

## RIEMANNIAN MANIFOLDS ADMITTING A CONFORMAL TRANSFORMATION GROUP

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### 1. Introduction

The purpose of the present paper is to generalize some of the known results on Riemannian manifolds with constant scalar curvature admitting a group of nonisometric conformal transformations.

Let  $M$  be a connected Riemannian manifold of dimension  $n$ , and  $g_{ji}$ ,  $\nabla_i$ ,  $K_{kji}{}^h$ ,  $K_{jt} = K_{tji}{}^t$  and  $K = K_{jt}g^{jt}$ , respectively, the positive definite fundamental metric tensor, the operator of covariant differentiation with respect to the Levi-Civita connection, the curvature tensor, the Ricci tensor and the scalar curvature of  $M$ , where and in the sequel the indices  $h, i, j, k, \dots$  run over the range  $1, \dots, n$ .

If we put

$$(1.1) \quad G_{ji} = K_{ji} - \frac{K}{n}g_{ji},$$

$$(1.2) \quad Z_{kji}{}^h = K_{kji}{}^h - \frac{K}{n(n-1)}(\delta_k^h g_{ji} - \delta_j^h g_{ki}),$$

we have

$$(1.3) \quad Z_{tji}{}^t = G_{ji}, \quad G_{ji}g^{ji} = 0.$$

When  $M$  admits an infinitesimal transformation  $v^h$ , we denote by  $\mathcal{L}$  the operator of Lie derivation with respect to  $v^h$ . Thus, if  $M$  admits an infinitesimal conformal transformation  $v^h$ , we have

$$(1.4) \quad \mathcal{L}g_{ji} = \nabla_j v_i + \nabla_i v_j = 2\rho g_{ji}, \quad \mathcal{L}g^{ih} = -2\rho g^{ih}$$

for a certain scalar field  $\rho$ . We denote the gradient of  $\rho$  by  $\rho_i = \nabla_i \rho$ .

For an infinitesimal conformal transformation  $v^h$  in  $M$ , we have [5]

$$(1.5) \quad \mathcal{L}K_{kji}{}^h = -\delta_k^h \nabla_j \rho_i + \delta_j^h \nabla_k \rho_i - \nabla_k \rho^h g_{ji} + \nabla_j \rho^h g_{ki},$$

$$(1.6) \quad \mathcal{L}K_{jt} = -(n-2)\nabla_j \rho_t - \Delta \rho g_{jt},$$

$$(1.7) \quad \mathcal{L}K = -2(n-1)\Delta\rho - 2\rho K,$$

where

$$(1.8) \quad \Delta\rho = g^{ij}\nabla_j\nabla_i\rho.$$

Thus, in  $M$  with  $K = \text{const.}$  we have

$$(1.9) \quad \Delta\rho = -\frac{K}{n-1}\rho.$$

We also have

$$(1.10) \quad \mathcal{L}G_{ji} = -(n-2)\left(\nabla_j\rho_i - \frac{1}{n}\Delta\rho g_{ji}\right),$$

$$(1.11) \quad \begin{aligned} \mathcal{L}Z_{kji}{}^h &= -\delta_k^h\nabla_j\rho_i + \delta_j^h\nabla_k\rho_i - \nabla_k\rho^h g_{ji} + \nabla_j\rho^h g_{ki} \\ &+ \frac{2}{n}\Delta\rho(\delta_k^h g_{ji} - \delta_j^h g_{ki}). \end{aligned}$$

Thus, in  $M$  with  $K = \text{const.}$  we have

$$(1.12) \quad \mathcal{L}G_{ji} = -(n-2)\left[\nabla_j\rho_i + \frac{K}{n(n-1)}\rho g_{ji}\right].$$

We denote by  $C_0(M)$  the largest connected group of conformal transformations of  $M$  and by  $I_0(M)$  that of isometries of  $M$ .

We first state here known results on Riemannian manifolds with  $K = \text{const.}$  admitting a conformal transformation group, and then try to generalize them.

