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REMARKS ON THE FIRST MAIN THEOREM IN EQUIDISTRIBUTION THEORY. II

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To Ambrose in gratitude and esteem

1. The purpose of this paper is to integrate the non-integrated First Main Theorem ((i) of the Theorem in Part I [12]) for the case of a holomorphic map $f: V \to M$, where V is an open complex manifold and M is compact kählerian and of the same dimension n as that of V. The formal aspect of the integration is trivial, but the more delicate part is to attend to the convergence of the improper integrals which thus arise. In general, the convergence question is quite untractable, as contrasted with the case of Riemann surfaces where such matters can be adequately treated without exception [8]. The principal results of this work are Theorem 5.1 and Corollary 5.2; the simplicity and the very geometric nature of the hypothesis in these results justify the approach adopted here.

I take this opportunity to point out that the proof of the Theorem in [12] is incomplete at two places; the last section (§8) of this paper is devoted to filling in these gaps.

2. Definition 2.1. Let V be an open complex manifold. *A* C^∞ function $\tau: V \to [0, \infty)$ is called an *exhaustion function* iff

(i) τ is proper, i.e. τ^{-1} (compact set) is compact,

(ii) τ has only isolated critical points in $\tau^{-1}[r_0, \infty)$ for some r_0.

A first remark is that every open manifold, real or complex, always admits an exhaustion function (in fact one with only nondegenerate critical points). *We will always work in the range* $[r_0, \infty)$. Let $\mathscr{C}$ be the set of critical values of τ in $[r_0, \infty)$. Then (ii) says that $\mathscr{C}$ is discrete. If $t \notin \mathscr{C}$, then $\tau^{-1}(t)$ is a compact submanifold of V by (i). If $t \in \mathscr{C}$, then $\tau^{-1}(t)$ is a compact set which, with a finite number of points deleted, is a submanifold of V. We will consistently employ the notation:

$$V[t] = \tau^{-1}[0, t],$$
$$\partial V[t] = \tau^{-1}(t) .$$

It should be emphasized that the parameter value of the exhaustion functions

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in this paper goes to $+\infty$. Now suppose that $f: V \to M$ is holomorphic, dim $V = \dim M = n$, and M is compact kählerian. Let $\mathscr{D}$ be the set of critical values of f in M. By Sard's theorem, $\mathscr{D}$ is then of measure zero. Suppose further that

$$M' = M - \mathscr{D}.$$

For each a $\in M'$, df is nonsingular at each point of $f^{-1}(a)$. Hence $f^{-1}(a)$ is discrete, and consequently $f^{-1}(a) \cap V[t]$ is finite. Suppose now:

(a) t is not a critical value of τ.

(b) $f^{-1}(a) \cap \partial V[t] = \varnothing$.

Then the Theorem in [12] guarantees that if $a \in M'$, then

$$v(t) = n(t, a) + \int_{\partial V[t]} f^* d^c \lambda_a, \tag{1}$$

where $v(t)$ now stands for $v(V[t])$, and similarly $n(t, a) \equiv n(V[t], a)$. The notation in [12] is of course used throughout this paper without further comment.

Let $[t_1, t_2]$ be an interval in $[r_0, \infty)$ such that each $t \in [t_1, t_2]$ satisfies (a) and (b). Then (1) holds for all such t so that

$$\int_{t_1}^{t_2} v(t)dt = \int_{t_1}^{t_2} n(t, a)dt + \int_{t_1}^{t_2} dt \int_{\partial V[t]} f^* d^c \lambda_a, \tag{2}$$

where $a \in M'$ as usual. Now since each $\partial V[t]$ is a submanifold provided that at most a finite number of points F are deleted, in all matters concerning integration, such a set F can be ignored by standard approximation techniques. Hence (2) continues to hold even if (a) is violated. To get around (b), more work is required. The following lemma will be proved in §7.

Lemma A. *If $a \in M'$, then $\int_{V[r]} d\tau \wedge f^* d^c \lambda_a$ is finite for all r.*

Granting this, Fubini's theorem implies that

$$\int_{V[t_2]-V[t_1]} d\tau \wedge f^* d^c \lambda_a = \int_{t_1}^{t_2} dt \int_{\partial V[t]} f^* d^c \lambda_a.$$

In particular, the iterated integral on the right is a continuous function of t_1, t_2. Now the other two integrals in (2) are obviously also continuous functions of t_1, t_2. Thus we may extend (2) to all sub-intervals of $[r_0, \infty)$ regardless of whether (a) and (b) are in force or not. Defining

$$T(r) = \int_{r_0}^{r} v(t)dt,$$

$$N(r, a) = \int_{r_0}^{r} n(t, a)\, dt$$

for $a \in M'$, we therefore have

$$T(r) = N(r, a) + \int_{V[r]-V[r_0]} d\tau \wedge f^* d^c \lambda_a \,. \tag{3}$$

If now $f^{-1}(a) \cap$ closure $(V[r] - V[r_0]) = \{p_1, \cdots, p_l\}$, let D_ε^α be a ball of radius ε around each p_α relative to some coordinate system, and let $D_\varepsilon = \bigcup_{\alpha=1}^{\alpha=l} D_\varepsilon^\alpha$. Denoting $V[r] - V[r_0] - D_\varepsilon$ by V_ε, we have by way of Lemma A that

$$\int_{V[r]-V[r_0]} d\tau \wedge f^* d^c \lambda_a = \lim_{\varepsilon \to 0} \int_{V_\varepsilon} d\tau \wedge f^* d^c \lambda_a \,.$$

