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ON A PROBLEM OF NOMIZU-SMYTH ON A NORMAL CONTACT RIEMANNIAN MANIFOLD

KENTARO YANO & SHIGERU ISHIHARA

The study of complex Einstein hypersurfaces of Kählerian manifolds of constant holomorphic sectional curvature has been initiated by Smyth [12] and continued by Nomizu and Smyth [7]. (See also, Ako [1], Chern [2], Kobayashi [5], Smyth [13], Takahashi [14], Yano and Ishihara [17]).

The main purpose of the present paper is to study the so-called invariant C-Einstein submanifolds of codimension 2 in a normal contact Riemannian manifold. We call a problem of this kind a problem of Nomizu-Smyth.

First of all we recall in §1 the definition and properties of contact Riemannian manifolds, and in §2 the fundamental formulas for submanifolds of codimension 2 in a Riemannian manifold.

In §§3, 4 we obtain the fundamental formulas respectively for submanifolds and invariant submanifolds of codimension 2 in a contact Riemannian manifold.

In the last §5, we study the problem of Nomizu-Smyth, that is, the problem of determining invariant C-Einstein submanifolds of codimension 2 in a normal contact Riemannian manifold of constant curvature.

1. Contact Riemmannian manifolds

First of all for later use we recall the definition and some properties of a contact Riemannian manifold. A $(2n+1)$-dimensional differentiable manifold M is said to admit a *contact structure* if there exists on M a 1-form $E = E_i dx^i$ such that the rank of the tensor field

$$F_{ji} = \frac{1}{2}(\partial_j E_i - \partial_i E_j) \tag{1.1}$$

is $2n$ everywhere on M, where ∂_i denotes the operator $\partial/\partial x^i$, (x^h) are the local coordinates of M, the indices $h, i, j, k, \cdots$ run over the range $\{1, \cdots \cdots, 2n+1\}$, and the so-called Einstein's summation convention is used with respect to this system of indices. A manifold admitting a contact structure is called a *contact manifold*.

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If a contact manifold M is orientable, we can find a vector field E^h on M such that

$$F_{ji}E^i = 0 , \qquad E_iE^i = 1 . \tag{1.2}$$

It is now well-known that there exists on M a positive definite Riemannian metric G_{ji} such that

$$\begin{aligned} E_i &= G_{ih}E^h , \\ F_t{}^hF_i{}^t &= -\delta_i^h + E_iE^h , \\ F_j{}^tF_i{}^sG_{ts} &= G_{ji} - E_jE_i , \end{aligned} \tag{1.3}$$

where

$$F_i{}^h = F_{is}G^{sh} , \tag{1.4}$$

(G^{sh}) being the inverse of the matrix (G_{ji}) (cf. [3]). A differentiable manifold admitting such a structure $(F_i{}^h, E_i, E^h, G_{ji})$ is called a *contact Riemannian manifold.*

We denote by $N_{ji}{}^h$ the Nijenhuis tensor formed with $F_i{}^h$, i.e.,

$$N_{ji}{}^h = F_j{}^t\partial_tF_j{}^h - F_i{}^t\partial_tF_j{}^h - (\partial_jF_i{}^t - \partial_iF_j{}^t)F_t{}^h .$$

If the tensor field

$$S_{ji}{}^h = N_{ji}{}^h + (\partial_jE_i - \partial_iE_j)E^h$$

vanishes identically, the contact Riemannian manifold is said to be *normal* (cf. [9], [10]). A contact Riemmanian manifold is normal if and only if

$$\nabla_jE_i = F_{ji} , \tag{1.5}$$

$$\nabla_jF_i{}^h = -G_{ji}E^h + \delta_j^hE_i , \tag{1.6}$$

∇_j denoting the covariant differentiation with respect to the Riemannian connection $\{_j{}^h{}_i\}$ determined by G_{ji} (cf. [4]).

