J. DIFFERENTIAL GEOMETRY
3 (1969) 103-110

SYMMETRIES OF SURFACES OF CONSTANT WIDTH

JAY P. FILLMORE

1. Introduction

A *hypersurface of constant width* in Euclidean space E^n is a compact continuous $(n-1)$-dimensional submanifold enclosing an open convex set and having the property that the distance between the two supporting hyperplanes having a given unit vector ξ as normals is independent of ξ.

Examples of such hypersurface in E^2 are the Reuleux triangle and a parallel curve of constant distance from a Reuleux triangle [4, p. 199]; these curves are continuous and of class C^1 respectively.

Assume that the coordinate origin of E^n is inside the region enclosed by the hypersurface. Given a unit vector ξ, let $h(\xi)$ denote the distance from the origin to the supporting hyperplane having ξ as normal and such that the vector from the origin to the point of contact makes an acute angle with ξ. $h(\xi)$ is the Minkowski support function and

$$h(\xi) + h(-\xi) = \text{constant} = 2a$$

is necessary and sufficient for the hypersurface to be of constant width.

In E^2 we may take $\xi = (\cos\theta, \sin\theta)$ and $h = a + b\cos 3\theta$ $(0 < 8b < a)$. The resulting curve

$$x_1 = h\cos\theta - \frac{dh}{d\theta}\sin\theta\,, \qquad x_2 = h\sin\theta + \frac{dh}{d\theta}\cos\theta$$

is analytic and of constant width—an analytic version of the Reuleux triangle. If we rotate this curve in E^n about an $(n-2)$-dimensional axis perpendicular to the line $\theta = 0$, we obtain an analytic surface of constant width in E^n, which is not a sphere.

The hypersurfaces just described admit many symmetries—orthogonal transformations (including the improper ones) which, when combined with a translation, carry the hypersurface into itself. In this paper we show that there exist in E^n analytic hypersurfaces of constant width, which do not admit any symmetries other than the identity.

In the final section we establish which closed subgroups of the proper and

Received February 28, 1968.

the full rotation groups in E^3 can occur as groups of symmetries of surfaces of constant width and which cannot.

2. Christoffel's theorem, spherical harmonics

Let $x = (x_1, \cdots, x_n)$ denote coordinates in E^n, and set $r = \sqrt{x_1^2 + \cdots + x_n^2}$. Define $H(x)$ for $x \neq 0$ by $H(x) = r\,h(x_1/r, \cdots, x_n/r)$, where $h(\xi)$ is the support function of a compact convex hypersurface in E^n. $H(x)$ is homogeneous of degree 1 with respect to positive constants. If the hypersurface is of differentiability class C^α, then $H(x)$ is of class $C^{\alpha-1}$ on E^n less the origin. Given such an $H(x)$, $\partial H/\partial x_i$ is homogeneous of degree 0 and thus depends only on $\xi = x/r$. The vector grad $H = (\partial H/\partial x_1, \cdots, \partial H/\partial x_n)$ furnishes the coordinates of the hypersurface as a function of ξ on the $(n-1)$-sphere S^{n-1} in E^n [1, p, 135].

Let $-\varphi(\xi)$ denote the sum of the principal radii of curvature at the point of the hypersurface having normal ξ. Define $\Phi(x)$ in E^n less the orgin by $\Phi(x) = r^{-1}\varphi(x/r)$, so $\Phi(x)$ is homogeneous of degree -1. A consequence of Rodrigues' formula is that $\Phi(x) = \Delta H(x)$ where Δ is the Laplacian in E^n [1, p. 134].

The *theorem of Christoffel* ([1, p. 136] and [3]) states that a function $\varphi(\xi)$ on S^{n-1} arises from a conpact convex hypersurface (of class C^4) as described above, if and only if, first: the integral

$$\int_{S^{n-1}} f(\xi)\varphi(\xi)d\xi = 0$$

for every $f(\xi)$ which is the restriction of one of the n functions $F(x) = x_i$, and second: the integral

$$\int_{S^{n-1}} (\xi\cdot\xi'')\theta(\xi\cdot\xi')[(\text{grad}\,\Phi)(\xi)\cdot\xi'']d\xi \geq 0$$

for all $\xi', \xi'' \in S^{n-1}$, which are orthogonal ($\xi'\cdot\xi'' = 0$), and strict inequality holds for some such ξ', ξ''. Here $d\xi$ denotes rotation invariant measure on S^{n-1}, $(\text{grad}\,\Phi)(\xi)$ is the restriction to S^{n-1} of the gradient of $\Phi(x)$, and $\theta(t)$ is a certain function of one variable which need not be explicitly written down here (See [3]). Furthermore, $\varphi(\xi)$ determines the hypersurface uniquely up to translations, and the hypersurface is C^∞ or analytic as $\varphi(\xi)$ is C^∞ or analytic.