Since f is holomorphic, $f^* d^c \lambda_a = d^c f^* \lambda_a$. (This is the first time the holomorphy of f is needed.) As $d\tau \wedge d^c f^* \lambda_a$ is C^∞ in V_ε, Proposition 7.10 of Bott & Chern [9, p. 99] implies that

$$\begin{aligned}\int_{V_\varepsilon} d\tau \wedge d^c f^* \lambda_a &= \int_{V_\varepsilon} d(d^c \tau \wedge f^* \lambda_a) - \int_{V_\varepsilon} f^* \lambda_a \wedge dd^c \tau \\ &= \int_{\partial V_\varepsilon} d^c \tau \wedge f^* \lambda_a - \int_{V_\varepsilon} f^* \lambda_a \wedge dd^c \tau \,.\end{aligned}$$

We now need a second lemma.

Lemma B. (i) $\displaystyle\int_{V[r]} f^* \lambda_a \wedge dd^c \tau$ *is finite.*

(ii) $\displaystyle\lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} d^c \tau \wedge f^* \lambda_a = 0 \,.$

This Lemma, to be proved in §7 along with Lemma A, assures us that

$$\begin{aligned}\lim_{\varepsilon \to 0} \int_{V_\varepsilon} d\tau \wedge d^c f^* \lambda_a = \int_{\partial V[r]} d^c \tau \wedge f^* \lambda_a &- \int_{\partial V[r_0]} d^c \tau \wedge f^* \lambda_a \\ &- \int_{V[r]-V[r_0]} f^* \lambda_a \wedge dd^c \tau \,.\end{aligned}$$

If we define for each $\alpha \in M'$.

$$S(r, a) = \int_{V[r]-V[r_0]} f^* \lambda_a \wedge dd^c \tau - \int_{\partial V[r]} d^c \tau \wedge f^* \lambda_a \,,$$

we have proved:

$$N(r, a) = T(r) + \int_{\partial V[r_0]} d^c\tau \wedge f^*\lambda_a + S(r, a) \tag{4}$$

for each $a \in M'$.

3. Equation (4) will be used to derive some results to the effect that $M - f(V)$ is a set of measure zero. Before doing that, let us note the special restriction on (4), namely, that $a \in M'$. If $a \in M - M'$, then it is not clear that $\int_{\partial V[r_0]} d^c\tau \wedge i^*\lambda_a$ exists. This is a serious problem because one would like to have an inequality:

$$N(r, a) \leq T(r) + S(r, a) + C, \tag{5}$$

where C is some constant *independent of* a and r. If this were true, then the conclusion in Theorem 4.1 about $M - f(V)$ being a set of measure zero could be strengthened to the statement that $\delta = 0$ a.e. on M, where as usual

$$\delta(a) = \liminf_{r \to \infty} \left(1 - \frac{N(r, a)}{T(r)}\right).$$

Now what happens in the case of Riemann surfaces is this: $\int_{\partial V[r_0]} d^c\tau \wedge f^*\lambda_a$ is a continuous function of a, and so we may let C be the maximum of this function on M (see [8, §9]).

As we do not even know the convergence of $\int_{\partial V[r_0]} d^c\tau \wedge f^*\lambda_a$ in general, this program is out of reach.

4. We will now proceed to put some restrictions on the Levi form $dd^c\tau$ of τ. In view of the important theorems of Andreotti-Grauert, it would be desirable to be able to deal with the cases where $dd^c\tau$ has both negative and positive eigenvalues. Unfortunately, this has not been accomplished.

Definition 4.1. An exhaustion function τ is called *concave* iff $dd^c\tau \leq 0$ on $\tau^{-1}[r, \infty)$ for some r_0, i.e. iff the eigenvalues of the Levi form $dd^c\tau$ are nonpositive from r_0 upward. It is called *convex* iff $dd^c\tau \geq 0$ on $\tau^{-1}[r, \infty)$ for some r_0, i.e. iff the eigenvalues of the Levi form $dd^c\tau$ are nonnegative from r_0 upward.

In the sequel, we will make the harmless assumption that the r_0 of Definition 4.1 is the same as that of Definition 2.1. Suppose then τ is concave. Because λ_a is a positive form on M ((ii) of the Theorem in [12]) and holomorphic mappings preserve positivity (we have again invoked the holomorphy of f), $(- f^*\lambda_a \wedge dd^c\tau)$ is a positive form on V in the sense that it induces a nonnegative measure on V. Because $d^c\tau \wedge f^*\lambda_a$ always induces a positive measure on $\partial V[t]$ (for all these, see [9, p. 100]), we have

$$S(r, a) = \int_{V[r]-V[r_0]} f^*\lambda_a \wedge dd^c\tau - \int_{\partial V[r]} d^c\tau \wedge f^*\lambda_a \leq 0\,.$$

Thus (4) implies that

$$N(r, a) \leq T(r) + \int_{\partial V[r_0]} d^c\tau \wedge f^*\lambda_a\,. \tag{6}$$