Differentiating (1.5) covariantly and taking account of (1.3) and (1.6), we have

$$\nabla_k\nabla_jE^h = -G_{kj}E^h + \delta_k^hE_j ,$$

which gives

$$K_{kji}{}^hE^i = \delta_k^hE_j - \delta_j^hE_k , \tag{1.7}$$

where $K_{kji}{}^h = K_{kjis}G^{sh}$ denotes the curvature tensor of G_{ji}. Transvecting (1.7) with arbitrary vectors X^k and Y_h, we find

$$(E^i Y^h K_{ihk}{}^j)X^k = (Y_s X^s)E^j - (E_s X^s)Y^j ,$$

which shows that there exists a vector Y^h satisfying

$$(E^i Y^h K_{ihk}{}^j)X^k = A^j$$

for arbitrarily given vectors X^h and A^h. Thus we have

Lemma 1. *Any normal contact Riemannian manifold is irreducible as a Riemannian manifold* [15].

When the Ricci tensor $K_{ji} = K_{sji}{}^s$ has components of the form

$$K_{ji} = aG_{ji} + bE_j E_i \tag{1.8}$$

with constants a and b, the contact Riemannian manifold M is said to be a *C-Einstein manifold.* When $b = 0$ in (1.8), the manifold M is an Einstein manifold.

Differentiating (1.8) covariantly, by virtue of (1.5) we have

$$\nabla_k K_{ji} = b(F_{kj}E_i + F_{ki}E_j) , \tag{1.9}$$

when the contact manifold M is normal. Conversely, if we assume that the normal contact Riemannian manifold satisfies the condition (1.9), by virtue of (1.5) we find

$$\nabla_k(K_{ji} - bE_j E_i) = 0 . \tag{1.10}$$

On the other hand, according to Lemma 1, the normal contact Riemannian manifold M is irreducible. Thus, taking account of (1.10), we have

$$K_{ji} - bE_j E_i = aG_{ji}$$

with a constant a, since the left hand side is a symmetric tensor. That is to say, the manifold M is a C-Einstein manifold. Therefore, we have

Lemma 2. *In order that a normal contact Riemannian manifold M is a C-Einstein manifold, it is necessary and sufficient that M satisfies the condition* (1.9).

2. Submanifolds of codimension 2 in a Riemannian manifold

We consider a submanifold V of codimension 2 on a differentiable manifold M of dimension $2n + 1$ with positive definite Riemannian metric G_{ji}, and denote the parameter representation of the submanifold V by

$$x^h = x^h(u^a)$$

where (u^a) are the local coordinates of V, and the indices a, b, c, d, e, f run over the range $\{1, \cdots, 2n-1\}$.

Put

$$B_b{}^h = \partial_b x^h ,$$

∂_b denoting the operator $\partial/\partial u^b$, and denote a pair of mutually orthogonal unit vector fields normal to V by C^h and D^h, which are locally defined in each coordinate neighborhood of V. Then the Riemannian metric induced on V is given by

$$g_{cb} = G_{ji} B_c{}^j B_b{}^i , \tag{2.1}$$

and we have

$$\begin{gathered} G_{ji} C^j B_b{}^i = 0 , \qquad G_{ji} D^j B_b{}^i = 0 , \\ G_{ji} C^j C^i = 1 , \quad G_{ji} D^j C^i = 0 , \quad G_{ji} D^j D^i = 1 . \end{gathered} \tag{2.2}$$

If we denote by ∇_c the so-called van der Waerden-Bortolotti covariant differentiation on V, i.e., if we put

$$\nabla_c B_b{}^h = \partial_c B_b{}^h + \left\{ {}_j{}^h{}_i \right\} B_c{}^j B_b{}^i - \left\{ {}_c{}^a{}_b \right\} B_a{}^h , \tag{2.3}$$

$$\nabla_c C^h = \partial_c C^h + \left\{ {}_j{}^h{}_i \right\} B_c{}^j C^i , \quad \nabla_c D^h = \partial_c D^h + \left\{ {}_j{}^h{}_i \right\} B_c{}^j D^i , \tag{2.4}$$

