $K = 0$ by (II.10). Thus $d\omega_1^0 = 0$, $\omega_1^0 \not\equiv 0$, and again $\omega_1^0 = 0$ defines a subgroup of G, which acts simply transitively on M.

If $a = c = 0$, then we have

$$\text{(II.11)} \qquad d\theta^0 = -\omega_1^0 \wedge \theta^1,\ d\theta^1 = -\omega_0^1 \wedge \theta^0,\ \Omega_1^0 = d\omega_1^0 = -2K\theta^0 \wedge \theta^1\ ,$$

$$\text{(II.12)} \qquad d\theta^2 = -r\theta^2 \wedge \theta^3,\ d\theta^3 = -t\theta^2 \wedge \theta^3,\ \Omega_3^2 = -(t^2 + r^2)\theta^2 \wedge \theta^3\ ,$$

and all other $\Omega_j^i = 0$. Hence $M = M_1 \times M_2$, where M_1 and M_2 are two-dimensional spaces with structure equations given by (II.11) and (II.12), respectively. Since K is arbitrary, M_1 is $C(+,2)$, $C(-,2)$ or $C(0,2)$. But $t^2 + r^2 \geq 0$ implies that M_2 can be $C(0,2)$ or $C(-,2)$.

Case VIII. $f_*\mathfrak{h} = \{0\}$. G acts simply transitively on M.

We summarize the results of this chapter in the following theorem. A Riemannian space which has a group of isometries acting simply transitively on it is called a group manifold.

Theorem. *Let M be a four-dimensional, simply connected homogeneous Riemannian manifold. Then M must be one of the following spaces:* $C(0,4)$, $C(+,4)$, $C(-,4)$, $P(2,C)$, $H(2,C)$, $R^1 \times C(+,3)$, $R^1 \times C(-,3)$, $C(\overset{+}{\underset{-}{0}},2) \times C(\overset{+}{\underset{-}{0}},2)$, *or a group manifold. In this list the first five spaces are Einstein spaces, the sixth and seventh spaces are not Einstein spaces, and the last space is an Einstein space if and only if both factors have the same sectional curvature.*

In Chapter III we shall determine which group manifolds are Einstein spaces.

Remark. Suppose that M has a group G, of isometries which acts simply transitively on M. Fix $p_0 \in M$, and define $F\colon G \to M$ by $F(g) = g(p_0)$. Then F is a diffeomorphism of G onto M. Let $\langle , \rangle$ denote the metric tensor on M. Then $F^*\langle , \rangle$ is a metric tensor induced on G, and F is an isometry with respect to these metrics. The important fact is that $F^*\langle , \rangle$ is a left-invariant metric on G. In fact, let l_g denote left multiplication by g in G. Then $F \circ l_g = g \circ F$ implies the third equality in the chain:

$$\begin{aligned} F^*\langle X, Y\rangle_g &= \langle F_* X, F_* Y\rangle_{g(p_0)} = \langle (g^{-1} \circ F)_* X, (g^{-1} \circ F)_* Y\rangle_{p_0} \\ &= \langle (F \circ l_{g^{-1}})_* X, (F \circ l_{g^{-1}})_* Y\rangle_{p_0} = F^*\langle l_{g^{-1}*} X, l_{g^{-1}*} Y\rangle_e , \end{aligned}$$

where $X, Y \in T_g(G)$.

Chapter III

By virtue of the theorem at the end of Chapter II, the problem of determining all simply connected homogeneous four-dimensional Einstein spaces M, is solved up to the case when M is a group manifold. Due to the remark at the end of Chapter II, we may regard a group manifold M as a Lie group G with a left-invariant Riemannian metric. Indeed, due to the results of Chapter I, we need only consider solvable groups G.

In this chapter we shall determine all such G which are Einstein spaces. This chapter is divided into two sections. The first section contains the detailed determination of all solvable group manifolds which are Einstein spaces, and the second a summary of the main results obtained in this chapter.

1. In this chapter G is a four-dimensional solvable Lie group, $\mathfrak{g}$ is its Lie algebra and $\langle , \rangle$ is an inner product on $\mathfrak{g}$. This inner product defines a left-invariant metric on G. Let ω and Ω be the Riemannian connection and curvature forms, respectively, defined on G and taking values in $so(4)$. As with all left-invariant linear froms on G, we regard ω and Ω as linear forms on g. Let $X_1, \cdots, X_4$ be an orthonormal basis of $\mathfrak{g}$, and let $A_k = \omega(X_k) \in so(4)$, $1 \le k \le 4$. Write $A_k = (A^i_{jk})$, with $A^i_{jk} + A^j_{ik} = 0$. Let C^i_{jk} be the structure constants of $\mathfrak{g}$ with respect to $X_1, \cdots, X_4$, i.e., $[X_j, X_k] = \sum_{i=1}^{4} C^i_{jk} X_i$. Then $d\omega(X_i, X_j) = -\frac{1}{2} \sum_{k=1}^{4} C^k_{ij} A_k$, since ω and the X_i are left-invariant. Furthermore,

$$\omega \wedge \omega(X_k, X_l) = \frac{1}{2}(A_k A_l - A_l A_k) = \frac{1}{2}[A_k, A_l] ,$$

where the last expression is the bracket in $so(4)$. Thus

$$\Omega(X_k, X_l) = d\omega(X_k, X_l) + \omega \wedge \omega(X_k, X_l) = \frac{1}{2}([A_k, A_l] - \sum_i C^i_{kl} A_i) .$$

(III.1) Let $R_{ijkl} = 2\Omega_{ij}(X_k, X_l)$ be the curvature tensor.

The Ricci tensor is defined by

$$R_{jl} = \sum_i R_{ijil} = \sum_i [A_i, A_l]_{ij} + \sum_i (A_i C^i)_{jl} , \tag{III.2}$$

where $C^i = (C^i_{jk})$ is a 4×4 skew symmetric matrix.

The scalar curvature is

$$R = \sum_{i,j} [A_i, A_j]_{ij} + \sum_i \operatorname{Trace}(A_i C^i) . \tag{III.3}$$

The $\{A^i_{jk}\}$ and $\{C^i_{jk}\}$ are related, in fact, each set is determined by the other. For, let $\theta^1, \cdots, \theta^4$ be the forms dual to $X_1, \cdots, X_4$. Then $d\theta^i = \sum_k -\omega^i_k \wedge \theta^k$. But $d\theta^i = -\frac{1}{2}C^i_{jk}\theta^j \wedge \theta^k$ and $\omega^i_k = \sum_j A^i_{kj}\theta^j$. Thus $-\frac{1}{2}C^i_{jk}\theta^j \wedge \theta^k = -\sum_k \frac{1}{2}(A^i_{kj} - A^i_{jk})\theta^j \wedge \theta^k$, which implies that $C^i_{jk} = A^i_{kj} - A^i_{jk}$. Conversely, permuting the indices cyclically and adding we get $C^i_{jk} - C^k_{ij} + C^j_{ki} = -2A^i_{jk}$.

The method of this chapter is a direct one. The Lie algebra $\mathfrak{g}$ is determined by specifying the structure constants $\{C^i_{jk}\}$ with respect to some basis $\{X_i\}$. At the same time the inner product on $\mathfrak{g}$ is specified by taking $\{X_i\}$ to be orthonormal. The $\{C^i_{jk}\}$ are determined by the $\{A^i_{jk}\}$. In order for $\mathfrak{g}$ to be a Lie Algebra the Jacobi identities must be satisfied by the C^i_{jk}; and in order for the metric to be Einsteinian the Ricci tensor must satisfy the equations $R_{ij} = R\delta_{ij}/4$. The method of this chapter is to take the $\{A^i_{jk}\}$ as unknowns, to set up the Jacobi and Einstein equations and to find all possible solutions for the A^i_{jk}.

Let $\mathfrak{g}$ be a four-dimensional vector space. Our problem now is to determine all possible ways of making $\mathfrak{g}$ into a solvable Lie algebra with an inner product, such that the Riemannian structure determined by this inner product is Einsteinian. For any Lie algebra structure on $\mathfrak{g}$ we define $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}'' = [\mathfrak{g}', \mathfrak{g}']$, etc. With any solvable Lie algebra structure and inner product on $\mathfrak{g}$ it is possible to choose an orthonormal basis on $\mathfrak{g}$ in one of the following seven ways. Bases chosen in the way of type n) below will be called *admissible* for any $\mathfrak{g}$ of that type for $n = 1, \cdots, 7$.

1) If $\dim \mathfrak{g}' = 3$, $\dim \mathfrak{g}'' = 2$, $\dim \mathfrak{g}''' = 1$, then $X_4 \in \mathfrak{g}'''$, $X_3 \in \mathfrak{g}''$, $X_2 \in \mathfrak{g}'$ and $X_1 \perp \mathfrak{g}'$.
2) If $\dim \mathfrak{g}' = 3$, $\dim \mathfrak{g}'' = 2$, $\mathfrak{g}''' = 0$, then $X_3, X_4 \in \mathfrak{g}''$, $X_2 \in \mathfrak{g}'$ and $X_1 \perp \mathfrak{g}'$.
3) If $\dim \mathfrak{g}' = 3$, $\dim \mathfrak{g}'' = 1$, then $X_4 \in \mathfrak{g}''$, $X_3, X_2 \in \mathfrak{g}'$, $X_1 \perp \mathfrak{g}'$.
4) If $\dim \mathfrak{g}' = 3$, $\mathfrak{g}'' = 0$, then $X_4, X_3, X_2 \in \mathfrak{g}'$, $X_1 \perp \mathfrak{g}'$.
5) If $\dim \mathfrak{g}' = 2$, $\dim \mathfrak{g}'' = 1$, then $X_4 \in \mathfrak{g}''$, $X_3 \in \mathfrak{g}'$, $X_1, X_2 \perp \mathfrak{g}'$.
6) If $\dim \mathfrak{g}' = 2$, $\mathfrak{g}'' = 0$, then $X_4, X_3 \in \mathfrak{g}'$, $X_1, X_2 \perp \mathfrak{g}'$.