We now integrate (6) with respect to a over M. For this, we need

Lemma C. *$n(t, a)$ is a locally integrable function on $[r_0, r] \times M$.*

This Lemma will also be proved in §7. For the moment, the holomorphy of f implies that $n(t, a) \geq 0$ a.e. (this is the third place where we need f to be holomorphic), so that Fubini's theorem may be applied. Thus, $\int_A N(r, a)\Psi_a = \int_{r_0}^r dt \int_A n(t, a)\Psi_a$ for any measurable subset A of M. We may recall that Ψ is the volume element of M and $\int_M \Psi = 1$ by choice. To avoid confusion, let M_1 denote the image of V in M under f. Now suppose V has a concave exhaustion and $M - M_1$ has positive measure ε ; then we will see that this leads to a contradiction. Since $n(t, a) = 0$ if $a \in M - M_1$,

$$\begin{aligned}\int_{M_1} N(r, a)\Psi_a &= \int_M N(r, a)\Psi_a = \int_{r_0}^r dt \int_M n(t, a)\Psi_a \\ &= \int_{r_0}^r dt \int_{V[t]} f^*\Psi = \int_{r_0}^r v(t)\,dt \equiv T(r)\,.\end{aligned}$$

In other words,

$$\int_{M_1} N(r, a)\Psi_a = T(r)\,, \tag{8}$$

where we have made use of Lemma 6.2 in [8]. On the other hand,

$$\begin{aligned}&\int_{M_1}\left(T(r) + \int_{\partial V[r_0]} d^c\tau \wedge f^*\lambda_a\right)\Psi_a \\ &\qquad = (1-\varepsilon)\,T(r) + \int_{M_1}\Psi_a \int_{\partial V[r_0]} d^c \wedge f^*\lambda_a\,.\end{aligned} \tag{9}$$

We will now prove

$$\int_{M_1} \Psi_a \int_{\partial V[r_0]} d^c\tau \wedge f^*\lambda_a \leq C_1\,, \qquad C_1 \text{ independent of } a \text{ and } r\,. \tag{10}$$

The argument leading to (10) is not dependent on any specific assumption on $dd^c\tau$ and will be used again in §5. Also note once and for all that although the integral $\int_{\partial V[r_0]} d^c\tau \wedge f^*\lambda_a$ is defined only for $a \in M'$. it makes sense to integrate it over M because $M-M' = \mathscr{D}$ is the critical values of f, and is a set of measure zero. So let us recall that $\lambda_a(y) = g(a, y)\, \Lambda\Psi_y$, where Λ denotes the interior product by the kähler form of M [7, p. 21], and g is C^∞ on $M \times M$ except that along the diagonal it is $0\left(\frac{1}{r^{2n-2}}\right)$ in the terminology of de Rham [5, p. 135]. Recall also that g has been chosen to be positive on $M \times M$ everywhere in accordance with (iii) of the Theorem in [12]. Thus f^*g (or more precisely, $(1 \times f)^*g$) is a positive (and obviously) measurable function on $M \times \partial V[r_0]$. But Ψ_a is a positive measure on M, and $d^c\tau \wedge f^*(\Lambda\Psi_y)$ is a positive measure on $\partial V[r_0]$. Therefore, we may again invoke Fubini's theorem (alias Tonnelli's theorem). Thus

$$\begin{aligned}\int_{M_1} \Psi_a \int_{\partial V[r_0]} d^c\tau \wedge f^*\lambda_a &= \int_{\partial V[r_0]} d^c\tau \wedge f^*(\Lambda\Psi_y) \cdot f^*\left(\int_{M_1} g(a, y)\, \Psi_a\right) \\ &\leq \int_{\partial V[r_0]} d^c\tau \wedge f^*(\Lambda\Psi_y) \cdot f^*\left(\int_{M} g(a, y)\, \Psi_a\right).\end{aligned}$$

Now because of the singularity of $g(a, y)$ (see [12, equation (8)]), $\int_M g(a, y)\, \Psi_a$ is a continuous function of y by virtue of Lemma 5 of de Rham [5, p. 140], and so the last integral in the above inequality is a finite constant C_1 (we have of course made use of the compactness of $\partial V[r_0]$). This C_1 is clearly independent of r and a, and so (10) is proved.

Combining (8), $\cdots$, (10), we have

$$T(r) \leq (1 - \varepsilon)\, T(r) + C_1\,.$$

If we assume df to be nonsingular at some point of V, then $v(t)$ becomes a positive nondecreasing function of t, so that $T(r) \to \infty$ as $r \to \infty$, and therefore $1 \leq (1 - \varepsilon)$, which is impossible. Hence we have proved

Theorem 4.1. *Let $f: V \to M$ be holomorphic, and df nonsingular at some point of V. If V has a concave exhaustion, and M is compact kählerian and of the same dimension as that of V, then $M - f(V)$ is of measure zero.*

It should be stressed that in our definition the parameter τ goes to $+\infty$.