A *spherical harmonic* $f(\xi)$ of degree k is the restriction to S^{n-1} of a homogeneous polynomial $F(x)$ of degree k which is harmonic: $\Delta F(x) = 0$. $F(x)$ and $f(\xi)$ are related by $F(x) = r^k f(x/r)$. For fixed k, the spherical harmonics of degree k are a finite dimensional vector space. Any two spherical harmonics of different degrees are orthogonal with respect to integrating the product over S^{n-1}. If $f(\xi)$ is any square integrable function on S^{n-1}, then there is a *unique* expansion $f(\xi) = \sum_{k=0}^{\infty} f_k(\xi)$, where $f_k(\xi)$ is a spherical harmonic of degree k and the series converges in mean to $f(\xi)$, [2]. A spherical harmonic of degree k is an odd function, i.e., satisfies $f(-\xi) = -f(\xi)$ if and only if k is odd.

If σ is an orthogonal transformation of E^n (proper or not), then σ sends S^{n-1} onto itself. If $f(\xi)$ is a spherical harmonic of degree k, then so is $f(\sigma\xi)$; this follows from $\Delta(F(\sigma x)) = (\Delta F)(\sigma x)$, i.e., the invariance of Δ under orthogonal transformations.

3. Construction of analytic hypersurfaces of constant width without symmetries

Let $-\varphi(\xi)$ be the sum of the principal radii of curvature of a compact convex hypersurface. If σ is an orthogonal transformation in E^n, and σ maps the hypersurface onto itself, then σ leaves invariant the support function and thus $\varphi(\xi)$, i.e., $\varphi(\sigma\xi) = \varphi(\xi)$ for all ξ. Since $\varphi(\xi)$ determines the hypersurface up to translation, the group of symmetries of the hypersurface consists of those σ preserving $\varphi(\xi)$. If we expand $\varphi(\xi) = \sum_{k=0}^{\infty} f_k(\xi)$ into spherical harmonics $f_k(\xi)$ of degree k, the uniqueness of the expansion implies that $\varphi(\sigma\xi) = \varphi(\xi)$ for all ξ if and only if $f_k(\sigma\xi) = f_k(\xi)$ for all ξ and k. Thus the determination of symmetries of a hypersurface is reduced to studying invariance of spherical harmonics under orthogonal maps.

We denote by $O(n)$ and $SO(n)$ the groups of all orthogonal and proper orthogonal transformations of E^n respectively.

Lemma 1. *Let $k \geq 1$ be odd and $\sigma \in O(n)$ different from the identity. Then there exists a spherical harmonic $f(\xi)$ of degree k which is not left invariant by σ, i.e., $f(\sigma\xi) \neq f(\xi)$ for some ξ.*

This lemma is simply the statement that the representation of $O(n)$ on the space of spherical functions of degree n given by sending σ into the left translation $f(\xi) \to f(\sigma^{-1}\xi)$ is faithful. We give here a proof of the lemma based on elementary properties of Gegenbauer polynomials [2].

Suppose $\sigma \in O(n)$ has the property that $f(\sigma\xi) = f(\xi)$ for all ξ for every spherical harmonic $f(\xi)$ of degree k. We must show σ is the identity. Let $f_1(\xi), \cdots, f_d(\xi)$ be an orthonormal basis of the space of spherical harmonics of degree k. Then $\sum_{i=1}^{d} f_i(\xi)f_i(\eta)$ depends only on the inner product $\xi\cdot\eta$, and in fact this sum can be expressed as $C_k^\lambda(\xi\cdot\eta)$ where $C_k^\lambda(h)$ is the Gegenbauer polynomial (coefficients of t^k in the expansion of $(1-2ht + t^2)^{-\lambda})$ with $\lambda = (n/2)-1$. Since $f_i(\sigma\xi) = f_i(\xi)$, $C_k(\sigma\xi\cdot\xi) = C_k(1)$ for all ξ. Now $C_k(h)$ has a trigonometric expansion

$$C_k^\lambda(\cos\theta) = \sum_{i=0}^{k} a_i a_{k-i} \cos(k - 2i)\theta$$

with

$$a_i = \frac{\lambda(\lambda+1)\cdots(\lambda+i-1)}{1\cdot 2\cdot \cdots \cdot(i-1)i} .$$

From this we conclude that $|C_k(\cos\theta)| \leq C_k(1)$, and equality holds if and only if $\cos(k-2i)\theta = 1$ for $i = 0, 1, \cdots, k$. If $C_k(\cos\theta) = C_k(1)$ and k is odd, then $\cos\theta = 1$. Hence from $C_k(\sigma\xi\cdot\xi) = C_k(1)$ we have $\sigma\xi\cdot\xi = 1$ for all ξ. Thus σ is the identity as desired.