$\left\{ {}_j{}^h{}_i \right\}$ and $\left\{ {}_c{}^a{}_b \right\}$ being the Christoffel symbols formed respectively with G_{ji} and g_{cb}, then, taking account of (2.2), we have

$$\nabla_c B_b{}^h = h_{cb} C^h + k_{cb} D^h , \tag{2.5}$$

$$\nabla_c C^h = - h_c{}^a B_a{}^h + l_c D^h , \quad \nabla_c D^h = - k_c{}^a B_a{}^h - l_c C^h , \tag{2.6}$$

where h_{cb} and k_{cb} are the second fundamental tensors, and l_c the third fundamental tensor with respect to C^h and D^h. As is well-known, we have

$$\begin{aligned} h_{cb} &= h_{bc} , & k_{cb} &= k_{bc} , \\ h_c{}^a &= h_{cb} g^{ba} , & k_c{}^a &= k_{cb} g^{ba} , \end{aligned}$$

where (g^{cb}) is the inverse of the matrix (g_{cb}). (2.5) are equations of Gauss, and (2.6) equations of Weingarten. We also have

$$K_{kjih} B_d{}^k B_c{}^j B_b{}^i B_a{}^h = R_{dcba} - (h_{da} h_{cb} - h_{ca} h_{db} + k_{da} k_{cb} - k_{ca} k_{db}) , \tag{2.7}$$

$$\begin{aligned} K_{kjih} B_d{}^k B_c{}^j B_b{}^i C^h &= (\nabla_d h_{cb} - \nabla_c h_{db}) - (l_d k_{cb} - l_c k_{db}) , \\ K_{kjih} B_d{}^k B_c{}^j B_b{}^i D^h &= (\nabla_d k_{cb} - \nabla_c k_{db}) + (l_d h_{cb} - l_c h_{db}) , \end{aligned} \tag{2.8}$$

$$K_{kjih} B_d{}^k B_c{}^j C^i D^h = \nabla_d l_c - \nabla_c l_d + h_d{}^a k_{ca} - h_c{}^a k_{da} , \tag{2.9}$$

where K_{kjih} and R_{dcba} are the curvature tensors of the enveloping manifold

M and the submanifold V respectively. (2.7) are equations of Gauss, (2.8) equations of Codazzi, and (2.9) equations of Ricci.

When the enveloping manifold M is of constant curvature c, that is, when K_{kjih} is of the form

$$K_{kjih} = c(G_{kh}G_{ji} - G_{jh}G_{ki}),$$

equations (2.7), (2.8) and (2.9) become respectively

$$R_{dcba} = c(g_{da}g_{cb} - g_{ca}g_{db}) + (h_{da}h_{cb} - h_{ca}h_{db} + k_{da}k_{cb} - k_{ca}k_{db}), \tag{2.10}$$

$$\begin{aligned} (\nabla_d h_{cb} - l_d k_{cb}) - (\nabla_c h_{db} - l_c k_{db}) = 0, \\ (\nabla_d k_{cb} + l_d h_{cb}) - (\nabla_c k_{db} + l_c h_{db}) = 0, \end{aligned} \tag{2.11}$$

$$\nabla_d l_c - \nabla_c l_d + h_d{}^a k_{ca} - h_c{}^a k_{da} = 0. \tag{2.12}$$

Transvecting (2.10) with g^{da}, we have

$$R_{cb} = 2(n-1)cg_{cb} + (h_e{}^e h_{cb} + k_e{}^e k_{cb}) - h_{ca}h_b{}^a - k_{ca}k_b{}^a, \tag{2.13}$$

where $R_{cb} = g^{da}R_{dcba}$ is the Ricci tensor of the submanifold V.

Equations (2.11) imply

Lemma 3. *For any submanifold of codimension 2 in a Riemannian manifold of constant curvature, the tensor fields*

$$h_{dcb} = \nabla_d h_{cb} - l_d k_{cb}, \qquad k_{dcb} = \nabla_d k_{cb} + l_d h_{cb}$$

are symmetric in all their indices d, c, b.