This theorem is similar to the equidistribution theorem in [9]. Note also the following consequence:

Corollary 4.2. *If V is obtained from a compact kähler manifold by removing a set of positive measure, then V admits no concave exhaustion.*

5. The case of convex exhaustion will now be examined (see Definition 4.1). Note the well-known fact that Stein manifolds are exactly those, which admit an exhaustion function without critical points and whose Levi form has *everywhere* strictly positive eigenvalues; so they form a sub-class of the complex manifolds with convex exhaustion. In general, we again integrate (4) to arrive at the analogue of Theorem 4.1. The precise statement is:

Theorem 5.1. *Let $f: V \to M$ be a holomorphic mapping between complex manifolds of the same dimension n such that df is nonsingular at one point of V. Assume V has a convex exhaustion function (i.e. eventually $dd^c\tau \leq 0$), and M is compact kählerian with kähler form κ. If*

$$\liminf_{r\to\infty} \frac{\int_{V[r]-V[r_0]} f^*\kappa^{n-1} \wedge dd^c\tau}{T(r)} = 0\,,$$

then $M - f(V)$ is a set of measure zero.

Proof. Let us denote the image of V in M under f by M_1. Assume the theorem to be false, then $M - M_1$ has measure $\varepsilon > 0$. Integrating (4) over M_1, from §4 (especially (10)) we obtain

$$T(r) \leq (1-\varepsilon)\,T(r) + \int_{M_1} S(r,a)\,\Psi_a + C_1\,. \tag{11}$$

Now, $d^c\tau \wedge f^*\lambda_a$ induces a positive measure on $\partial V[r]$ (we again refer to Bott-Chern [9, p. 100] for details), so that

$$\int_{M_1} S(r,a)\,\Psi_a \leq \int_{M_1} \Psi_a \int_{V[r]-V[r_0]} f^*\lambda_a \wedge dd^c\tau\,.$$

At this point, the condition $dd^c\tau > 0$ enters crucially. Recall from §4 that $f^*\lambda_a = f^*g(a,\cdot\,)\cdot f^*\Lambda\Psi$. Because both $f^*\Lambda\Psi$ and $dd^c\tau$ are positive forms, $f^*\Lambda\Psi \wedge dd^c\tau$ induces a positive measure on V. Since Ψ is also a positive measure on M and $f^*g(a,\cdot\,)$ is a positive function measurable on $M \times V$, Fubini's Theorem is again applicable.

Hence,

$$\int_{M_1} \Psi_a \int_{V[r]-V[r_0]} f^*\lambda_a \wedge dd^c\tau = \int_{M_1} \Psi_a \left(\int_{V[r]-V[r_0]} (f^*g(a,y)) \cdot f^*\Lambda\Psi_y \wedge dd^c\tau \right)$$

$$= \int_{V[r]-V[r_0]} f^*\Lambda\Psi_y \wedge dd^c\tau \cdot f^*\left(\int_{M_1} g(a, y)\Psi_a\right)$$

$$\leq \int_{V[r]-V[r_0]} f^*\Lambda\Psi_y \wedge dd^c\tau \cdot f^*\left(\int_M g(a, y)\Psi_a\right).$$

Again by Lemma 5 of de Rham [5, p. 160], $\int_M g(a, y)\Psi_a$ is a continuous function of y and thus attains a positive maximum C_2, where C_2 is independent of r and a. Hence the right side of the above inequality is not greater than

$$C_2 \int_{V[r]-V[r_0]} f^*\Lambda\Psi \wedge dd^c\tau .$$

Now observe that $\Psi = \kappa^n/n!$ and so a simple computation gives $\Lambda\Psi = \kappa^{n-1}/(n-1)!$. Thus

$$\int_{M_1} \Psi_a \int_{V[r]-V[r_0]} f^*\lambda_a \wedge dd^c\tau \leq \frac{C_2}{(n-1)!} \int_{V[r]-V[r_0]} f^*\kappa^{n-1} \wedge dd^c\tau .$$

Combining this with (11), we get

$$T(r) \leq (1-\varepsilon)T(r) + C_1 + \frac{C_2}{(n-1)!} \int_{V[r]-V[r_0]} f^*\kappa^{n-1} \wedge dd^c\tau .$$

If the assumption of Theorem 5.1 is in force, then, as $r \to \infty$, $1 \leq (1-\varepsilon)$, which gives a contradiction. q.e.d.

The most important special case of Theorem 5.1, without doubt, is that of $V = C^n$. In this case, we let $\tau = \sum_i z_i \bar{z}_i$. Then $dd^c\tau = 2\sqrt{-1}\ \sum_i dz_i \wedge d\bar{z}_i$ and $dd^c\tau > 0$ everywhere. We may therefore let r_0 be 0. On the other hand, the kähler form ω_0 associated with the flat metric on C^n is

$$\omega_0 = \frac{\sqrt{-1}}{2} \sum_i dz_i \wedge d\bar{z}_i ,$$

and therefore $dd^c\tau = 4\omega_0$. Hence

Corollary 5.2. *Let $f: C^n \to M$ be holomorphic with df nonsingular at a point of C^n. If M is compact kählerian and of dimension n, and κ is its kähler form, then*

$$\liminf_{r\to\infty} \frac{\int_{C^n[r]} f^*\kappa^{n-1} \wedge \omega_0}{T(r)} = 0 \tag{12}$$

implies that $M - f(C^n)$ is of measure zero.