**Theorem A** (Lichnerowicz [3]). *If  $M$  is a compact Riemannian manifold of dimension  $n > 2$ ,  $K = \text{const.}$ ,  $K_{ji}K^{ji} = \text{const.}$ , and  $C_0(M) \neq I_0(M)$ , then  $M$  is isometric to a sphere.*

**Theorem B** (Lichnerowicz [3], Yano & Obata [7]). *If a compact Riemannian manifold  $M$  of dimension  $n \geq 2$  with  $K = \text{const.}$  admits an infinitesimal nonisometric conformal transformation  $v^h: \mathcal{L}g_{ji} = 2\rho g_{ji}$ ,  $\rho \neq \text{const.}$ , and if one of the following conditions is satisfied, then  $M$  is isometric to a sphere:*

- (1) *The vector field  $v^h$  is a gradient of a scalar.*
- (2)  *$K_i{}^h\rho^i = k\rho^h$ ,  $k$  being a constant.*
- (3)  *$\mathcal{L}K_{ji} = \alpha g_{ji}$ ,  $\alpha$  being a scalar field.*

**Theorem C** (Hsiung [1]). *If  $M$  is compact and of dimension  $n > 2$ ,  $K = \text{const.}$ ,  $K_{kji}{}^h K^{kji}{}^h = \text{const.}$ , and  $C_0(M) \neq I_0(M)$ , then  $M$  is isometric to a sphere.*

**Theorem D** (Yano [6]). *If  $M$  is compact orientable and of dimension  $n > 2$  with  $K = \text{const.}$ , and admits an infinitesimal nonisometric conformal*

transformation  $v^h: \mathcal{L}g_{ji} = 2\rho g_{ji}$ ,  $\rho \neq \text{const.}$ , such that  $\int_M G_{ji} \rho^j \rho^i dV$  is non-negative,  $dV$  being the volume element of  $M$ , then  $M$  is isometric to a sphere.

**Theorem E** (Yano [6]). *If  $M$  is a compact and of dimension  $n > 2$  with  $K = \text{const.}$ , and admits an infinitesimal nonisometric conformal transformation  $v^h: \mathcal{L}g_{ji} = 2\rho g_{ji}$ ,  $\rho \neq \text{const.}$ , such that  $\mathcal{L}(G_{ji}G^{ji}) = \text{const.}$  or  $\mathcal{L}(Z_{kjih}Z^{kjih}) = \text{const.}$ , then  $M$  is isometric to a sphere.*

**Theorem F** (Hsiung [2]). *Suppose that a compact Riemannian manifold  $M$  of dimension  $n > 2$  with  $K = \text{const.}$  admits an infinitesimal nonhomothetic conformal transformation  $v^h$ . If*

$$(1.13) \quad a^2 \mathcal{L}(Z_{kjih}Z^{kjih}) + (2a + nb)b \mathcal{L}(G_{ji}G^{ji}) = \text{const.} ,$$

where  $a$  and  $b$  are constants such that

$$(1.14) \quad c \equiv 4a^2 + 2(n - 2)ab + n(n - 2)b^2 > 0 ,$$

then  $M$  is isometric to a sphere.

To prove and generalize these theorems, we need the following

**Theorem G** (Obata [4]). *If a complete Riemannian manifold of dimension  $n \geq 2$  admits a nonconstant function  $\rho$  such that*

$$(1.15) \quad \nabla_j \nabla_i \rho = -c^2 \rho g_{ji} ,$$

where  $c$  is a positive constant, then  $M$  is isometric to a sphere of radius  $1/c$  in  $(n + 1)$ -dimensional Euclidean space.

We also need following integral formulas proved in [6].

If a compact orientable Riemannian manifold  $M$  of dimension  $n > 2$  with  $K = \text{const.}$  admits an infinitesimal nonhomothetic conformal transformation  $v^h: \mathcal{L}g_{ji} = 2\rho g_{ji}$ ,  $\rho \neq \text{const.}$ , then we have

$$(1.16) \quad \int_M G_{ji} \rho^j \rho^i dV = \frac{1}{n - 2} \int_M \left[ 2\rho^2 G_{ji} G^{ji} + \frac{1}{2} \rho \mathcal{L}(G_{ji} G^{ji}) \right] dV ,$$

$$(1.17) \quad \int_M G_{ji} \rho^j \rho^i dV = \int_M \left[ \frac{1}{2} \rho^2 Z_{kjih} Z^{kjih} + \frac{1}{8} \rho \mathcal{L}(Z_{kjih} Z^{kjih}) \right] dV .$$

### 2. Generalization of Theorem B, (2), (3)

**Theorem 2.1.** *If a compact orientable  $M$  of dimension  $n > 2$  with  $K = \text{const.}$  admits an infinitesimal nonhomothetic conformal transformation  $v^h: \mathcal{L}g_{ji} = 2\rho g_{ji}$ ,  $\rho \neq \text{const.}$ , such that*

$$(2.1) \quad \mathcal{L}(G^{ji} \mathcal{L}G_{ji}) \leq 0,$$

then  $M$  is isometric to a sphere.