Lemma 2. *Let $C > 0$, and $f_2(\xi), f_3(\xi), \cdots,$ be spherical harmonics on S^{n-1}, $f_k(\xi)$ of degree k, such that $\sum_{k=2}^{\infty} f_k(\xi)$ is of class C^1. Then for $|\varepsilon|$ sufficiently small, the function $-\varphi(\xi)$, where*

$$\varphi(\xi) = C + \varepsilon\sum_{k=2}^{\infty} f_k(\xi) ,$$

is the sum of the principal radii of curvature of a compact convex hypersurface in E^n. A support function for such a surface is given by

$$h(\xi) = \frac{C}{n-1} - \varepsilon\sum_{k=2}^{\infty}\frac{f_k(\xi)}{(k-1)(n+k-3)} .$$

The first condition of Christoffel's theorem is satisfied since spherical harmonics of degree 1 are absent from $\varphi(\xi)$. That the inequality in the second condition is satisfied, is seen as follows. The integral is strictly >0 for some orthogonal pair ξ', ξ'' when $\varphi(\xi)$ is constant since this corresponds to a sphere. By symmetry on the sphere ($O(n)$ moves such ξ', ξ'' transitively), the integral is strictly >0 for all orthogonal pairs ξ', ξ''. Since the integral is linear in $\varphi(\xi)$, the choice for $\varphi(\xi)$ as in the lemma will yield an integral which is strictly >0 for sufficiently small $|\varepsilon|$.

That the support function yields the given $\varphi(\xi)$ is seen by computing that

$$\begin{aligned}\Delta\left(rF_k\left(\frac{x}{r}\right)\right) &= \Delta(r^{1-k}F_k(x)) \\ &= -(k-1)(n+k-1)r^{-1-k}F_k(x) \\ &= -(k-1)(n+k-1)r^{-1}F_k\left(\frac{x}{r}\right) ,\end{aligned}$$

using Euler's identity for the homogeneous function $F_k(x)$ and the fact that $F_k(x)$ is harmonic.

Theorem 1. *There exists an analytic hypersurface of constant width in E^n which admits no symmetries other than the identity.*

Choose $\sigma_1 \in O(n)$, σ_1 not the identity. Let $f_{k_1}(\xi)$ be a spherical harmonic of odd degree $k_1 \geq 3$, which is not left invariant by σ_1, and suppose

$G_1 = \{\sigma \in O(n) \mid f_{k_1}(\sigma\xi) = f_{k_1}(\xi) \text{ all } \xi\}$. Then G_1 is a closed subgroup of $O(n)$, therefore a Lie group, and a proper subgroup of $O(n)$. If G_1 is not the identity, choose $\sigma_2 \in G_1$, σ_2 not the identity. Let $f_{k_2}(\xi)$ be a spherical function of odd degree $k_2 > k_1$ which is not left invariant by σ_2. Put $G_2 = \{\sigma \in O(n) \mid f_k(\sigma\xi) = f_k(\xi) \text{ for all } \xi \text{ and for } k = k_1, k_2\}$. G_2 is a closed subgroup of $O(n)$ and a proper subgroup of G_1. Continue in this fashion to obtain a chain $O(n) \supsetneqq G_1 \supsetneqq G_2 \supsetneqq \cdots$. At each step, either the dimension of the Lie group or the number of its components decreases, so after finitely many steps we reach the identity group. Thus we have spherical functions $f_{k_1}(\xi), \cdots, f_{k_s}(\xi)$ of odd degrees k_i, $3 \leq k < \cdots < k_s$, such that if $\sigma \in O(n)$ and $f_{k_i}(\sigma\xi) = f_{k_i}(\xi)$ for all ξ and $i = 1, \cdots, s$, then σ is the identity.