It should be noted that the integrand $f\,\kappa^{n-1} \wedge \omega_0$ has a very simple geometric meaning. Let us fix a point p of C^n and let $f(p) = q \in M$. Then the unit sphere S in the tangent space C_p^n of p is mapped by df onto a (possibly degenerate) hyper-ellipsoid in the tangent space M_q of q. This hyper-ellipsoid has $2n$ principal axes of length $\lambda_1, \lambda_1, \lambda_2, \lambda_2, \cdots, \lambda_n, \lambda_n$. Let S_{n-1} be the $(n-1)$th elementary symmetric function on $\lambda_1, \cdots, \lambda_n$. Then it is a routine exercise to show that

$$(f^*\kappa^{n-1} \wedge \omega_0)(p) = \frac{1}{n} S_{n-1} \cdot \omega_o^n(p) .$$

Next, let us recall that, since $\tau = \sum_i z_i \bar{z}_i$, $C^n[r]$ are the balls of radius $\sqrt{r}$, and $T(r) = \int_0^r v(t)dt$ where $v(t) = \int_{C^n[t]} f^*\Psi$. Since it is a bit awkward to consider balls of radius $\sqrt{r}$ instead of balls of radius r, we rephrase the above corollary slightly. Let therefore D_r denote the ball in C^n of radius r. In conformity with the definitions of $v(t)$ and $T(r)$ above, let

$$v^\sharp(t) = \int_{D_t} f^*\Psi ,$$

$$T^\sharp(r) = \int_0^r v^\sharp(t)\, dt .$$

Then we have the following result weaker than Corollary 5.2:

Corollary 5.3. *Under the same hypothesis as in Corollary 5.2, if we have, instead of* (12),

$$\liminf_{r\to\infty} \frac{\int_{D_r} f^*\kappa^{n-1} \wedge \omega_0}{T^\sharp(r)} = 0 ,$$

then $M - f(C^n)$ is a set of measure zero.

Proof. Since $C^n[t] = D_{\sqrt{t}}$, $v^\sharp(t) = v(t^2)$. Thus $T(r) = \int_0^{\sqrt{r}} 2\, v^\sharp(s)s\, ds$.

Consequently, (12) is equivalent to:

$$\liminf_{r\to\infty} \frac{\int_{D_{\sqrt{r}}} f^*\kappa^{n-1} \wedge \omega_0}{\int_0^{\sqrt{r}} s v^\sharp(s)\, ds} = 0 ,$$

or equivalently,

$$\liminf_{r\to\infty} \frac{\displaystyle\int_{D_r} f^*\kappa^{n-1} \wedge \omega_0}{\displaystyle\int_0^r v^*(s)s\,ds} = 0 . \tag{13}$$

Now obviously, $T^*(r) = \int_0^r v^*(s)ds$ is dominated asymptotically by $\int_0^r v^*(s)s\,ds$. So if the hypothesis of Corollary 5.3 is true, then (13) must hold. Hence by Corollary 5.2 our proof is complete.

Note that the quantity $\int_{D_r} f^*\kappa^{n-1} \wedge \omega_0$ was first introduced by Chern in [1, p. 537], where it was denoted by $v_1(t)$. Now Chern's theorem (ibid.) will be deduced from Corollary 5.2 and the following lemmas.

First calculus lemma. *Suppose h is continuous on $[a, \infty)$ and*

$$\limsup_{r\to\infty} \int_a^r h = +\infty .$$

Then for every real number $n \geq 1$, $\limsup_{r\to\infty} \int_a^r t^n h(t)\,dt = +\infty$.

Proof. By the Tietze extension theorem, we may consider h as a continuous function defined on the real line. Let $l(t) = \int_1^t h$. Then the hypothesis obviously implies $\limsup l(r) = +\infty$. Now, it suffices to prove $\limsup_{r\to\infty} \int_1^r t^n h = +\infty$. Precisely, if $r_0 > 1$, $N > 0$ are given, we will produce a number $u \geq r_0$ such that $\int_1^u t^n h \geq N$. For this purpose, let S be the subset of $[r_0, \infty)$ such that for every $s \in S$, $l(s) \geq \max\{N, \text{maximum of } |l| \text{ in } [1, r_0]\}$. S is nonempty because $\limsup_{r\to\infty} l(r) = +\infty$, and S is of course closed. We let u be the *mnimum* member of S. Then by choice, $l(u) \geq \max\{N, \text{maximum of } |l| \text{ in } [1, u]\}$. We proceed to use integration by parts. Since $l' = h$, we get

$$\int_1^u t^n h = u^n l(u) - \int_1^u l(t) n t^{n-1}\,dt .$$

Now nt^{n-1} is positive on $[1, u]$; so the mean value theorem for integrals may be applied. Hence for some $t_0 \in [1, u]$,

$$\int_1^u t^n h = u^n l(u) - l(t_0) \int_1^u n t^{n-1}\,dt$$

$$
\begin{aligned}
&= u^n l(u) - l(t_0)(u^n - 1) \\
&\geq u^n l(u) - |\, l(t_0) \,| (u^n - 1) \\
&\geq u^n l(u) - l(u)(u^n - 1) \geq l(u) \geq N .
\end{aligned}
$$