7) If $\dim \mathfrak{g}' = 1$, then $X_4 \in \mathfrak{g}'$, $X_1, X_2, X_3 \perp \mathfrak{g}'$.

The importance of admissible bases is the fact that they satisfy the conditions

$$\begin{aligned} &[X_1, X_2] \in \operatorname{span}(X_2, X_3, X_4)\,, && [X_2, X_3] \in \operatorname{span}(X_3, X_4)\,, \\ &[X_1, X_3] \in \operatorname{span}(X_2, X_3, X_4)\,, && [X_2, X_4] \in \operatorname{span}(X_3, X_4)\,, \\ &[X_1, X_4] \in \operatorname{span}(X_2, X_3, X_4)\,, && [X_3, X_4] \in \operatorname{span}(X_4)\,. \end{aligned}$$

Thus, with respect to an admissible basis, the structure constants C^1_{12}, C^1_{13}, C^1_{14}, C^1_{23}, C^1_{24}, C^1_{34}, C^2_{23}, C^2_{24}, C^2_{34} and C^3_{34} are all 0. Consequently, since $C^i_{jk} = A^i_{kj} - A^i_{jk}$, we have that A^1_{21}, A^1_{31}, A^1_{41}, A^2_{32}, A^2_{42}, $A^3_{43} = 0$; and that $A^1_{32} = A^1_{23}$, $A^1_{42} = A^1_{24}$, $A^1_{43} = A^1_{34}$, $A^2_{43} = A^2_{34}$. Hence we may assume that the A^i_{jk} satisfy these relations. In order to facilitate identification of the 14 remaining unknowns let

$$\begin{aligned} &A^2_{31} = \delta, \quad A^2_{41} = \varepsilon, \quad A^3_{41} = \mu, \quad A^1_{22} = a, \quad A^1_{32} = b\,, \\ &A^1_{42} = c, \quad A^3_{42} = f, \quad A^1_{33} = \varphi, \quad A^1_{43} = \sigma, \quad A^2_{33} = \rho\,, \\ &A^2_{43} = \chi, \quad A^1_{44} = \omega, \quad A^2_{44} = y, \quad A^3_{44} = s\,. \end{aligned}$$

Then

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \delta & \varepsilon \\ 0 & -\delta & 0 & \mu \\ 0 & -\varepsilon & -\mu & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & 0 & 0 \\ -b & 0 & 0 & f \\ -c & 0 & -f & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & b & \varphi & \sigma \\ -b & 0 & \rho & \chi \\ -\varphi & -\rho & 0 & 0 \\ -\sigma & -\chi & 0 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & c & \sigma & \omega \\ -c & 0 & \chi & y \\ -\sigma & -\chi & 0 & s \\ -\omega & -y & -s & 0 \end{bmatrix}.$$

The structure constants are

$$C^1 = 0\,, \qquad C^2 = \begin{bmatrix} 0 & a & b+\delta & c+\varepsilon \\ -a & 0 & 0 & 0 \\ -b-\delta & 0 & 0 & 0 \\ -c-\varepsilon & 0 & 0 & 0 \end{bmatrix},$$

$$C^3 = \begin{bmatrix} 0 & b-\delta & \varphi & \mu+\sigma \\ \delta-b & 0 & \rho & \chi+f \\ -\varphi & -\rho & 0 & 0 \\ -\mu-\sigma & -\chi-f & 0 & 0 \end{bmatrix}, \quad C^4 = \begin{bmatrix} 0 & c-\varepsilon & \sigma-\mu & \omega \\ \varepsilon-c & 0 & \chi-f & y \\ \mu-\sigma & f-\chi & 0 & s \\ -\omega & -y & -s & 0 \end{bmatrix}.$$

The bracket operation on $\mathfrak{g}$ is given explicitly by the equations

$$[X_1, X_2] = aX_2 + (b - \delta)X_3 + (c - \varepsilon)X_4\,, \qquad [X_2, X_4] = (\chi + f)X_3 + yX_4\,,$$
$$[X_1, X_3] = (b + \delta)X_2 + \varphi X_3 + (\sigma - \mu)X_4\,, \qquad [X_3, X_4] = sX_4\,,$$
$$[X_1, X_4] = (c + \varepsilon)X_2 + (\mu + \sigma)X_3 + \omega X_4\,,$$
$$[X_2, X_3] = \rho X_3 + (\chi - f)X_4\,.$$

Any four-dimensional solvable Lie algebra with an inner product has an admissible basis $X_1, \cdots, X_4$, and consequently with respect to such a basis the structure constants are of the above form.

Proposition. *The Jacobi identities on* $\mathfrak{g}$ *are equivalent to the first Bianchi identities on the curvature form.*

Proof. This proposition is true for any Lie algebra $\mathfrak{g}$. Let $\theta^1, \cdots, \theta^n$ be an orthonormal coframe of $\mathfrak{g}$, and ω, Ω be the connection form and curvature form, respectively, Then $d^2\theta = 0$ is the Jacobi identity on $\mathfrak{g}$ and $\Omega \wedge \theta = 0$ is the first Bianchi identity. But $d\theta = -\omega \wedge \theta$ and $d\omega = -\omega \wedge \omega + \Omega$, and therefore $d^2\theta = -\Omega \wedge \theta$.

It is convenient to set up the Bianchi identities on the A^i_{jk}, rather than the Jacobi identities. The following procedure will be followed in setting up these equations and the Einstein equations, and for solving them.

I. 1. Compute $[A_i, A_j]$.
   2. Compute R_{ijkl} from formula (III.1).
   3. Compute R_{ij} from the R_{ijkl}; obtain Einstein equations.
   4. Compute R from the R_{ij}.
   5. Set up the Bianchi identities: $R_{ijkl} = R_{klij}$ and
   $$R_{1234} + R_{1423} + R_{1342} = 0\,.$$

II. Solve the equations in each of the following cases.
   1. $R = 0$.
   2. $R \neq 0$.
      Case A. $\dim \mathfrak{g}' = 3$.
         i) $\dim \mathfrak{g}'' = 2$, $\dim \mathfrak{g}''' = 1$.
         ii) $\dim \mathfrak{g}'' = 2$, $\mathfrak{g}''' = 0$.
         iii) $\dim \mathfrak{g}'' = 1$.
         iv) $\mathfrak{g}'' = 0$.
      Case B. $\dim \mathfrak{g}' = 2$.
      Case C. $\dim \mathfrak{g}' = 1$.

I.1. All matrices are skew-symmetric.

$$[A_1, A_2] = \begin{bmatrix} 0 & b\delta + c\varepsilon & c\mu - a\delta & -a\varepsilon - b\mu \\ & 0 & -\varepsilon f & \delta f \\ & & 0 & 0 \\ & & & 0 \end{bmatrix},$$

$$[A_1, A_3] = \begin{bmatrix} 0 & \varphi\delta + \sigma\varepsilon & \sigma\mu - b\delta & -b\varepsilon - \varphi\mu \\ & 0 & \chi\mu & -\rho\mu \\ & & 0 & \rho\varepsilon - \delta\chi \\ & & & 0 \end{bmatrix},$$

$$[A_1, A_4] = \begin{bmatrix} 0 & \sigma\delta + \omega\varepsilon & \omega\mu - c\delta & -c\varepsilon - \sigma\mu \\ & 0 & \mu y - \varepsilon s & \delta s - \chi\mu \\ & & 0 & \chi\varepsilon - \delta y \\ & & & 0 \end{bmatrix},$$

$$[A_2, A_3] = \begin{bmatrix} 0 & -b\rho - c\chi & a\rho + \sigma f & a\chi - \varphi f \\ & 0 & b^2 + \chi f - a\varphi & bc - \rho f - a\sigma \\ & & 0 & \varphi c - b\sigma \\ & & & 0 \end{bmatrix},$$

$$[A_2, A_4] = \begin{bmatrix} 0 & -b\chi - cy & a\chi + \omega f - cs & ay + bs - \sigma f \\ & 0 & cb + yf - a\sigma & c^2 - \chi f - a\omega \\ & & 0 & \sigma c - b\omega \\ & & & 0 \end{bmatrix},$$

$$[A_3, A_4] = \begin{bmatrix} 0 & \sigma\rho + \omega\chi - \varphi\chi - \sigma y & b\chi - \sigma s - c\rho & by + \varphi s - c\chi \\ & 0 & c\varphi - b\sigma - \chi s & c\sigma + \rho s - b\omega \\ & & 0 & \sigma^2 + \chi^2 - \varphi\omega - \rho y \\ & & & 0 \end{bmatrix},$$

I.2.