3. Submanifolds of codimension 2 in a contact Riemannian manifold

We now assume that the enveloping manifold M is a contact Riemannian manifold of dimension $2n+1$ with structure $(F_i{}^h, E_i, E^h, G_{ji})$, and that there is given in M a submanifold V of codimension 2. Then, for the transforms of $B_b{}^h$, C^h and D^h by $F_i{}^h$, due to the relations $F_{ji}C^jC^i = F_{ji}D^jD^i = 0$ and $F_{ji}C^jD^i = -F_{ji}D^jC^i$ we have equations of the form

$$F_i{}^h B_b{}^i = f_b{}^a B_a{}^h + p_b C^h + q_b D^h, \tag{3.1}$$

$$\begin{aligned} F_i{}^h C^i &= -p^a B_a{}^h + rD^h, \\ F_i{}^h D^i &= -q^a B_a{}^h - rC^h, \end{aligned} \tag{3.2}$$

where p^a and q^a are defined by

$$p^a = p_b g^{ba}, \qquad q^a = q_b g^{ba}$$

respectively, $f_b{}^a$ define a global tensor field of type (1, 1) in V, independent of the choice of C^h and D^h, p^a and q^a are two local vector fields, and r is a global scalar field in V, independent of the choice of C^h and D^h. On the submanifold V the vector field E^h has the form

$$E^h = e^a B_a{}^h + \alpha C^h + \beta D^h , \tag{3.3}$$

where e^a define a global vector field in V and α, β two local scalar fields.

Considering the transform of (3.1) by $F_i{}^h$ and taking account of (1.2), (3.1), (3.2) and (3.3), we find

$$\begin{aligned} f_c{}^a f_b{}^c &= -\delta_b^a + e_b e^a + p_b p^a + q_b q^a , \\ f_b{}^a p_a &= \alpha e_b + r q_b , \\ f_b{}^a q_a &= \beta e_b - r p_b , \end{aligned} \tag{3.4}$$

where

$$e_b = g_{ba} e^a . \tag{3.5}$$

Similarly, we have from (3.2)

$$p_a p^a = 1 - \alpha^2 - r^2 , \quad q_a q^a = 1 - \beta^2 - r^2 , \quad p_a q^a = -\alpha\beta . \tag{3.6}$$

Taking the transform of (3.3) by $F_i{}^h$ and using (3.1) and (3.2), we find

$$f_b{}^a e^b = \alpha p^a + \beta q^a , \quad p_a e^a = \beta r , \quad q_a e^a = -\alpha r . \tag{3.7}$$

On the other hand, due to $g_{ji} E^j E^i = 1$, from (3.3) it follows

$$e_a e^a = 1 - \alpha^2 - \beta^2 . \tag{3.8}$$

Now differentiating (3.1) covariantly on the submanifold V and using (2.5), (2.6) we obtain

$$\begin{aligned} &(\nabla_j F_i{}^h) B_c{}^j B_b{}^i + F_i{}^h (h_{cb} C^i + k_{cb} D^i) \\ &= (\nabla_c f_b{}^a) B_a{}^h + f_b{}^a (h_{ca} C^h + k_{ca} D^h) \\ &\quad + (\nabla_c p_b) C^h + p_b (- h_c{}^a B_a{}^h + l_c D^h) \\ &\quad + (\nabla_c q_b) D^h + q_b (- k_c{}^a B_a{}^h - l_c C^h) . \end{aligned} \tag{3.9}$$

If we assume that the enveloping manifold M is normal, then we have, from (1.6) and (3.9),

$$\begin{aligned} \nabla_c f_b{}^a &= - g_{cb} e^a + \delta_c^a e_b - h_{cb} p^a + h_c{}^a p_b - k_{cb} q^a + k_c{}^a q_b , \\ \nabla_c p_b &= - \alpha g_{cb} - r k_{cb} - h_{ca} f_b{}^a + l_c q_b , \\ \nabla_c q_b &= - \beta g_{cb} + r h_{cb} - k_{ca} f_b{}^a - l_c p_b . \end{aligned} \tag{3.10}$$