Second calculus lemma. *Let f, g be nonnegative functions continuous on $[a, \infty)$ such that* $\liminf_{r\to\infty} \dfrac{\int_a^r f}{\int_a^r g} = 0$. *Then for every real number $n \geq 1$,*

$$\liminf_{r\to\infty} \frac{\int_a^r t^n f}{\int_a^r t^n g} = 0 .$$

Proof. If f is identically zero, there is nothing to prove. So we assume $f > 0$ somewhere, then for the hypothesis to hold, it is necessarily true that $\limsup_{r\to\infty} \int_a^r g = +\infty$. Consequently, $\limsup_{r\to\infty} \int_a^r (\varepsilon g - f) = +\infty$, where ε is some positive constant to be specified. By the first calculus lemma, $\limsup_{r\to\infty} \int_a^r (\varepsilon g - f)\, t^n = +\infty$. In particular, on an infinite sequence $\{r_i\}$, $r_i \to +\infty$, we must have

$$\int_a^{r_i} t^n f < \int_a^{r_i} \varepsilon\, t^n g .$$

Now, suppose the lemma were false, then $\liminf_{r\to\infty} \dfrac{\int_a^r t^n f}{\int_a^r t^n g} = \varepsilon > 0$. Hence in a neighborhood $[u, \infty)$ of ∞,

$$\frac{\int_a^r t^n f}{\int_a^r t^n g} \geq \varepsilon \qquad \text{for all } r \in [u, \infty] .$$

In other words, $\int_a^r t^n f \geq \int_a^r \varepsilon\, t^n g$ for all $r \in [u, \infty)$. This clearly contradicts the above inequality.

Corollary 5.4 (*Chern's theorem*). *Let $f: C^n \to M$ be the same as in Corollary 5.2. Define $v_k(r) = \int_{D_r} f^*\kappa^{n-k} \wedge \omega_0^k$, $0 \leq k \leq n$, and let*

$$T_0(r) = \int_1^r \frac{v_0(t)dt}{t^{2n-1}}, \qquad T_1(r) = \int_1^r \frac{v_1'(t)}{t^{2n}}\, dt\, .$$

Suppose for $r \to \infty$,

$$\text{(a)} \quad T_0(r) \to \infty\, ,$$

$$\text{(b)} \quad \liminf_{r\to\infty} \frac{T_1(r)}{T_0(r)} = 0\, ;$$

then $M - f(C^n)$ is a set of measure zero.

Proof. Obviously all the $v_k(r)$ are nondecreasing functions of r, so that $v_1'(t)$ is a nonnegative function. We may therefore apply the second calculus lemma to (b) to conclude that (13) and hence (12) hold.

6. The author would like to make some comments on Theorem 5.1 and Corollary 5.2 in connection with Chern's Theorem (Corollary 5.4). This theorem of Chern's for $M = P_nC$ has so far been the central result in n-dimensional equidistribution theory around which all later developments of the subject have revolved. But the fact that Theorem 5.1 and Corollary 5.2 generalize this theorem by allowing for more general domain and image manifolds, by itself, hardly justifies their existence. Of course it gives one satisfaction of sorts to have this generality, but the fact remains that we still do not understand the special case of $f: C^n \to P_nC$. So the author feels the main points about Theorem 5.1 and Corollary 5.2 are that they show more transparently how Chern's $v_1(r)$ comes about, that their proof is simple and straight-forward and that one can finally dispense with condition (1) of Corollary 5.4, namely, $T_0(r) \to \infty$. For in order that the latter be valid, $v_0(r)$ roughly speaking must grow as rapidly as r^{2n-2}. This growth restriction is not satisfied for such simple mappings as polynominal mappings of $C^n \to C^n \subseteq P_nC$, as is self-evident. In other words, the gap between necessary and sufficient conditions for $P_nC - f(C^n)$ to be of measure zero is too wide in the presence of condition (a). As a first step towards a better understanding of holomorphic mappings, this gap should be narrowed. It was this desire to remove the hypothesis of $T_0(r) \to \infty$ that had led to the writing of this series of papers.

7. We now prove Lemmas A, B, C. We first dispose of Lemmas A and B. Since $a \in M'$, $f^{-1}(a) \cap V[r]$ is a finite set $\{p_1, \cdots, p_l\}$. Recall that by the very definition of M', $df(p_i)$ is nonsingular for all i. Thus we may choose a neighborhood U of a so small that (i) $f^{-1}(U)$ is a disjoint union $U_1, \cdots, U_l$,

(ii) $p_i \in U_i$ and (iii) f is a diffeomorphism on each U_i. Let normal coordinates $x_1, \cdots, x_{2n}$ be chosen around a as in §3 of [12] so that $x_i(a) = 0$ and $\left\{\frac{\partial}{\partial x_i}(0)\right\}$ is orthonormal. We may as well assume $U = \{\sum_i x_i^2 < \varepsilon^2\}$. Let us fix our attention on U_1, say, and let $y_i = f^*x_i$ be the coordinate functions on U_1. Thus $y_i(p) = 0$, $i = 1, \cdots, 2n$. Now $f^*\lambda_a = f^*g\,(a, \cdot)f^*\Lambda\Psi$, and $f^*\Lambda\Psi$ is C^∞. To prove that the improper integrals in Lemmas A and B converge, we need only focus our attention on the singularity of $f^*g(a, \cdot)$. Let us assume $n > 1$, as the case $n = 1$ is contained in [8]. By (8) in Part I [12], the singularity of $f^*g(a, \cdot)$ around p_1 is

$$\frac{1}{(2n-2)S_{2n}} \frac{1}{(\sum_i y_i^2)^{n-1}} + \eta ,$$

where η is a function such that both $(\sum_i y_i^2)^{\left(\frac{2n-3}{2}\right)} \eta$ and $(\sum_i y_i^2)^{(n-1)}\, d\eta$ are bounded (see (i) of §8 of the present paper). From this, Lemma A and (i) of Lemma B are immediate. For (ii) of Lemma B, let us choose D_t^1 to be the open ball $\{\sum_i y_i^2 < \varepsilon^2\}$; then the desired conclusion is also obvious by employing polar coordinates.