$$\begin{aligned}
R_{1212} &= 2(b\delta + c\varepsilon) - a^2 - b^2 - c^2 \,, \\
R_{1313} &= 2(\sigma\mu - b\delta) - b^2 - \varphi^2 - \sigma^2 \,, \\
R_{1414} &= -2(c\varepsilon + \sigma\mu) - c^2 - \sigma^2 - \omega^2 \,, \\
R_{2323} &= 2\chi f - a\varphi + b^2 - \rho^2 - \chi^2, \\
R_{2424} &= -2\chi f - a\omega + c^2 - \chi^2 - y^2 \,, \\
R_{3434} &= \sigma^2 + \chi^2 - \varphi\omega - \rho y - s^2 \,, \\
R_{1213} &= \varphi(\delta - b) + \sigma(\varepsilon - c) - a(b + \delta) + c\mu \,, \\
R_{1312} &= c\mu - a(\delta + b) + \varphi(\delta - b) - \sigma(c - \varepsilon) \,, \\
R_{1214} &= \sigma\delta + \omega(\varepsilon - c) - a(c + \varepsilon) - b(\mu + \sigma) \,, \\
R_{1412} &= -b\mu - a(\varepsilon + c) - \sigma(b - \delta) - \omega(c - \varepsilon) \,, \\
R_{1223} &= -2b\rho - 2c\chi + cf \,, \\
R_{2312} &= -\varepsilon f - \rho(b - \delta) - \chi(c - \varepsilon) \,, \\
R_{1224} &= -b(2\chi + f) - 2cy \,, \\
R_{2412} &= \delta f - \chi(b - \delta) - y(c - \varepsilon) \,, \\
R_{1234} &= \sigma(\rho - y) + \chi(\omega - \varphi) - cs \,,
\end{aligned}$$

$$R_{3412} = -fa - s(c - \varepsilon) ,$$
$$R_{1314} = \omega(\mu - \sigma) - \varphi(\mu + \sigma) - b(c + \varepsilon) - c\delta ,$$
$$R_{1413} = -b\varepsilon - \varphi(\mu + \sigma) - c(b + \delta) - \omega(\sigma - \mu) ,$$
$$R_{1323} = \rho(a - \varphi) + \sigma(2f - \chi) ,$$
$$R_{2313} = -\rho\varphi + \chi(2\mu - \sigma) ,$$
$$R_{1324} = a\chi + \omega f - cs - \sigma y - \varphi(\chi + f) ,$$
$$R_{2413} = -\rho\mu - \chi\varphi - y(\sigma - \mu) ,$$
$$R_{1334} = b\chi - c\rho - 2\sigma s ,$$
$$R_{3413} = \rho\varepsilon - \delta\chi - f(b + \delta) - s(\sigma - \mu),$$
$$R_{1423} = a\chi - \varphi f - \sigma\rho - \omega(\chi - f) ,$$
$$R_{2314} = \mu y - \varepsilon s - \chi\omega - \rho(\mu + \sigma) ,$$
$$R_{1424} = y(a - \omega) - \sigma(\chi + 2f) + bs ,$$
$$R_{2414} = \delta s - y\omega - \chi(2\mu + \sigma) ,$$
$$R_{1434} = by - c\chi + s(\varphi - \omega) ,$$
$$R_{3414} = \chi\varepsilon - \delta y - s\omega - f(c + \varepsilon) ,$$
$$R_{2324} = cb - a\sigma + y(f - \chi) - \rho(\chi + f) ,$$
$$R_{2423} = bc - a\sigma - \rho(\chi + f) + y(f - \chi) ,$$
$$R_{2334} = c\varphi - b\sigma - 2\chi s .$$
$$R_{3423} = c\varphi - b\sigma - s(\chi - f) ,$$
$$R_{2434} = c\sigma - b\omega + s(\rho - y) ,$$
$$R_{3424} = c\sigma - b\omega - sy .$$

I.3. Einstein equations: $R_{ij} = R\,\delta_{ij}/4$.

1) $R_{11} = -(a^2 + 2b^2 + 2c^2 + 2\sigma^2 + \varphi^2 + \omega^2)$,

2) $R_{22} = 2(b\delta + c\varepsilon) - a(\varphi + \omega) - a^2 - \rho^2 - 2\chi^2 - y^2$,

3) $R_{33} = 2(\sigma\mu - b\delta + \chi f) - \rho y - \varphi(a + \omega) - \varphi^2 - \rho^2 - s^2$,

4) $R_{44} = -2(c\varepsilon + \sigma\mu + \chi f) - py - \omega(a + \varphi) - \omega^2 - y^2 - s^2$,

5) $R_{12} = \rho(a - \varphi) + y(a - \omega) - 2\sigma\chi + bs$,

6) $R_{13} = b(2\rho + y) + c(\chi - f) + s(\varphi - \omega)$,

7) $R_{14} = b(\chi + f) + c(2y + \rho) + 2\sigma s$,

8) $R_{23} = \varphi(\delta - b) + \sigma\varepsilon - a(b + \delta) + c\mu - b\omega + s(\rho - y)$,

9) $R_{24} = \sigma\delta + \omega(\varepsilon - c) - a(c + \varepsilon) - b\mu - c\varphi + 2\chi s$,

10) $R_{34} = \omega(\mu - \sigma) - \rho(\chi + f) + y(f - \chi) - \varphi(\mu + \sigma) - c\delta - b\varepsilon - a\sigma$.

I.4. $R = -((a+\varphi)^2 + 2b^2 + (\rho + y)^2 + (a+\omega)^2 + (\varphi+\omega)^2 + 2c^2 + 2\sigma^2 + 2\chi^2 + \rho^2 + 2s^2 + y^2)$.

I.5. Bianchi identities:

11) $0 = (f-\chi)(c+\varepsilon) - \rho(b+\delta)$,

12) $0 = (\chi + f)(b+\delta) + y(c+\varepsilon)$,

13) $0 = \sigma(\rho - y) + \chi(\omega - \varphi) + af - s\varepsilon$,

14) $0 = a\rho + 2\sigma f - 2\chi\mu$,

15) $0 = a\chi + f(\omega - \varphi) - cs + \mu(\rho - y)$,

16) $0 = (\chi + f)(b+\delta) - s(\sigma + \mu) - \rho(c+\varepsilon)$,

17) $0 = a\chi - f(\varphi - \omega) + \mu(\rho - y) + \varepsilon s$,

18) $0 = ay + s(b-\delta) - 2\sigma f + 2\chi\mu$,

19) $0 = y(b+\delta) + \varphi s + (f-\chi)(c+\varepsilon)$,

20) $0 = s(\chi + f)$,

21) $0 = \rho s$.

II. We now find all possibe solutions up to isomorphic Lie algebras with isometric Riemannian structures. Each solution is described in detail as it arises. The results will be summarized in a theorem at the end of the chapter. In the rest of this chapter equations numbered 1) through 21) refer to the numbered equations in the above steps I.3 and I.5.

II.1. $R = 0$. From the above equation I.4, we get

$$a, \varphi, b, y, \omega, c, \sigma, \chi, \rho, s = 0 .$$

The Einstein equations all become identities, $0 = 0$, and the Bianchi equations become (listing only non-identities)

11) $0 = f\varepsilon$, 16) $0 = f\delta$,

12) $0 = f\delta$, 19) $0 = f\varepsilon$.

These equations have two sets of solutions:

a) $f = 0$, ε and δ arbitrary,

b) $\varepsilon = \delta = 0$, f arbitrary.

In either case there are no conditions on μ, so μ is arbitrary.

In case a) the bracket operations become

$$\text{(III.4)}\qquad \begin{aligned} [X_1, X_2] &= -\delta X_3 - \varepsilon X_4 , & [X_2, X_3] &= 0 , \\ [X_1, X_3] &= \delta X_2 - \mu X_4 , & [X_2, X_4] &= 0 , \\ [X_1, X_4] &= \varepsilon X_2 + \mu X_3 , & [X_3, X_4] &= 0 . \end{aligned}$$

In case b) they become

$$
\text{(III.5)}\qquad
\begin{aligned}
&[X_1, X_2] = 0\,, && [X_2, X_3] = -fX_4\,,\\
&[X_1, X_3] = -\mu X_4\,, && [X_2, X_4] = fX_3\,,\\
&[X_1, X_4] = \mu X_3\,, && [X_3, X_4] = 0\,.
\end{aligned}
$$

We observe that the solutions of type (III.4) can be transformed into solutions of type (III.5) by an admissible change of basis. To see this, consider $Ad(X_1)$ restricted to span (X_2, X_3, X_4) for arbitrarily given ε, δ and μ. The matrix of $Ad(X_1)$ with respect to X_2, X_3, X_4 is

$$
\begin{bmatrix} 0 & \delta & \varepsilon \\ -\delta & 0 & \mu \\ -\varepsilon & -\mu & 0 \end{bmatrix}.
$$

Consider the standard inner product $\langle A, B\rangle = -\operatorname{Trace} AB$ on $so(3)$. Then the adjoint representation of $SO(3)$ has $SO(3)$ acting on $so(3)$ as orthogonal transformations with respect to this inner product, and $SO(3)$ acts transitively on the unit sphere in $so(3)$. Hence there exists an $A \in SO(3)$ such that

$$
A\begin{bmatrix} 0 & \delta & \varepsilon \\ -\delta & 0 & \mu \\ -\varepsilon & -\mu & 0 \end{bmatrix}A^{-1} = \sqrt{\delta^2 + \varepsilon^2 + \mu^2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.
$$

Thus the desired change of basis is given by A. If $X_1, \cdots, X_4$ are now the new admissible basis, then

$$
\text{(III.6)}\qquad
\begin{aligned}
&[X_1, X_2] = 0\,, && [X_2, X_3] = 0\,,\\
&[X_1, X_3] = -tX_4\,, && [X_2, X_4] = 0\,,\\
&[X_1, X_4] = tX_3\,, && [X_3, X_4] = 0\,,
\end{aligned}
$$

where $t = \sqrt{\delta^2 + \varepsilon^2 + \mu^2}$. This is a solution of type (III.5) with $\mu = t$, $f = 0$.