Differentiating (3.2), (3.3) covariantly on the submanifold V and taking account of (1.5), (1.6), (3.1) and (3.2), for normal M we find

$$(3.11) \qquad \nabla_c r = -h_{cb}q^b + k_{cb}p^b ,$$

$$(3.12) \qquad \begin{gathered} \nabla_b e^a = f_b{}^a + \alpha h_b{}^a + \beta k_b{}^a , \\ \nabla_b{}^\alpha = p_b - h_{ba}e^a + \beta l_b , \qquad \nabla_b\beta = q_b - k_{ba}e^a - \alpha l_b . \end{gathered}$$

4. Invariant submanifolds of codimension 2 in a contact Riemannian manifold

We now assume that the tangent space of the submanifold V of codimension 2 in a contact Riemannian manifold M is invariant under the action of $F_i{}^h$ at every point, and we call such a submanifold an *invariant submanifold*.

For an invariant submanifold, we obtain

$$(4.1) \qquad F_i{}^h B_b{}^i = f_b{}^a B_a{}^h ,$$

that is,

$$(4.2) \qquad p_b = 0 , \qquad q_b = 0$$

in (3.1). Thus we have

$$F_i{}^h C^i = rD^h , \qquad F_i{}^h D^i = -rC^h$$

from (3.2),

$$(4.3) \qquad \begin{gathered} f_c{}^a f_b{}^c = -\delta_b^a + e_b e^a , \\ \alpha e_b = 0 , \qquad \beta e_b = 0 \end{gathered}$$

from (3.4),

$$(4.5) \qquad 1 - \alpha^2 - r^2 = 0 , \quad 1 - \beta^2 - r^2 = 0 , \quad \alpha\beta = 0$$

from (3.6), and finally

$$(4.6) \qquad f_b{}^a e^b = 0 , \quad \beta r = 0 , \quad \alpha r = 0$$

from (3.7). Moreover, equations (4.5) imply

$$\alpha = \beta = 0 , \qquad r^2 = 1 .$$

Conversely, if $r^2 = 1$, then equations (3.6) show that $p^a = 0$, $q^a = 0$, $\alpha = 0$, $\beta = 0$, and consequently V is invariant because of (3.1) and the Riemannian metric g_{cb} being positively definite.

Thus, in order that *a submanifold V of codimension 2 in a contact Riemannian manifold M be invariant, it is necessary and sufficient that $r^2 = 1$ in* (3.2) (*cf.* [8]).

In the sequal, we always consider invariant submanifolds and hence may assume that $r = 1$. We then have, for an invariant submanifold V,

$$F_i{}^h B_b{}^i = f_b{}^a B_a{}^h \,, \quad F_i{}^h C^i = D^h \,, \quad F_i{}^h D^i = -C^h \,; \tag{4.7}$$

$$E^h = e^a B_a{}^h \,; \tag{4.8}$$

$$\begin{aligned} f_c{}^a f_b{}^c &= -\delta_b^a + e_b e^a \,, \\ f_b{}^a e^b &= 0 \,, \qquad e_a e^a = 1 \,. \end{aligned} \tag{4.9}$$

Transvecting (4.8) with $G_{ih} B_b{}^i$ and taking account of (2.1), (3.5) and (4.1), we find

$$E_i B_b{}^i = e_b \,. \tag{4.10}$$

If we transvert the last equation of (1.3) with $B_c{}^j B_b{}^i$ and take account of (2.1), (4.7) and (4.10), then we obtain

$$f_c{}^e f_b{}^d g_{ed} = g_{cb} - e_c e_b \,. \tag{4.11}$$

On the other hand, we have, from (1.1) and (1.4),

$$F_j{}^h G_{ih} = \frac{1}{2}(\partial_j E_i - \partial_i E_j) \,.$$

Transvecting this equation with $B_c{}^j B_b{}^i$, and taking account of (2.1), (4.7), (4.10) and $\partial_c B_b{}^h = \partial_b B_c{}^h$, we find

$$f_c{}^a g_{ab} = \frac{1}{2}(\partial_c e_b - \partial_b e_c) \,. \tag{4.12}$$

Thus equations (3.5), (4.9), (4.11) and (4.12) show that *any invariant submanifold of codimension 2 in a contact Riemannian manifold is also a contact Riemannian manifold.*