We need the following

**Lemma 2.1.** *If a compact orientable  $M$  admits an infinitesimal conformal transformation  $v^h: \mathcal{L}g_{ji} = 2\rho g_{ji}$ , then we have*

$$(2.2) \quad \int_M \rho F dV = -\frac{1}{n} \int_M \mathcal{L}F dV$$

for any function  $F$ .

*Proof.* Since  $\rho = \frac{1}{n} \nabla_s v^s$ , we have, by Green's theorem,

$$\begin{aligned} \int_M \rho F dV &= \frac{1}{n} \int_M (\nabla_s v^s) F dV \\ &= -\frac{1}{n} \int_M v^s \nabla_s F dV \\ &= -\frac{1}{n} \int_M \mathcal{L}F dV. \end{aligned}$$

*Proof of the Theorem.* Substituting

$$(2.3) \quad \mathcal{L}(G_{ji} G^{ji}) = 2G^{ji} \mathcal{L}G_{ji} - 4\rho G^{ji} G_{ji}$$

into integral formula (1.16), we find

$$(2.4) \quad \int_M G_{ji} \rho^j \rho^i dV = \frac{1}{n-2} \int_M \rho G^{ji} \mathcal{L}G_{ji} dV.$$

Consequently, by Lemma 2.1 and the assumption of the theorem, we have

$$\int_M G_{ji} \rho^j \rho^i dV = -\frac{1}{n(n-2)} \int_M \mathcal{L}(G^{ji} \mathcal{L}G_{ji}) dV \geq 0.$$

Thus  $M$  is isometric to a sphere by Theorem D.

**Remark 2.1.** Since

$$(2.5) \quad Z^{kji}{}_h \mathcal{L}Z_{kji}{}^h = \frac{4}{n-2} G^{ji} \mathcal{L}G_{ji},$$

the condition (2.1) of the theorem can be replaced by

$$(2.6) \quad \mathcal{L}(Z^{kji}{}_h \mathcal{L}Z_{kji}{}^h) \leq 0.$$

**Remark 2.2.** As the proof of the theorem shows, condition (2.1) can be replaced by

$$(2.7) \quad \mathcal{L}(G^{ji} \mathcal{L}G_{ji}) = \lambda, \quad \int_M \lambda dV \leq 0.$$

The same remark applies to Theorems 4.1, 4.3, 5.1, 6.1, 6.2 and 6.4.

**Remark 2.3.** Theorem 2.1 generalizes Theorem B, (2). In fact, using  $K_i^h \rho^i = k \rho^h$ ,  $\nabla_j K^{ji} = 0$ ,  $\nabla_j v_i + \nabla_i v_j = 2\rho g_{ji}$  and  $\nabla_i v^i = n\rho$ , we have

$$\begin{aligned} \nabla_j(K^{ji} \rho v_i) &= K^{ji} \rho_j v_i + K^{ji} \rho \nabla_j v_i \\ &= k \rho_i v^i + \frac{1}{2} K^{ji} \rho (\nabla_j v_i + \nabla_i v_j) \\ &= k \nabla_i(\rho v^i) - k \rho \nabla_i v^i + K \rho^2 \\ &= k \nabla_i(\rho v^i) - nk \rho^2 + K \rho^2, \end{aligned}$$

from which, by integration,

$$\int_M (K - nk) \rho^2 dV = 0,$$

and consequently  $k = K/n$ .

Thus, from  $K_i^h \rho^i = k \rho^h$  we have

$$\begin{aligned} \left(K^{ji} - \frac{K}{n} g^{ji}\right) \rho_i &= 0, \\ \left(K^{ji} - \frac{K}{n} g^{ji}\right) \nabla_j \rho_i &= 0, \end{aligned}$$

and consequently, by virtue of (1.10),

$$G^{ji}(\mathcal{L}G_{ji}) = 0.$$

**Remark 2.4.** Theorem 2.1 generalizes Theorem B, (3). In fact, from (1.6) and  $\mathcal{L}K_{ji} = \alpha g_{ji}$  we find

$$-(n - 2) \nabla_j \rho_i - \Delta \rho g_{ji} = \alpha g_{ji},$$

from which

$$\alpha = -2(n - 1) \Delta \rho / n,$$

and consequently

$$-(n - 2) \left( \nabla_j \rho_i - \frac{1}{n} \Delta \rho g_{ji} \right) = 0,$$

that is,  $\mathcal{L}G_{ji} = 0$ .