Let $C > 0$ and $\varepsilon \neq 0$ sufficiently small so that

$$\varphi(\xi) = C + \varepsilon \sum_{i=1}^{s} f_{k_i}(\xi)$$

is the negative of the sum of the principal radii of curvature of a compact convex hypersurface. If $\sigma \in O(n)$ is a symmetry of the surface, then σ leaves invariant $\varphi(\xi)$ and thus all $f_{k_i}(\xi)$, and therefore is the identity. Since $\varphi(\xi)$ is analytic, the hypersurface is analytic. Finally, from the form of the support function in Lemma 2, $h(\xi) + h(-\xi) = 2C/(n-1)$. Hence the hypersurface has constant width.

In a like fashion one can show

Theorem 2. *Let G be the group of symmetries of a compact convex hypersurface of class C^4 in E^n, then G is a closed subgroup of $O(n)$ and there exists an analytic compact convex hypersurface in E^n having G as its group of symmetries. If the original hypersurface is of constant width, then the analytic hypersurface may be taken to be of constant width.*

The hypothesis that the hypersurface be C^4 may be relaxed to C^1. For the support function of the hypersurface may be smoothed to a C^∞ function without changing the group of symmetries while maintaining the form constant plus an odd function.

If we imitate the construction of a Reuleux triangle in n-dimension, a parallel hypersurface at constant distance from this hypersurface is of class C^1 and has the symmetry group (a finite group) of the regular n-simplex. Thus:

Corollary. *There exists an analytic hypersurface of constant width in E^n having the same group of symmetries as a regular n-simplex.*

For $n \geq 3$ quite different hypersurfaces of constant width can have the same symmetry groups. See, for example, [7, p. 81].

4. Symmetry groups of analytic surfaces of constant width

Let R and R' denote $SO(3)$ and $O(3)$ respectively. We wish to determine which of the closed subgroups of these groups can occur as the group of symmetries of a C^4 or analytic surface of constant width. In view of the previous paragraphs, this is equivalent to determining whether or not there exist odd spherical harmonics on S^2 or harmonic (homogeneous) polynomials in three variables, which are invariant under a given closed subgroup. The latter question is answered for the finite subgoups by Polya and Meyer in [5].

We begin with R and its 1-dimensional subgroups. Any 1-dimensional subgroup of R is conjugate to either G_1, the group of rotations of E^3 about the x_3-axis, or G_2, the rotations of E^3 which either fix or reverse the x_3-axis. These groups have one and two connected components respectively.

The spherical harmonics of degree k on S^2, which are left invariant by G_1, form a 1-dimensional real vector space. Indeed, they are constant on the intersection of S^2 with planes perpendicular to the x_3-axis and are the so-called zonal harmonics [6]. These arise as solutions of Legendre's equation and the analytic solutions of this equation are a 1-dimensional space. If a spherical harmonic of degree k is invariant under G_2, it is invariant under G_1, as well as a rotation of π about the x_1-axis. The corresponding solution of the Legendre equation must be left invariant, so k is necessarily even.

G_2 cannot occur as the group of symmetries of an analytic surface of constant width, since $\varphi(\xi) = C +$ (a sum of spherical harmonics of odd degree) implies $\varphi(\xi) \equiv C$, and the surface would be a sphere and admit R for its symmetries. G_1 can occur as such a group of symmetries, since $\varphi(\xi) = C + \varepsilon f_3(\xi)$, where $C > 0$, $f_3(\xi) \not\equiv 0$ is a spherical harmonic of degree 3, and $\varepsilon \neq 0$ is sufficiently small, is left invariant by G_1 but by no larger group (necessarily G_2 or R). $\varphi(\xi)$ defines the surface as in Christoffel's theorem.

The case of G_1 and G_2 are geometrically obvious: G_1 occurs as symmetries of an analytic Reuleux triangle rotated about an axis of symmetry, and G_2 is excluded by considering the intersection of the surface with a plane containing the x_3-axis.

We turn to the finite subgroups of R. These are [8]: the tetrahedral T, octahedral O, icosahedral I, cyclic $C_n (n \geq 1)$, and dihedral $D_n (n \geq 2)$ groups of orders 12, 24, 60, n, and $2n$ respectively.