In a solution of type (III.5), let $\mu^2 + f^2 > 0$ be given. Let $X = (\mu^2 + f^2)^{-1/2}(\mu X_1 + fX_2)$, $Y = (\mu^2 + f^2)^{-1/2}(fX_1 - \mu X_2)$, where $X_1, \cdots, X_4$ are an admissible basis for this solution. Then X, Y, X_3, X_4 are also an admissible basis for the same solution, but with respect to this basis the bracket operations are the same as (III.6) with $t = \sqrt{\mu^2 + f^2}$. Thus we need consider only solutions of type (III.6).

Let $X_1, \cdots, X_4$ be an admissible basis for the Lie algebra $\mathfrak{g}$ defined by (III.6) with $t = 1$. For any $t > 0$ equations (III.6) are satisfied by tX_1, X_2, X_3, X_4. Hence equations (III.6) define a 1-parameter family of inner products on a single Lie algebra $\mathfrak{g}$.

Proposition 1. *The left-invariant Riemannian metric defined by any of these inner products is flat.*

Proof. X_2 is a central element of $\mathfrak{g}$, i.e., $\mathfrak{z}(\mathfrak{g}) \neq 0$. Now apply Corollary 1 from Chapter I.

Let G be the simply connected Lie group with Lie algebra $\mathfrak{g}$. Then the above 1-parameter family of inner products on $\mathfrak{g}$ induces a 1-parameter family of left-invariant Riemannian metrics on G, each of which is flat by Proposition 1. Hence this is an example of a simply connected Lie group G for which distinct inner products on its Lie algebra $\mathfrak{g}$ define isometric left-invariant Riemannian metrics on G.

A matrix representation of $\mathfrak{g}$ is given by

$$X_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_2 = \left[\begin{array}{ccc|c} & & & 1 \\ & \bigcirc & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right],$$

$$X_3 = \left[\begin{array}{ccc|c} & & & 0 \\ & \bigcirc & & 1 \\ & & & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right], \quad X_4 = \left[\begin{array}{ccc|c} & & & 0 \\ & \bigcirc & & 0 \\ & & & 1 \\ \hline 0 & 0 & 0 & 0 \end{array}\right].$$

II.2. Recall that the equations numbered 1) through 21) refer to the equations listed in the above steps I.3 and I.5.

Case A. i) $\dim \mathfrak{g}' = 3$, $\dim \mathfrak{g}'' = 2$, $\dim \mathfrak{g}''' = 1$. $\mathfrak{g}''' \neq 0 \Rightarrow s \neq 0$. Thus $\rho = 0$ by 21), and then $f = -\chi$ by 20). But this implies that $\dim \mathfrak{g}'' \leq 1$. Hence there are no solvable Lie algebras satisfying these conditions.

ii) $\dim \mathfrak{g}' = 3$, $\dim \mathfrak{g}'' = 2$, $\mathfrak{g}''' = 0$. For convenience we rewrite the bracket operations from page 31.

(III.7):

$$[X_1, X_2] = aX_2 + (b - \delta)X_3 + (c - \varepsilon)X_4\,, \quad [X_2, X_3] = \rho X_3 + (\chi - f)X_4$$
$$[X_1, X_3] = (b + \delta)X_2 + \varphi X_3 + (\sigma - \mu)X_4\,, \quad [X_2, X_4] = (\chi + f)X_3 + yX_4\,,$$
$$[X_1, X_4] = (c + \varepsilon)X_2 + (\mu + \sigma)X_3 + \omega X_4\,, \quad [X_3, X_4] = 0\,.$$

We may choose $X_3 \in \mathfrak{g}'$ so that $[X_1, X_2] \in \operatorname{span}(X_2, X_3)$, that is, we may assume that $\varepsilon = c$. Now

14) + 18) is $0 = a(\rho + y)$,

19) − 11) is $0 = (b + \delta)(\rho + y)$,

12) − 16) is $0 = (c + \varepsilon)(\rho + y)$.

If $\rho + y \neq 0$, then a, $b + \delta$, $c + \varepsilon = 0$, and $\dim \mathfrak{g}' \leq 2$. Thus $\rho + y = 0$, $\rho = -y$.

Claim. $y \neq 0$.

Proof. Suppose $y = 0$. Then $\dim \mathfrak{g}'' = 2$ implies that $\chi = f \neq 0$ and $\chi - f \neq 0$. (Look at equations (III.7).) Then $b + \delta = 0$ by 12), and $\dim \mathfrak{g}' = 3$ requires that $a \neq 0$.

6) becomes: $0 = c(\chi - f)$, and 7) becomes: $0 = b(\chi + f)$.

Thus $b = c = 0$, and $\delta = 0$ since $b + \delta = 0$. Hence we now have:

$$\begin{aligned} &[X_1, X_2] = aX_2 , && [X_2, X_3] = (\chi - f)X_4 , \\ &[X_1, X_3] = \varphi X_3 + (\sigma - \mu)X_4 , && [X_2, X_4] = (\chi + f)X_3 , \\ &[X_1, X_4] = (\mu + \sigma)X_3 + \omega X_4 , && [X_3, X_4] = 0 . \end{aligned}$$

$0 = \sigma\chi$ by 5). If $\chi = 0$, then $f \neq 0$, and 18) implies that $\sigma = 0$. But 13) implies that $0 = af$ which would be impossible. Thus $\chi \neq 0$ and $\sigma = 0$. Thus $\mu = 0$ by 14). Consider

15) $0 = a\chi + f(\omega - \varphi)$,

13) $0 = (\omega - \varphi)\chi + fa$,

which are linear in a and $\omega - \varphi$, and $a \neq 0$. Thus $\chi^2 - f^2 = 0$, which contradicts that $\chi + f \neq 0$ and $\chi - f \neq 0$. This proves the claim that $y \neq 0$. Now $\delta = -b$ by 19), and then $c = 0$ by 16), and $b = 0$ by 6).

Of the remaining equations consider the following four, which are linear in a, σ, $\omega - \varphi$ and μ:

$$\begin{aligned} &5) \quad 0 = -y(\omega - \varphi) - 2\chi\sigma, \\ &13) \quad 0 = \chi(\omega - \varphi) \quad - 2y\sigma + fa, \\ &15) \quad 0 = f(\omega - \varphi) \qquad\qquad + \chi a - 2y\mu, \\ &18) \quad 0 = \qquad\qquad\quad - 2f\sigma + ya + 2\chi\mu. \end{aligned}$$

Again consider the table of bracket operations:

$$\text{(III.8)} \quad \begin{aligned} &[X_1, X_2] = aX_2 , && [X_2, X_3] = -yX_3 + (\chi - f)X_4 , \\ &[X_1, X_3] = \varphi X_3 + (\sigma - \mu)X_4 , && [X_2, X_4] = (\chi + f)X_3 + yX_4 , \\ &[X_1, X_4] = (\mu + \sigma)X_3 + \omega X_4 , && [X_3, X_4] = 0 . \end{aligned}$$

Since $\dim \mathfrak{g}' = 3$ it follows that $a \neq 0$. Thus

$$0 = \det \begin{bmatrix} -y & -2\chi & 0 & 0 \\ \chi & -2y & f & 0 \\ f & 0 & \chi & -2y \\ 0 & -2f & y & 2\chi \end{bmatrix} = 4(\chi^2 + y^2)(\chi^2 + y^2 - f^2) ,$$

and $f^2 = \chi^2 + y^2$. But, looking at (III.8), this implies that $\dim \mathfrak{g}'' \leq 1$. Hence there are no solutions in Case A, ii).

iii) $\dim \mathfrak{g}' = 3$, $\dim \mathfrak{g}'' = 1$.

In this case we may assume that $\rho = 0$ and $\chi + f = 0$, merely by taking $X_4 \in \mathfrak{g}''$, $X_2, X_3 \in \mathfrak{g}'$, and $X_1 \perp \mathfrak{g}'$, which is certainly an admissible basis. Then the following four equations are reduced to

12) $0 = y(c + \varepsilon)$, 16) $0 = s(\sigma + \mu)$,

11) $0 = \chi(c + \varepsilon)$, 19) $0 = y(b + \delta) + \varphi s$.

We shall consider two cases. Case (a): $s \neq 0$ and Case (b): $s = 0$.