Now to Lemma C. Let the singular locus of df in the closure of $V[r] - V[r_0]$ be $\mathcal{S}$, i.e. $\mathcal{S}$ is the subset on which df is singular. $\mathcal{S}$ is a lower dimensional variety in a compact set and hence is itself compact. So $M - f(\mathcal{S}) - f(\partial V[r_0]) - f(\partial V[r])$ to be called N is open dense. Let $\mathcal{C}$ be the set of critical values of τ in $[r_0, r]$. Since $\mathcal{C}$ is a finite set, the set $Q \equiv \{r_0, r] - \mathcal{C}\} \times N$ is open dense in $[r_0, r] \times M$. It suffices then to prove that $n(t, a)$ is locally integrable in Q. From this point on, the details are similar to those of Lemma 6.1 in [8]. The main point is that for each $a \in N$, $f^{-1}(a) \cap V[r] - V[r_0]$ is a finite set $\{p_1, \cdots, p_l\}$ and f maps a neighborhood U_i of p_i diffeomorphically onto a fixed neighborhood U of a in N. Then $n(t, a)$ becomes locally constant in Q.

8. We now come to the two errors committed in Part I [12]. Both are concerned with the careless handling of the symbol $0(1/r^s)$, and are precisely as follows. (We assume throughout this section that the dimension d of M exceeds 2.)

(i) Recall first that from equation (8), we have, for y near a,

$$g(a, y) = \frac{1}{(d-2)s_d} \frac{1}{r_a^{d-2}(y)} + \eta ,$$

where $r_a(y)$ denotes the geodesic distance from y to a, and η is a function such that (r_n^{d-3}) is bounded. Then in the computation of the explicit expression for $\mu_a(= *dg(a, \cdot))$, it was assumed without comment that $(r_a^{d-2})\, d\eta$ *was bounded.*

This is of course false in general, but true in this special case; now we are going to prove this validity. To facilitate the discussion, let us take over all the notation in §§27–31 of de Rham's book [5]. (Therefore, we change M to V, d to n, and $r_a(y)$ to $r(a, y)$, etc.) From [5, p. 157] we know that the de Rham-Green operator G is given by

$$G = \Omega - (\Omega K - F + H\Omega - H\Omega K + HF) \overset{\text{def}}{=\!=} \Omega - \Omega_1 ,$$

and g is just the degree zero portion of the kernel of this integral operator. What was designated η above is exactly the contribution from the kernel of Ω_1. It should be noted once and for all that all the kernels which ever come up are C^∞ off the diagonal of $V \times V$. In the following, we only worry about the behavior near the diagonal; so all the statements refer only to a neighborhood of the diagonal. Now to be precise, $r_a^{d-4}\eta$ is already bounded. It is then a fact that a closer examination of Ω_1 will give the boundedness of $r_a^{d-3}d\eta$ as desired. The details are straightforward but tedious; so let it suffice to pinpoint the crucial facts. The first fact is that the kernels of F and H are C^∞ on all of $V \times V$, and the kernel of Ω is the product of a C^∞ function with a power of $r(x, y)$. Therefore Lemma 1 in [5, p. 136] is applicable. Next, $K = -H - \Delta F - P + PH + P\Delta F$, and the only wearisome member here is P; we must have control of its kernel. Recall how P is defined on [5, p. 156]; it is the resolvent operator of $(I - Q)\xi = \alpha$, i.e. $\xi = (I - P)\alpha$ should provide a solution. Now the kernel $q(x, y)$ of Q is $0(r^{2-n})$ and is a product of a C^∞ function with a power of $r(x, y)$ and therefore is known as having a weak singularity (Mikhlin, [10]). The standard way to handle $(I - Q)\xi = \alpha$ is to consider instead the *equivalent* equation

$$(I - Q^m)\xi = (I + Q + \cdots + Q^{m-1})\alpha , \tag{14}$$

where m is some positive integer [10, pp. 62–64]. By Lemma 6 on p. 140 and Lemma 4 on p. 138 of [5], we may as well assume m so large that Q^m has a C^1 kernel. Then the classical theory of Fredholm applies, and it tells us that the resolvent operator of (14) has the form: $I - P = (I + P_1)(I + Q + \cdots + Q^{m-1}) + P_2$, where the kernels of P_1 and P_2 are C^1 near the diagonal of $V \times V$, and C^∞ off the diagonal as usual. Therefore the only contribution to the singularity of the kernel $p(x, y)$ of P as well as to the partial derivatives of $p(x, y)$ comes from $(I + Q + \cdots + Q^{m-1})$. But the latter is manageable precisely because Q is a product of a C^∞ function with a power of $r(x, y)$. On the basis of these facts, a repeated application of Lemma 1 (p. 136), Lemma 4 (p. 138) and Lemma 6 (p. 140) of de Rham [5] gives us the desired boundedness of $r_a^{d-3}\, d\eta$.