We now assume that the enveloping contact Riemannian manifold M is normal and the submanifold V is invariant. From the first equations of (3.12) and (3.10) we then have, respectively,

$$\begin{aligned} \nabla_b e^a &= f_b{}^a \,, \\ \nabla_c f_b{}^a &= -g_{cb} e^a + \delta_c^a e_b \end{aligned} \tag{4.13}$$

by virtue of $p^a = 0$, $q^a = 0$, $\alpha = 0$, $\beta = 0$.

Equations (4.13) show that *any invariant submanifold of codimension* 2 *in a normal contact Riemannian manifold is also a normal contact Riemannian manifold.*

When the enveloping manifold M is normal and the submanifold V is invariant, from the second and third equations of (3.10) and (3.12), by virtue of $p_b = 0$, $q_b = 0$, $\alpha = 0$, $\beta = 0$, $r = 1$ we obtain, respectively,

$$k_{cb} = - h_{ca}f_b{}^a\,, \qquad h_{cb} = k_{ca}f_b{}^a\,, \tag{4.14}$$

$$h_{ba}e^a = 0\,, \qquad k_{ba}e^a = 0\,. \tag{4.15}$$

Since $f_{cb} = f_c{}^d g_{db}$ is skew-symmetric, and h_{cb}, k_{cb} are symmetric, equations (4.14) give

$$h_{ca}f_b{}^a - h_{ba}f_c{}^a = 0\,, \qquad k_{ca}f_b{}^a - k_{ba}f_c{}^a = 0\,, \tag{4.16}$$

$$h_c{}^c = h_{cb}g^{cb} = 0\,, \qquad k_c{}^c = h_{cb}g^{cb} = 0\,, \tag{4.17}$$

which thus show that *any invariant submanifojd of codimension* 2 *in a normal contact Riemannian manifold is minimal* (cf. [8]).

Denote the tensor fields $h_b{}^a$, $k_b{}^a$ and $f_b{}^a$ of type (1, 1) by h, k and f respectively. Then (4.14), (4.6) are respectively equivalent to the conditions

$$h = kf\,, \qquad k = - hf\,, \tag{4.18}$$

$$hf + fh = 0\,, \qquad kf + fk = 0\,. \tag{4.19}$$

From (4.18) and (4.19), we thus have $h^2 = h(kf) = - h(fk) = - (hf)k = k^2$, or

$$h^2 = k^2\,, \tag{4.20}$$

and also $hk = (kf)k = k(fk) = - k(kf) = - kh$, or

$$hk + kh = 0\,. \tag{4.21}$$

5. Invariant C-Einstein submanifolds of codimension 2 in a normal contact Riemannian manifold

We assume that the enveloping manifold M is a normal contact Riemannian manifold of constant curvature, which necessarily equals to 1 (cf. [6], [10], [11], [16]), and the invariant submanifold V of codimension 2 imbedded in M is a C-Einstein manifold. Taking account of (2.13) with $c = 1$ and (4.17), we then see that the Ricci tensor of V has the form

$$R_{cb} = 2(n - 1)g_{cb} - h_{ca}h_b{}^a - k_{ca}k_b{}^a\,.$$

On the other hand, since V is a C-Einstein manifold, we have

$$R_{cb} = ag_{cb} + be_c e_b$$

with constants a and b. Thus

$$ag_{cb} + be_c e_b = 2(n-1)g_{cb} - h_{ca}h_b{}^a - k_{ca}k_b{}^a . \tag{5.1}$$