Case (a). $s \neq 0$. Then $c + \varepsilon = 0$, for otherwise, $\chi = 0 = y$ by 11) and 12), and then $\varphi = 0$ by 19). But then $c = 0$ by 15) and $\varepsilon = 0$ by 17); i.e., $c + \varepsilon = 0$, a contradiction. Note $\sigma + \mu = 0$ by 16). The bracket operation table has become:

$$\begin{aligned} &[X_1, X_2] = aX_2 + (b - \delta)X_3 + 2cX_4\,, && [X_2, X_3] = 2\chi X_4\,, \\ &[X_1, X_3] = (b + \delta)X_2 + \varphi X_3 + 2\sigma X_4\,, && [X_2, X_4] = yX_4\,, \\ &[X_1, X_4]\omega X_4\,, && [X_3, X_4] = sX_4\,. \end{aligned}$$

Let $V = \operatorname{span}(X_2, X_3)$. Then $\mathfrak{g}' = \operatorname{span}([X_1, V], X_4)$ has dimension 3, and consequently $\dim [X_1, V] = 2$ and $X_4 \notin [X_1, V]$. Thus $\dim\{V \cap [X_1, V]\} \geq 1$. So we may pick a unit vector $Y_3 \in V$ such that $[X_1, Y_3] \in V$, and a unit vector $Y_2 \in V$ such that $Y_2 \perp Y_3$. Then X_1, Y_2, Y_3, X_4 are again an admissible basis. Notice that $[Y_2, Y_3] \in \operatorname{span}(X_4)$, $[Y_2, X_4] \in \operatorname{span}(X_4)$ and $[Y_3, X_4] \in \operatorname{span}(X_4)$ since $[V, V]$ and $[V, X_4]$ are both contained in $\operatorname{span}(X_4)$. Thus, if we relabel this basis as $X_1, \cdots, X_4$, then the structure constants become:

$$\text{(III.9)}\quad \begin{aligned} &[X_1, X_2] = aX_2 + (b - \delta)X_3 + 2cX_4\,, && [X_2, X_3] = 2\chi X_4, \\ &[X_1, X_3] = (b + \delta)X_2 + \varphi X_3\,, && [X_2, X_4] = yX_4\,, \\ &[X_1, X_4] = \omega X_4\,, && [X_3, X_4] = sX_4\,. \end{aligned}$$

In particular we have $\sigma = 0$. Notice, however, that the condition $s \neq 0$ may no longer hold.

Consider 18) $0 = ay + s(b - \delta)$,

19) $0 = (b + \delta)y + \varphi s$.

Looking at the first two equations in (III.9) we see that

$$\det \begin{bmatrix} a & b - \delta \\ b + \delta & \varphi \end{bmatrix} \neq 0\,,$$

since $\dim \mathfrak{g}' = 3$. Thus $y = 0$, $s = 0$, and $\chi \neq 0$ since $\mathfrak{g}'' \neq 0$. Therefore $a + \varphi - \omega = 0$ by 17), and $c = 0$ by 6).

From 2) + 3) $- R/2 = 4\chi^2 + 2(a + \varphi)^2$, and using 4) we get that $\chi^2 = -R/24$. We may set $R = -24$ with no loss of generality. Then $\chi^2 = 1$, and

$(a + \varphi)^2 = 4$. Squaring the equation on the last third line on p. 338 we get that $\omega^2 = 4$. Then

$$0 = 2b\delta + \varphi^2 - a^2 \quad \text{from 3)-2)};$$
$$0 = \delta(\varphi - a) - 2b(\varphi + a) \quad \text{by 8)}.$$

Multiplying the last equation by $(\varphi + a)$, and using the equation above it we get that $0 = b(\delta^2 + 4)$. Hence $b = 0$.

Now $a^2 + \varphi^2 = 2$ by 1). Thus $2a\varphi = (a + \varphi)^2 - (a^2 + \varphi^2) = 2$, and $a^2 = 1$ by 2), and consequently $\varphi^2 = 1$. But $a\varphi = 1$, and $a^2 = 1$, $\varphi^2 = 1$ implies that $a = \varphi$.

All 21 equations are now satisfied. The solutions are: $b, c, \varepsilon, \mu, \sigma, y, s, \rho = 0$, δ arbitrary, $\chi^2 = 1$, $a^2 = 1$, $a = \varphi$, $\omega = 2a$, $f = -\chi$ with $R = -24$. Thus

$$\text{(III.10)} \qquad \begin{aligned} &[X_1, X_2] = aX_2 - \delta X_3\,, && [X_2, X_3] = 2\chi X_4\,,\\ &[X_1, X_3] = \delta X_2 + aX_3\,, && [X_2, X_4] = 0\,,\\ &[X_1, X_4] = 2aX_4\,, && [X_3, X_4] = 0\,. \end{aligned}$$

Remark. It suffices to take $a = \chi = 1$. In fact, let $\varepsilon_i = \pm 1$, $i = 1, 2$. Then

$$\begin{aligned} &[\varepsilon_1 X_1, \varepsilon_2 X_2] = (\varepsilon_1 a)(\varepsilon_2 X_2) - (\varepsilon_1\varepsilon_2\delta)X_3\,, && [\varepsilon_2 X_2, X_3] = 2(\varepsilon_2\chi)X_4\,,\\ &[\varepsilon_1 X_1, X_3] = (\varepsilon_1\varepsilon_2\delta)(\varepsilon_2 X_2) + (a\varepsilon_1)X_3\,, && [X_2, X_4] = 0\,,\\ &[\varepsilon_1 X_1, X_4] = 2(\varepsilon_1 a)X_4\,, && [X_3, X_4] = 0\,. \end{aligned}$$

Thus by an admissible change of basis we may make $a = \chi = 1$, although δ may have been changed to $-\delta$. If so then $\varepsilon_1 X_1, X_3, \varepsilon_2 X_2, -X_4$ are an admissible basis with structure constants $a = \chi = 1$, $+\delta$.

Proposition 2. *All of the solutions defined by* (III.10) *are, as Riemannian spaces, isometric to a hermitian hyperbolic space.*

Proof. By a hermitian hyperbolic space we mean the symmetric space $M = SU(2, 1)/S(U_2 \times U_1)$ defined by the involutive automorphism φ on $su(2, 1)$, where

$$su(2,1) = \left\{ \begin{bmatrix} Z & u \\ {}^t\bar{u} & ia \end{bmatrix} \,\middle|\, \begin{array}{l} Z \text{ is } 2 \times 2 \text{ skew hermitian, } u \in C^2, \\ a \in R,\ a + \operatorname{Trace} Z = 0. \end{array} \right\},$$

$$\varphi \begin{bmatrix} Z & u \\ {}^t\bar{u} & ia \end{bmatrix} = \begin{bmatrix} Z & -u \\ -{}^t\bar{u} & ia \end{bmatrix},$$

and with the bilinear form B on $su(2, 1)$ given by $B(X, Y) = -\frac{1}{2}\operatorname{Trace} XY$.

Let $\mathfrak{g}(\delta)$ be the Lie algebra defined by the structure constants $a = \chi = 1$ and δ as in equations (III.10) with respect to a given basis $X_1, \cdots, X_4$, and take the inner product on $\mathfrak{g}(\delta)$ defined by making $X_1, \cdots, X_4$ orthonormal.

For convenience we define $\mathfrak{h} \subset su(2,1)$ to be the $+1$ eigenspace of φ, and let $\mathscr{P}$ be the -1 eigenspace of φ.

Consider the faithful matrix representation of $\mathfrak{g}(\delta)$:

$$X_1 = \begin{bmatrix} -2i\dfrac{\delta}{3} & 0 & 0 \\ 0 & i\dfrac{\delta}{3} & 1 \\ 0 & 1 & i\dfrac{\delta}{3} \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

$$X_3 = \begin{bmatrix} 0 & i & -i \\ i & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \quad X_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & i & -i \\ 0 & i & -i \end{bmatrix}.$$

Regard this representation as a monomorphism $D\colon \mathfrak{g}(\delta) \to su(2,1)$, and let $G(\delta)$ be the simply connected Lie group with algebra $\mathfrak{g}(\delta)$. Then D induces a homomorphism $\hat{D}\colon G(\delta) \to SU(2,1)$, and thus $G(\delta)$ acts on M via the composition of $\hat{D}$ and the projection $\pi\colon SU(2,1) \to M$. This action defines a map $f\colon G(\delta) \to M$ given by $f(x) = \pi \circ \hat{D}(x)$. Then $f_* = \pi_* D$ maps $\mathfrak{g}(\delta)$ onto $T_{\pi(1)}(M)$, (cf. proof of claim below). Thus f maps a neighborhood of $1 \in G(\delta)$ onto a neighborhood of $\pi(1) \in M$. Thus $f(G(\delta))$ is open and closed in M and, consequently, f is surjective. By dimension, f_* is injective. Thus $f^{-1}(\{\pi(1)\})$ is a discrete subgroup of $G(\delta)$, and so $f\colon G(\delta) \to M$ is a covering space. But M is simply connected. Hence f is a diffeomorphism, i.e., $G(\delta)$ acts simply transitively on M.

Claim. *f is an isometry.*

Proof. It suffices to prove that $f_*\colon \mathfrak{g}(\delta) \to T_{\pi(1)}(M)$ is an isometry. The metric on $T_{\pi(1)}(M)$ is defined by identifying $T_{\pi(1)}(M)$ with $\mathscr{P}$ via π_* and then taking B restricted to $\mathscr{P}$. (Recall that $B(X, Y) = -\frac{1}{2}$ Trace XY.) The metric on $\mathfrak{g}(\delta)$ is defined by the orthonormal basis $X_1, \cdots, X_4$ above. Then

$$f_*X_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad f_*X_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

$$f_*X_3 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \quad f_*X_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

are an orthonormal basis of $\mathscr{P}$ with the inner product B.

Hence we have proven that the Riemannian space defined by $\mathfrak{g}(\delta)$ is isometric to a hermitian hyperbolic space.