Introduce spherical coordinates on S^2 by $\xi = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. Reflection through the origin corresponds to $(\theta, \varphi) \rightarrow (\pi - \theta, \pi + \varphi)$, and rotation through an angle of π about the x_1-axis corresponds to $(\theta, \varphi) \rightarrow (\pi - \theta, -\varphi)$. The functions $\cos n\varphi \, P_k^n(\cos\theta)$ and $\sin n\varphi \, P_k^n(\cos\theta)$, where $P_k^n(\cos\theta)$ is an associated Legendre function [6], are spherical harmonics of degree k, odd functions on S^2 if k is odd, and are left invariant under rotation through an angle of $2\pi/n$ around the x_3-axis. Letting k be odd, we easily check that $\cos n\varphi \, P_k^n(\cos\theta)$ for n odd and $\sin n\varphi \, P_k^n(\cos\theta)$ for n even are left invariant under rotation through π about the x_1-axis. Thus each $D_n (n \geq 2)$ is the group of rotations in R leaving invariant a spherical function of odd degree. Hence, as above, each D_n is the group of symmetries of an analytic surface of constant width. The Legendre function $P_{k+2}(\cos\theta)$, with k odd, is left invariant under

$C_n \subset D_n$ but not by rotation about the x_1-axis. Adding this to the functions above gives C_n as the group of rotations leaving invariant an odd analytic function. Hence each $C_n(n \geq 1)$ is the group of symmetries of a surface of constant width.

The group T was shown to be the group of symmetries of a surface of constant width in the last section. Let now G be O or I. In [5], Polya and Meyer determine the dimensions of the spaces of harmonic polynomials which are invariant under a given finite subgroup of $SO(3)$. From their results it follows that there exists a spherical harmonic $f_k(\xi)$ of degree k which is invariant under G. (The first such k is 9 and 15 for $G = O$ and I respectively.) G is contained in the group $\{\sigma \in R \mid f_k(\sigma\xi) = f_k(\xi) \text{ all } \xi\}$ which is a proper subgroup of R. It cannot be C_n, D_n, G_1, or G_2 since these send an axis into itself and G does not. Thus this subgroup is O or I. Since O and I have 24 and 60 elements respectively, we conclude $G = \{\sigma \in R \mid f_k(\sigma\xi) = f_k(\xi) \text{ all } \xi\}$ and that G is the group of symmetries of an analytic surface of constant width.

Thus we have

Theorem 3. *With the exception of subgroups conjugate to G_2, every closed subgroup of R is the group of symmetries of an analytic surface of constant width.*

In a like manner one can determine which closed subgroups of $R' = O(3)$ are of the form $\{\sigma \in R' \mid f(\sigma\xi) = f(\xi) \text{ all } \xi\}$ for some odd (analytic) function $f(\xi)$ on S^2.

We use the following notation [5]. If G is a subgroup of R, G' denotes the subgroup of R' generated by G and reflection through the origin. If G is a subgroup of R having a subgroup H of index 2, then (G, H) denotes the subgroup of R' consisting of H and the elements of G not in H composed with reflection through the origin. These two types of closed subgroups of R' along with the closed subgroups of R are, up to conjugacy, all closed subgroups of R'.

Theorem 4. *The following closed subgroups of R' are the groups of symmetries of an analytic surface of constant width:*

$$T, O, I, C_n(n \geq 1), D_n(n \geq 2), (O, T),$$
$$(D_n, C_n)(n \geq 1), (C_{2n}, C_n)(n \geq 1),$$
$$(D_{2n}, D_n)(n \geq 2), (G_2, G_1), R',$$

and the following are not:

$$T', O', I', C'_n\ (n \geq 1), D'_n(n \geq 2),$$
$$G_1, G_2, G'_1, G'_2, R.$$

References

[1] W. Blaschke, *Vorlesungen über Differentialgeometrie*, Vol. I, 2nd ed., Springer, Berlin, 1924.
[2] A. Erdelyi (Editor), *Higher transcendental functions*, Vols. 1 and 2, McGraw-Hill, New York, 1953.
[3] W. J. Firey, *The determination of convex bodies from their mean radius of curvature functions*, Mathematika **14** (1967) 1–13.
[4] D. Laugwitz, *Differential and Riemannian geometry*, Academic Press, New York, 1965.
[5] G. Polya & B. Meyer, *Sur les symétries des functions sphériques de Laplace*, C. R. Acad. Sci. Paris **228** (1950) 28–30, 1083–1084.
[6] E. T. Whitaker & G. N. Watson, *A course of modern analysis*, Cambridge University Press, Cambridge, 1927.
[7] I. M. Yaglom & V. G. Boltyanskiĭ, *Convex figures*, Holt, Rinehart and Winston, New York, 1949.
[8] H. Zassenhaus, *The theory of groups*, Chelsea New York, 1949.

UNIVERSITY OF CALIFORNIA, SAN DIEGO