If the submanifold V is an Einstein manifold, i.e., if $b = 0$ in (5.1), then from (4.20) and (5.1) we find

$$h^2 = k^2 = \lambda I$$

with constant λ and the identity tensor I. Since the induced metric of the submanifold is positive definite, the above equation, together with (4.15), implies

$$h = k = 0 .$$

Thus we have

Proposition 5.1. *Any invariant Einstein submanifold V in a normal contact Riemannian manifold of constant curvature is totally geodesic.*

Taking account of (4.20), from (5.1) we have

$$h_{ca}h_b{}^a = k_{ca}k_b{}^a = \left(n - 1 - \frac{a}{2}\right)g_{cb} - \frac{b}{2}e_c e_b ,$$

from which, taking account of (4.15), we find

$$h_{ca}h_b{}^a = k_{ca}k_b{}^a = \mu(g_{cb} - e_c e_b) \tag{5.2}$$

with a constant μ. Transvecting (5.2) with $f_d{}^b$ and taking account of (4.14), we obtain

$$h_{da}k_c{}^a = \mu f_{dc} , \qquad k_{da}h_c{}^a = -\mu f_{dc} . \tag{5.3}$$

Differentiating both equations of (4.14) covariantly and taking account of (4.13), (4.14) and (4.15), we find

$$\begin{aligned} h_{dcb} &= k_{dca}f_b{}^a + k_{dc}e_b , \\ k_{dcb} &= - h_{dca}f_b{}^a - h_{dc}e_b , \end{aligned} \tag{5.4}$$

where

$$h_{dcb} = \nabla_d h_{cb} - l_d k_{cb} , \qquad k_{dcb} = \nabla_d k_{cb} + l_d h_{cb} . \tag{5.5}$$

Transvecting (5.4) with e^b and taking account of (4.9), we have

$$h_{dcb}e^b = k_{dc}\,, \qquad k_{dcb}e^b = -\,h_{dc}\,. \tag{5.6}$$

If we differentiate (5.2) covariantly and take account of (4.13) and (5.3), then we find

$$\begin{aligned} h_{dcb}h_a{}^b + h_{dab}h_c{}^b &= -\,\mu(f_{dc}e_a + f_{da}e_c)\,, \\ k_{dcb}k_a{}^b + k_{dab}k_c{}^b &= -\,\mu(f_{dc}e_a + f_{da}e_c)\,. \end{aligned} \tag{5.7}$$

According to Lemma 3 stated in §2, we have $k_{cdb} = k_{cbd}$, which and the second equation of (5.4) imply

$$h_{dce}f_b{}^e + h_{dc}e_b = h_{cbe}f_d{}^e + h_{cb}e_d\,.$$

Transvecting the above equation with $f_a{}^b$ and taking account of Lemma 3, (4.9), (4.14) and (5.6), we have, after changing the indices,

$$h_{dcb} = -\,f_d{}^f f_c{}^e h_{feb} + k_{db}e_c + k_{cb}e_d\,.$$

If we substitute the equation above into the first equation of (5.7) written as

$$h_{dcb}h_a{}^b + h_{dba}h_c{}^b = -\,\mu(f_{dc}e_a + f_{da}e_c)\,,$$

and take account of (4.15) and (5.3), then we find

$$f_d{}^f\{f_b{}^e h_c{}^b h_{fea} + f_c{}^e h_{feb}h_a{}^b - \mu g_{fc}e_a\} = 0\,,$$

from which

$$f_b{}^e h_c{}^b h_{fea} + f_c{}^e h_{feb}h_a{}^b - \mu g_{fc}e_a = e_f l_{ca}\,, \tag{5.8}$$

where l_{ca} is a certain tensor field of type (0, 2), because $f_d{}^f e_f = 0$ and $f_d{}^f$ is of rank $2n-2$. Transvecting (5.8) with e^f and taking account of (5.6), we have

$$l_{ca} = f_b{}^e h_c{}^b k_{ea} + f_c{}^e k_{eb}h_a{}^b - \mu e_c e_a\,,$$

which reduces to

$$l_{ca} = \mu(2g_{ca} - 3e_c e_a)$$

because of (4.18), (4.19) and (5.2). If we substitute this in (5.8), then we obtain

$$f_b{}^e h_c{}^b h_{fea} + f_c{}^e h_{feb}h_a{}^b = 2\mu(g_{ca} - e_c e_a)e_f + \mu(g_{fc} - e_f e_c)e_a\,.$$