Proposition 3. *$\mathfrak{g}(\delta)$ is ismorphic, as Lie algebra, to $\mathfrak{g}(\mu)$ if and only if $|\delta| = |\mu|$.*

Proof. By the remark on page 339 it is clear that $\mathfrak{g}(\delta) \cong \mathfrak{g}(-\delta)$.

Let $X_1, \cdots, X_4$ be an admissible basis of $\mathfrak{g}(\delta)$ with structure constants $a = \chi = 1$ and δ, notation as in equations (III.10). Let X be any vector in $\mathfrak{g}(\delta)$, which is not contained in $\mathfrak{g}(\delta)'$. Then $X = \sum_{i=1}^{4} a_i X_i$, $a_1 \neq 0$. Regard Ad(X) as a linear transformation of $\mathfrak{g}(\delta)' \to \mathfrak{g}(\delta)'$. Then the matrix of Ad$(X)$ with respect to the basis X_2, X_3, X_4 is

$$\begin{bmatrix} a_1 & a_1\delta & 0 \\ -a_1\delta & a_1 & 0 \\ -2a_3 & 2a_2 & 2a_1 \end{bmatrix}.$$

The eigenvalues of Ad(X) are $2a_1$ and $a_1(1 \pm i\delta)$. Notice that $|a_1|$ = length of the component of $X \perp \mathfrak{g}(\delta)'$.

Now let Y be any vector in $\mathfrak{g}(\mu)$, $\not\subset \mathfrak{g}(\mu)'$. Regarding Ad$(Y)$ as a linear transformation of $\mathfrak{g}(\mu)' \to \mathfrak{g}(\mu)'$, its eigenvalues are $2b_1$, and $b_1(1 \pm i\mu)$, where $|b_1|$ = length of the components of Y normal to $\mathfrak{g}(\mu)'$. Note $|b_1| \neq 0$.

Suppose there exist such X and Y with equal eigenvalues. Then $a_1 = b_1$ and consequently $|\delta| = |\mu|$.

Recall that we are still in Case A, iii); i.e., $\dim \mathfrak{g}' = 3$, $\dim \mathfrak{g}'' = 1$. We have completed part a), which was headed by the assumption that $s \neq 0$.

b) $s = 0$. Thus $[X_3, X_4] = 0$. We may assume that $[X_2, X_4] = 0$, also, for otherwise by interchanging X_2 and X_3 we would again have an admissible basis, and with respect to it $s \neq 0$. Thus we have $y = 0$. Recall that we already have $\rho = 0$ and $f = -\chi$. Thus $\chi \neq 0$ since $\mathfrak{g}'' \neq 0$. Consequently $c + \varepsilon = 0$ by 11), and $\mu + \sigma = 0$ by 18). Now

$$\begin{aligned} &[X_1, X_2] = aX_2 + (b - \delta)X_3 + 2cX_4\,, && [X_2, X_3] = 2\chi X_4\,, \\ &[X_1, X_3] = (b + \delta)X_2 + \varphi X_3 + 2\sigma X_4\,, && [X_2, X_4] = 0\,, \\ &[X_1, X_4] = \omega X_4\,, && [X_3, X_4] = 0\,. \end{aligned}$$

But this is the same situation as arose in part a) above, for note that choosing a basis which made $\sigma = 0$ in no way depended on s being non-zero, at least not at the stage when it was chosen. That is, we may again choose a basis, as we did in part a), so that $\sigma = 0$, and then proceed exactly as we did then.

Case A, iv). $\dim \mathfrak{g}' = 3$, $\mathfrak{g}'' = 0$.

Now, with respect to an admissible basis, ρ, χ, f, y, $s = 0$. We may pick an admissible basis which will eliminate some more of the unknowns in the following way. Regard Ad(X_1) as a linear transformation of $\mathfrak{g}' \to \mathfrak{g}'$. Then $\dim \mathfrak{g}' = 3$ implies that Ad(X_1) has at least one real eigenvector, which may be assumed to be X_2. This makes $b - \delta = 0$ and $c - \varepsilon = 0$. Now let V equal the orthogonal complement of X_2 in $\mathfrak{g}'$. $\operatorname{Ad}(X_1)V$ must have dimension 2 since $\dim \mathfrak{g}' = 3$. Thus $V \cap \operatorname{Ad}(X_1)V \neq 0$ and we may pick $X_3 \in V$ such that $\operatorname{Ad}(X_1)X_3 \in V$, which makes $b + \delta = 0$. X_4 is now determined up to sign.

We now have

$$\begin{aligned} &[X_1, X_2] = aX_2 , && [X_2, X_3] = 0 , \\ &[X_1, X_3] = \varphi X_3 + (\sigma - \mu)X_4 , && [X_2, X_4] = 0 , \\ &[X_1, X_4] = 2cX_2 + (\mu + \sigma)X_3 + \omega X_4 , && [X_3, X_4] = 0 . \end{aligned} \tag{III.11}$$

The Bianchi equations are all satisfied.

Claim. $c = 0$.

Proof. Suppose $c \neq 0$. Then $\mu = -\sigma$, $\varphi = -2a$ by 8) and 9). If $\sigma \neq 0$, then $a = -2\omega$ by 10) and $R/4 = 2c^2 + 6\omega^2$ by 2), which is impossible since $R < 0$. Thus $\sigma = 0$. From 1) + 4) − 2) − 3),

$$c^2 = 3a^2 + R/4 . \tag{III.12}$$

Putting this into 1) gives $\omega^2 = -3R/4 - 11a^2$. Putting these into 4) gives $0 = a(5a + \omega)$. But $a \neq 0$, since $\dim \mathfrak{g}' = 3$ (look at equations (III.11)). Thus $\omega = -5a$. Putting this back into 1) gives $-R/2 = 24a^2$. Consequently, from (III.12) we get $c^2 = 3R/16 < 0$, which is impossible. Hence $c = 0$.

Now $0 = \mu(\omega - \varphi) - \sigma(\omega + \varphi + a)$ by 10); and 4) − 3) gives $0 = 4\sigma\mu + (\omega - \varphi)(a + \omega + \varphi)$. Multiply through by σ and use 10) above to get $0 = \mu(4\sigma^2 + (\omega - \varphi)^2)$.

Suppose $\mu \neq 0$. Then $\sigma = 0$, $\omega = \varphi$, and $0 = a(a - \varphi)$ by 1) − 3). Thus $a = \varphi$ since $a \neq 0$. Hence $a^2 = -R/12$ by 1), and all 21 equations are now satisfied.

Suppose $\mu = 0$. Then $0 = \sigma(\omega + \varphi + a)$ by 10). If $\omega + \varphi + a = 0$, then $R = 0$ by 2), which is not the case. Thus $\sigma = 0$, and $\omega = \varphi$ by 4) − 3). Hence we get the same solution as above, except with $\mu = 0$.

The solutions we have obtained in this case are: Set $R = -12$; then $a^2 = 1$, $\varphi = \omega = a$, μ arbitrary; $\rho, \chi, f, y, s, c, \sigma, b, \delta, \varepsilon = 0$. The bracket operations are

$$\begin{aligned} &[X_1, X_2] = aX_2 , && [X_2, X_3] = 0 , \\ &[X_1, X_3] = aX_3 - \mu X_4 , && [X_2, X_4] = 0 , \\ &[X_1, X_4] = \mu X_3 + aX_4 , && [X_3, X_4] = 0 . \end{aligned} \tag{III.13}$$

Proposition 4. *For any value of μ and any choice of $a = \pm 1$, the space defined by equations* (III. 13) *is a space of constant negative curvature equal to -1. Thus as Riemannian manifolds each of these solutions is isometric to real hyperbolic space.*

Proof. Directly compute the R_{ijkl} from pages 332 and 333.

The question remains as to which of the above solutions are non-isomorphic Lie algebras. Let $X_1, \cdots, X_4$ be a given basis with structure constants a and μ as in (III.13). Then the orthogonal change of basis to $-X_1, X_2, X_4, X_3$ changes a to $-a$ and leaves μ the same, while the orthogonal change of basis to X_1, X_2, X_4, X_3 leaves a the same but changes μ to $-\mu$. Hence, in order to determine non-isomorphic solutions, it suffices to take $a = 1$ and $\mu \geq 0$.

Let $\mathfrak{g}(\mu)$ be the Lie algebra defined by equations (III.13) for each value of $\mu \geq 0$.

Proposition 5. *$\mathfrak{g}(\mu)$ is isomorphic to $\mathfrak{g}(\delta)$ if and only if $\mu = \delta$.*

Proof. Let X be any element of $\mathfrak{g}(\mu)$ not contained in $\mathfrak{g}(\mu)'$, and regard $\operatorname{Ad}(X)$ as a linear transformation of $\mathfrak{g}(\mu)' \to \mathfrak{g}(\mu)'$. As in the proof of Proposition 3 the eigenvalues of $\operatorname{Ad}(X)$ are a_1 and $a_1(1 \pm i\mu)$, where $|a_1|$ is the length of the component of X normal to $\mathfrak{g}(\mu)'$. In the same way, if $Y \in \mathfrak{g}(\delta)$, but $Y \notin \mathfrak{g}(\delta)'$, and $|b_1| =$ length of the component of Y normal to $\mathfrak{g}(\delta)'$, then the eigenvalues of $\operatorname{Ad}(Y)\colon \mathfrak{g}(\delta)' \to \mathfrak{g}(\delta)'$ are b_1 and $b_1(1 \pm i\delta)$.