(ii) A second unproved statement occurs right after equation (12) in [12]. We had

$$\mu_a = \frac{-1}{S_d r_a^d} \Sigma_i (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_d + \xi \,,$$

where ξ is a form such that $r_a^{d-3}\,\xi$ is bounded near a. Then it was stated without proof that $\int_{\partial U_i} f^*\xi \to 0$ as $\varepsilon \to 0$ (ε is the radius of ∂U_i). This is not at all obvious; so a proof is also supplied now.

Observe first that $d\xi$ is an integrable form. One can see this most simply by noting $d\mu_a = d\xi$ and then by using the fact that $d\mu_a = \Omega$ in $M - \{a\}$. Thus $d\xi = \Omega$ in $M - \{a\}$ and the above assertion is obvious. Now it suffices to establish the following.

Suppose W', V' are closed coordinate neighborhoods of two arbitrary manifolds with coordinate functions $\{y_1, \cdots, y_d\}$ and $\{x_1, \cdots, x_d\}$ defined on them respectively. Suppose further that $y_1(p) = \cdots y_d(p) = 0$ and $x_1(a) = \cdots = x_d(a) = 0$. Let $f: W' \to V'$ be a C^∞ map such that $f^{-1}(a) = \{p\}$, and ξ be a form on V', C^∞ in $V' - \{a\}$ such that $r^{d-3}\xi$ is bounded in $V'(r = (\Sigma_i x_i^2)^{1/2})$ and $\int_{V'} d\xi$ is finite. Then $\int_{(\Sigma_i y_i^2) < \varepsilon^2} f^*\xi \to 0$ as $\varepsilon \to 0$.

Proof. Choose $V = \{\Sigma_i x_i^2 < \varepsilon_1^2\}$, and let ε_1 be so small that $V \cap f(W') = \varnothing$. Then for all $b \in V$, the degree of f at b is $n(W', b) =$ constant $(= n(W', a))$ (see, for instance, [11, p, 91]), Let $U = f^{-1}(V)$. We may further assume ε_1 so small that $\bar{V} \subseteq V'$, $\bar{U} \subseteq W'$. Now choose a C^∞ function ρ with support in V so that $\rho \equiv 1$ in $V'' \equiv \left\{\Sigma_i x_i^2 < \left(\frac{\varepsilon_1}{2}\right)^2\right\}$. Let $W_\varepsilon = \{\Sigma_i y_i^2 < \varepsilon^2\}$, where ε is so small that $\bar{W} \subseteq f^{-1}(V'')$. Now,

$$\begin{aligned}\int_U f^*d(\rho\xi) &= \int_V n(U, b) d(\rho\xi) \qquad ([8, \text{Lemma } 6.2]) \\ &= \int_V n(W', b) d(\rho\xi) \qquad (\text{def. of } U, V)\,.\end{aligned}$$

Since $n(W', b)$ is a constant, call it C. Then

$$\int_U f^*d(\rho\xi) = C\int_V d(\rho\xi) = \text{finite}\,. \tag{15}$$

We have used the assumption of $\int_{V'} |\, d\xi \,| < \infty$. Now

$$\int_U f^*d(\rho\xi) = \int_{U - W_\varepsilon} d(f^*\rho\xi) + \int_{W_\varepsilon} f^*d(\rho\xi)$$

$$= \int_{\partial U} f^*\rho\xi - \int_{\partial W_\varepsilon} f^*\rho\xi + \int_{W_\varepsilon} f^*d(\rho\xi)$$

$$= - \int_{\partial W_\varepsilon} f^*\xi + \int_{W_\varepsilon} f^*d(\rho\xi)$$

$$\to - \lim_{\varepsilon\to 0} \int_{\partial W_\varepsilon} f^*\xi$$

as $\varepsilon \to 0$, by (15). On the other hand, if we choose $\delta < \frac{1}{2}\varepsilon_1$ so that $V_\delta = \{\Sigma_i x_i^2 < \delta^2\} \subseteq V''$. Then

$$c\int_V d(\rho\xi) = c\int_{V-V_\delta} d(\rho\xi) + c\int_{V_\delta} d(\rho\xi)$$

$$= c\left(\int_{\partial V}\rho\xi - \int_{\partial V_\delta}\rho\xi + \int_{\partial V_\delta} d(\rho\xi)\right)$$

$$= c\left(-\int_{\partial V_\delta}\xi + \int_{V_\delta} d(\rho\xi)\right)$$

$$\to 0 \qquad \text{as } \delta \to 0\,,$$

because of (15) and the fact that $r^{d-3}\xi$ is bounded. Combination of these with (15) hence completes the proof.

Note. After the completion of this paper, Professor Wilhelm Stoll has kindly called my attention to the following two points:

(i) Theorem 5.1 has been proved for $M = P_nC$ in his paper [6, Theorem 5.1.3].

(ii) Dr. Hirschfelder in his thesis (University of Notre Dame, 1968) has considered a more general situation than what is done in this series of papers. His results seem to overlap considerably with those given here, while his method is different from ours.

References

(*Continuation of Part* I [12])

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