If we transvect the above equation with $f_d{}^c$ and take account of (4.9), (4.18), (4.19), (5.3) and (5.6), then we find

$$h_d{}^e h_{fea} - h_{fdb} h_a{}^b + \mu f_{af} e_d = \mu(2f_{da} e_f - f_{fd} e_a) ,$$

that is,

$$h_d{}^e h_{fea} - h_{fdb} h_a{}^b = \mu(2f_{da} e_f - f_{fd} e_a - f_{af} e_d) ,$$

from which and (5.7) it follows that

$$h_{fea} h_d{}^e = - \mu(f_{fd} e_a + f_{ad} e_f) .$$

Transvecting the above equation with $h_b{}^d$ and taking account of (4.14), (5.2) and (5.6), we find

$$h_{fba} = k_{fb} e_a + k_{af} e_b + k_{ba} e_f . \tag{5.9}$$

Similarly, we have

$$k_{fba} = - h_{fb} e_a - h_{af} e_b - h_{ba} e_f . \tag{5.10}$$

Thus from (5.5), (5.9) and (5.10) we arrive at

Proposition 5.2. *Let V be an invariant submanifold of codimension 2 in a normal contact Riemannian manifold of constant curvature. If V is a C-Einstein manifold, then*

(A)
$$\begin{aligned} \nabla_f h_{ba} - l_f k_{ba} &= k_{fb} e_a + k_{af} e_b + k_{ba} e_f , \\ \nabla_f k_{ba} + l_f h_{ba} &= - h_{fb} e_a - h_{af} e_b - h_{ba} e_f . \end{aligned}$$

Differentiating (2.10) covariantly and using the above condition (A) we obtain

Proposition 5.3. *Let V be an invariant submanifold of codimension 2 in a normal contact Riemannian manifold of constant curvature. If V is a C-Einstein manifold, then*

(B)
$$\nabla_e R_{dcba} = S_{edcb} e_a + S_{ecda} e_b + S_{ebad} e_c + S_{eabc} e_d ,$$

where

$$S_{edcb} = k_{ed} h_{cb} - k_{ec} h_{db} + h_{ec} k_{db} - h_{ed} k_{cb} . \tag{5.10}$$

If we transvect equation (B) with g^{da} and take account of (4.17), (5.3) and (5.10), then we have

Proposition 5.4. *Let V be an invariant submanifold of codimension 2 in a normal contact Riemannian manifold of constant curvature. If V is a C-Einstein manifold, then*

(C)
$$\nabla_e R_{cb} = b(f_{ec} e_b + f_{eb} e_c) ,$$

b being constant.

Any invariant submanifold in a normal contact Riemannian manifold is also a normal contact Riemannian manifold. Taking account of Lemma 2 stated in § 1, from Propositions 5.2, 5.3 and 5.4 we thus obtain

Theorem. *For an invariant submanifold V of codimension 2 in a normal contact Riemannian manifold of constant curvature, the condition that V be a C-Einstein manifold is equivalent to one of the conditions* (A), (B) *and* (C).

Transvecting (B) with e^a and taking account of (4.15) and (5.10), we find

$$S_{edcb} = (\nabla_e R_{dcba})e^a ,$$

substitution of which in the condition (B) gives immediately

Proposition 5.5. *If an invariant submanifold of codimension 2 in a normal contact Riemannian manifold of constant curvature is a C-Einstein manifold, then the identity*

$$\begin{aligned} \nabla_e R_{dcba} = {} & (\nabla_e R_{dcbf})e^f e_a + (\nabla_e R_{dcfa})e^f e_b \\ & + (\nabla_e R_{dfba})e^f e_c + (\nabla_e R_{fcba})e^f e_d \end{aligned}$$

holds.

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