Suppose there exist such X and Y with equal eigenvalues, as there must if $\mathfrak{g}(\mu) \cong \mathfrak{g}(\delta)$. Then necessarily $a_1 = b_1$ and consequently $\mu = \delta$.

A matrix representation of $\mathfrak{g}(\mu)$ is given by

$$X_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 \\ 0 & -\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \qquad X_2 = \left[\begin{array}{ccc|cc} & & & -1 & 1 \\ & \bigcirc & & 0 & 0 \\ & & & 0 & 0 \\ \hline 1 & 0 & 0 & & \\ 1 & 0 & 0 & \multicolumn{2}{c}{\bigcirc} \end{array}\right],$$

$$X_3 = \left[\begin{array}{ccc|cc} & & & 0 & 0 \\ & \bigcirc & & -1 & 1 \\ & & & 0 & 0 \\ \hline 0 & 1 & 0 & & \\ 0 & 1 & 0 & \multicolumn{2}{c}{\bigcirc} \end{array}\right], \qquad X_4 = \left[\begin{array}{ccc|cc} & & & 0 & 0 \\ & \bigcirc & & 0 & 0 \\ & & & -1 & 1 \\ \hline 0 & 0 & 1 & & \\ 0 & 0 & 1 & \multicolumn{2}{c}{\bigcirc} \end{array}\right].$$

Note that this representation represents $\mathfrak{g}(\mu)$ as a subalgebra of $so(4,1)$.

A simpler matrix representation of $\mathfrak{g}(0)$, good for any dimension n, is given by the set of all $n \times n$ real matrices which have all rows but the first equal to 0. The orthonormal basis $X_1, \cdots, X_n$ is defined to be

$$X_i = \begin{bmatrix} \delta_{i1} & \cdots & \delta_{in} \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, \qquad i = 1, \cdots, n\,,$$

and the corresponding group is

$$\left\{ \begin{bmatrix} \chi_1 & & & \chi_n \\ 0 & 1 & 0 \cdots & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix} \middle| \chi_1, \cdots, \chi_n \in R,\ \chi_1 > 0 \right\}.$$

Case B. $\dim \mathfrak{g}' = 2$.

In this case, with respect to an admissible basis, $a = 0$, $\delta = -b$, $\varepsilon = -c$.

$$\text{(III.14)}\quad \begin{aligned} &[X_1, X_2] = 2bX_3 + 2cX_4\,, && [X_2, X_3] = \rho X_3 + (\chi - f)X_4\,, \\ &[X_1, X_3] = \varphi X_3 + (\sigma - \mu)X_4\,, && [X_2, X_4] = (\chi + f)X_3 + yX_4\,, \\ &[X_1, X_4] = (\mu + \sigma)X_3 + \omega X_4\,, && [X_3, X_4] = sX_4\,. \end{aligned}$$

Claim. $s = 0$.

Proof. Suppose $s \neq 0$. Then $\rho = 0$ by 21), $\chi + f = 0$ by 20), $\varphi = 0$ by 19), $\mu + \sigma = 0$ by 16), and $b - \delta = 0$ by 18). But this means that $\dim \mathfrak{g}' \leq 1$ (look at equations (III.14)). Hence $s = 0$.

Now $\mathfrak{g}'$ is abelian, and X_3 and X_4 are determined only up to a rotation. Pick X_4 to be a multiple of $[X_1, X_2]$; then $b = 0$.

Claim. $c = 0$.

Proof. Suppose $c \neq 0$. Then $\chi - f = 0$ by 6), $2y + \rho = 0$ by 7), $\mu - \sigma = 0$ by 8), and $2\omega + \varphi = 0$ by 9). Thus 10) and 13) become

10) $y\chi + \omega\sigma = 0$,

13) $\omega\chi - y\sigma = 0$,

which have a non-trivial solution in χ and σ if and only if $\omega^2 + y^2 = 0$.

Suppose $\omega^2 + y^2 = 0$, i.e., $\omega = y = 0$. Then $\rho = \varphi = 0$ by 7) and 9), and $R/4 = 2(\sigma^2 + \chi^2)$ by 3), which is impossible since $R < 0$. Suppose $\omega^2 + y^2 > 0$. Then $\chi = 0 = \sigma$. Thus $y^2 = \omega^2$ by 1) and 2), and $y^2 = -R/16$ by 3). But then $16c^2 = R/2$ by 2), which is impossible. Hence $c = 0$.

The bracket operations now look as follows:

$$\begin{aligned} &[X_1, X_2] = 0\,, && [X_2, X_3] = \rho X_3 + (\chi - f)X_4\,, \\ &[X_1, X_3] = \varphi X_3 + (\sigma - \mu)X_4\,, && [X_2, X_4] = (\chi + f)X_3 + yX_4\,, \\ &[X_1, X_4] = (\mu + \sigma)X_3 + \omega X_4\,, && [X_3, X_4] = 0\,. \end{aligned}$$

Regard $Ad(X_1)$ and $Ad(X_2)$ as linear transformations of $\mathfrak{g}' \to \mathfrak{g}'$. Then the matrix of each with respect to the basis X_3, X_4 is:

$$\mathrm{Ad}\,(X_1) = \begin{bmatrix} \varphi & \sigma + \mu \\ \sigma - \mu & \omega \end{bmatrix}, \quad Ad(X_2) = \begin{bmatrix} \rho & \chi + f \\ \chi - f & y \end{bmatrix}.$$

Let $V = \mathrm{span}\,(X_1, X_2)$. It is possible to choose an orthonormal basis of V so that $\mu + \sigma = 0$. To see this, suppose $\mu + \sigma \neq 0$, and let

$$F(t) = \cos t + \frac{(f + \chi)}{(\mu + \sigma)} \sin t\,.$$

Then there exists a t_0, $0 < t_0 < \pi$, such that $F(t_0) = 0$. Let $Y = X_1 \cos t_0 + X_2 \sin t_0$. Then

$$\begin{aligned} Ad(Y) &= \cos t_0\, Ad(X_1) + \sin t_0\, Ad(X_2) \\ &= \begin{bmatrix} * & (\sigma + \mu)\cos t_0 + (\chi + f)\sin t_0 \\ * & * \end{bmatrix} = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}. \end{aligned}$$

Label a new basis $X_1 = Y$ and $X_2 \in V$, $\perp Y$, unit length. Then $[X_1, X_2] = 0$, and

$$Ad(X_1) = \begin{bmatrix} \varphi & 0 \\ 2\sigma & \omega \end{bmatrix}, \quad Ad(X_2) = \begin{bmatrix} \rho & \chi + f \\ \chi - f & y \end{bmatrix},$$

i.e., $\mu = -\sigma$.

Now $0 = \sigma(\chi + f)$ by 14). We must now consider two cases, namely, $\chi + f = 0$ and $\chi + f \neq 0$.

Case 1. $x + f = 0$.

Choose t, $0 \leq t < \pi$, such that $\sigma \cos t + \chi \sin t = 0$. Let $Y_1 = X_1 \cos t + X_2 \sin t$, $Y_2 = -X_1 \sin t + X_2 \cos t$. Then

$$Ad(Y_1) = \cos t\, Ad(X_1) + \sin t\, Ad(X_2) = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}, \quad Ad(Y_2) = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}.$$

Thus we may assume that $\sigma = 0$.

Claim. $\chi = 0$.

Proof. Suppose $\chi \neq 0$. Then $y = 0$ by 10), and $\omega = \varphi$ by 13). Thus $\varphi \neq 0$ by 1), and $\rho = 0$ by 5). But then $\varphi = 0$ by 2) and 3), a contradiction. Hence $\chi = 0$.

Only equations 1) through 5) remain to be solved. Combing 1), 2), 3) and 4) we get $0 = \rho y + \varphi\omega$. Equations 1) though 5) are then equivalent to:

1) $-R/4 = \varphi^2 + \omega^2$, 4) $-R/4 = \omega^2 + y^2$,

2) $-R/4 = \rho^2 + y^2$, 5) $0 = \varphi\rho + \omega y$,

3) $-R/4 = \varphi^2 + \rho^2$, 6) $0 = \rho y + \varphi\omega$.

Set $R = -4$. Then (φ, ρ), (ω, y), (ρ, y) and (φ, ω) are unit vectors, with $(\varphi, \rho) \perp (\omega, y)$ and $(\rho, y) \perp (\varphi, \omega)$. Therefore, all possible solutions are given by:

$$\varphi = \cos t\,, \quad \omega = \sin t\,, \quad \rho = \mp \sin t, \quad y = \pm \cos t, \quad \text{for} \quad 0 \leq t \leq 2\pi\,,$$

and with the signs related; and $a, b, \delta, c, \varepsilon, \sigma, \mu, \chi, f, s = 0$.

$$\text{(III.15)}\qquad \begin{aligned} &[X_1, X_2] = 0\,, &&[X_2, X_3] = -(\pm)X_3 \sin t\,,\\ &[X_1, X_3] = X_3 \cos t\,, &&[X_2, X_4] = \pm\, X_4 \cos t\,,\\ &[X_1, X_4] = X_4 \sin t\,, &&[X_3, X_4] = 0\,. \end{aligned}$$

Consider the solution for $t = 0$, and let t, $0 \leq t \leq 2\pi$, be given. Let $Y_1 = X_1 \cos t + X_2 \sin t$ and $Y_3 = -(\pm)\, X_1 \sin t + (\pm)\, X_2 \cos t$. Then

$$\begin{aligned} &[Y_1, Y_2] = 0\,, &&[Y_2, X_3] = -(\pm)\, X_3 \sin t\,,\\ &[Y_1, X_3] = X_3 \cos t\,, &&[Y_2, X_4] = (\pm)\, X_4 \cos t\,,\\ &[Y_1, X_4] = X_4 \sin t\,, &&[X_3, X_4] = 0\,. \end{aligned}$$

Hence, by an orthonormal change of basis, we obtain the solution for t. Thus we have only one solution, which we take to be for $t = 0$, and the "$+$" sign. The bracket operations are then:

$$\text{(III.16)}\qquad \begin{aligned} &[X_1, X_2] = 0\,, &&[X_2, X_3] = 0\,,\\ &[X_1, X_3] = X_3\,, &&[X_2, X_4] = X_4\,,\\ &[X_1, X_4] = 0\,, &&[X_3, X_4] = 0\,. \end{aligned}$$

Let $\mathscr{D}_1 = \operatorname{span}(X_1, X_3)$, and $\mathscr{D}_2 = \operatorname{span}(X_2, X_4)$. Then $\mathscr{D}_1$ and $\mathscr{D}_2$ are clearly ideals in $\mathfrak{g}$ and $\mathfrak{g} = \mathscr{D}_1 \oplus \mathscr{D}_2$. Examining the curvature tensor R_{ijkl} it is seen that the Riemannian metric is the product of the induced metrics on $\mathscr{D}_1$ and $\mathscr{D}_2$, and the induced metrics on $\mathscr{D}_i$, $i = 1, 2$, are of constant curvature equal to -1.

This completes Case 1, where we assumed that $\chi + f = 0$. We must now consider Case 2. However, we shall not get any new solutions.

Case 2. $\chi + f \neq 0$.

Now $\sigma = 0$ by 14). Thus $Ad(X_1)$ is already diagonal. $\omega = \varphi$ by 13) and 15) and the assumption that $\chi + f \neq 0$. Thus $\varphi^2 = -R/8$ by 1). In particular, $\varphi \neq 0$. Thus $\rho + y = 0$ by 5).

Now $0 = yf$ by 10), and $0 = \chi f$ by 4). But $\chi^2 + y^2 = -R/8$ by 2), and so χ and y cannot both be zero. Consequently, $f = 0$. Thus $\chi \neq 0$, since $\chi + f \neq 0$.

Set $R = -8$. Then $\varphi^2 = 1$ by 1), and $\chi^2 + y^2 = 1$ by 2). Hence in this case we have found all possible solutions which can be listed as follows: $R = -8$, $\varphi^2 = 1$, $\omega^2 = 1$, $\omega = \varphi$, $\chi = \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$, $\rho = -y$, $a, b, c, \delta, \varepsilon, f, \sigma, \mu, s = 0$.

The bracket operations are:

$$\text{(III.17)}\qquad \begin{aligned} &[X_1, X_2] = 0\,, &&[X_2, X_3] = -X_3 \sin t + X_4 \cos t\,,\\ &[X_1, X_3] = (\pm)X_3\,, &&[X_2, X_4] = X_3 \cos t + X_4 \sin t\,,\\ &[X_1, X_4] = (\pm)X_4\,, &&[X_3, X_4] = 0\,. \qquad \text{(Signs are related.)} \end{aligned}$$

Claim. These solutions are all the same as the solution given by equations (III.15) given on page 346. That is, by making an appropriate orthogonal change of basis in any of the solutions defined by (III.17), we will get the same solution defined by (III.16).

Proof. We show first that all solutions at (III.17) are the same as the one at (III.17) for $t = \pi/2$ and the "$+$" sign, and then that this solution is the same as the one at (III.15) for $t = \pi/4$ and the "$-$" sign, except for a scale change which comes from our choice of $R = -4$ at (III.15) and $R = -8$ at (III.17).

Let $X_1, \cdots, X_4$ be an admissible basis for the solution at (III.17) for $t = \pi/2$ and the "$+$" sign. Let $Y_1 = (\pm)X_1$, $Y_2 = X_2$, $Y_3 = X_3 \cos\beta + X_4 \sin\beta$, $Y_4 = -X_3 \sin\beta + X_4 \cos\beta$, where β is to be determined. Then

$$\text{(III.18)}\qquad \begin{aligned} &[Y_1, Y_2] = 0\,, && [Y_2, Y_3] = -Y_3 \cos 2\beta + Y_4 \sin 2\beta\,,\\ &[Y_1, Y_3] = (\pm)Y_3\,, && [Y_2, Y_4] = Y_3 \sin 2\beta + Y_4 \cos 2\beta\,,\\ &[Y_1, Y_4] = (\pm)Y_4\,, && [Y_3, Y_4] = 0\,. \end{aligned}$$

Given $t, 0 \leq t \leq 2\pi$, choose β so that $\cos 2\beta = \sin t$ and $\sin 2\beta = \cos t$. In fact, take $2\beta = \pi/2 - t$. Equations (III.18) now show that the solution at (III.17) for $t = \pi/2$ and the "$+$" sign is the same as any solution at (III.17). It is obvious that this solution is the same as the one at (III.15) for $t = \pi/4$ and the "$-$" sign.

Case C. $\dim \mathfrak{g}' = 1$.

In this case, with respect to an admissible basis, $a, b, \delta, \varphi, c + \varepsilon, \mu + \sigma, \rho, \chi + f = 0$. Thus:

$$\begin{aligned} &[X_1, X_2] = 2cX_4\,, && [X_2, X_3] = 2\chi X_4\,,\\ &[X_1, X_3] = 2\sigma X_4\,, && [X_2, X_4] = yX_4\,,\\ &[X_1, X_4] = \omega X_4\,, && [X_3, X_4] = sX_4\,. \end{aligned}$$

There are no solutions in this case.

Equations 7) and 8) are linear in y and σ, and in c and s: 7) $0 = cy + s\sigma$, 8) $0 = sy + 2c\sigma$. Consider the two cases: i) $2c^2 - s^2 = 0$ and ii) $2c^2 - s^2 \neq 0$.

i) $2c^2 - s^2 = 0$. Suppose also that $2\sigma^2 - y^2 = 0$. Then $0 = \omega^2 - 2\chi^2$ by 1) and 2). But $-R/4 = \omega^2 - 2\chi^2$ by 4). Thus $2\sigma^2 - y^2 \neq 0$. But then $\chi = \omega = 0$ by 5) and 6), and $s = c = 0$ by 7) and 8). Thus $0 = 2\sigma^2 - y^2$ by 1) and 2), a contradiction. Hence i) is impossible.

ii) $2c^2 - s^2 \neq 0$. Then $y = \sigma = 0$ by 7) and 8), and $\chi = \omega = 0$ by 6) and 9). Thus $2c^2 - s^2 = 0$ by 1) and 3), a contradiction. Hence there are no solutions when $\dim \mathfrak{g}' = 1$.

2. The following theorem summarizes the results obtained in § 1. A matrix representation for each Lie algebra is given in § 1 and in the introduction, and so is not repeated here.

Theorem. *Let G be a four-dimensional Lie group with a left-invariant Riemannian metric. Then G is an Einstein space if and only if its Lie algebra $\mathfrak{g}$ is one of the following solvable Lie algebras with the inner product defined, up to change in scale, by $X_1, \cdots, X_4$ being an orthonormal basis. Distinct values of t define non-isomorphic Lie algebras.*

1. $$\begin{aligned} &[X_1, X_2] = 0\,, && [X_2, X_3] = 0\,, \\ &[X_1, X_3] = X_4\,, && [X_2, X_4] = 0\,, \\ &[X_1, X_4] = -X_3\,, && [X_3, X_4] = 0\,. \end{aligned}$$

As a Riemannian space this is flat.

2. $$\begin{aligned} &[X_1, X_2] = X_2 - tX_3\,, && [X_2, X_3] = 2X_4\,, \\ &[X_1, X_3] = tX_2 + X_3\,, && [X_2, X_4] = 0\,, \\ &[X_1, X_4] = 2X_4\,, && [X_3, X_4] = 0\,, \quad 0 \leq t < \infty\,. \end{aligned}$$

As a Riemannian space each of these is a hermitian hyperbolic space with sectional curvature K satisfying $-1 \geq K \geq -4$.

3. $$\begin{aligned} &[X_1, X_2] = X_2\,, && [X_2, X_3] = 0\,, \\ &[X_1, X_3] = X_3 - tX_4\,, && [X_2, X_4] = 0\,, \\ &[X_1, X_4] = tX_3 + X_4\,, && [X_3, X_4] = 0\,, \quad 0 \leq t < \infty\,. \end{aligned}$$

As a Riemannian space each of these is a real hyperbolic space with constant curvature K equal to -1.

4. $$\begin{aligned} &[X_1, X_2] = 0\,, && [X_2, X_3] = 0\,, \\ &[X_1, X_3] = X_3\,, && [X_2, X_4] = X_4\,, \\ &[X_1, X_4] = 0\,, && [X_3, X_4] = 0\,. \end{aligned}$$

This Lie algebra is the direct sum of a two-dimensional Lie algebra with itself, and the Riemannian space is the direct product of a two-dimensional solvable group manifold, of constant curvature K equal to -1, with itself.

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