**Remark 2.5.** If  $G^{ji}\mathcal{L}G_{ji} = \text{const.}$ , then (2.1) is automatically satisfied, but under our assumption the constant must be zero. In fact, making use of (1.3) and  $\nabla_j G^{ji} = 0$ , from (1.10) we have

$$\begin{aligned} G^{ji}\mathcal{L}G_{ji} &= -(n-2)G^{ij}\nabla_j\rho_i \\ &= -(n-2)\nabla_j(G^{ji}\rho_i), \end{aligned}$$

and consequently by integration over  $M$  we find

$$\int_M G^{ji}\mathcal{L}G_{ji}dV = 0.$$

Thus, if  $G^{ji}\mathcal{L}G_{ji} = \text{const.}$  the constant must be zero.

### 3. Decomposition of a conformal Killing vector

**Theorem 3.1.** *If a compact orientable  $M$  of dimension  $n \geq 2$  with  $K = \text{const.}$  admits a conformal Killing vector field*

$$(3.1) \quad v^h = p^h + q^h,$$

where  $p^h$  is a Killing vector field and  $q^h = \nabla^h q$ ,  $q \neq \text{const.}$  is a gradient conformal Killing vector field, then  $M$  is isometric to a sphere.

Conversely, if a sphere of dimension  $n \geq 2$  admits a conformal Killing vector field  $v^h$ , then  $v^h$  is decomposed into the form (3.1) where  $p^h$  is a Killing vector field and  $q^h$  a gradient conformal Killing vector field.

*Proof.* Suppose that a compact orientable  $M$  with  $K = \text{const.}$  admits a conformal Killing vector  $v^h$ . Then we have

$$(3.2) \quad \mathcal{L}g_{ji} = \nabla_j v_i + \nabla_i v_j = 2\rho g_{ji},$$

and

$$(3.3) \quad \Delta\rho = -\frac{K}{n-1}\rho.$$

We note here that  $K$  is a positive constant [6]. If  $v^h$  is the sum of a Killing vector  $p^h$  and a gradient conformal Killing vector  $q^h = \nabla^h q$ , substituting (3.1) into (3.2), we find

$$(3.4) \quad \nabla_j \nabla_i q = \rho g_{ji},$$

from which

$$(3.5) \quad \Delta q = n\rho.$$

From (3.3) and (3.5), we find

$$\Delta\left(\rho + \frac{K}{n(n-1)}q\right) = 0,$$

from which, by Bochner's lemma,

$$(3.6) \quad \rho + \frac{K}{n(n-1)}q = \text{constant}.$$

Substituting (3.6) into (3.4), we find

$$\nabla_j \nabla_i (q + c) = -\frac{K}{n(n-1)}(q + c)g_{ji},$$

where  $c$  is a constant. Thus,  $q$  being not a constant,  $M$  is isometric to a sphere.

Conversely, suppose that  $M$ , isometric to a sphere, admits a conformal Killing vector  $v^h$ . It is known that  $v^h$  can be decomposed into

$$v^h = p^h + q^h,$$

where

$$(3.7) \quad \nabla_i p^i = 0, \quad q^h = \nabla^h q.$$

From

$$\mathcal{L}g_{ji} = \nabla_j v_i + \nabla_i v_j = 2\rho g_{ji},$$

we have

$$(3.8) \quad T_{ji} = \nabla_j p_i + \nabla_i p_j + 2\nabla_j \nabla_i q - 2\rho g_{ji} = 0.$$

Forming  $T_{ji}T^{ji}$ , we find

$$(3.9) \quad \begin{aligned} T_{ji}T^{ji} &= (\nabla_j p_i + \nabla_i p_j)(\nabla^j p^i + \nabla^i p^j) \\ &+ 4\left(\nabla_j \nabla_i q - \frac{1}{n}\Delta q g_{ji}\right)\left(\nabla^j \nabla^i q - \frac{1}{n}\Delta q g^{ji}\right) \\ &+ 8(\nabla^j \nabla^i q)(\nabla_j p_i) = 0. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \int_M (\nabla^j \nabla^i q)(\nabla_j p_i) dV &= \int_M (\nabla^i \nabla^j q)(\nabla_j p_i) dV \\ &= - \int_M (\nabla^j q)(\nabla^i \nabla_j p_i) dV \\ &= - \int_M K_{ji}(\nabla^j q)p^i dV \end{aligned}$$

because of

$$\nabla_i \nabla_j p^i - \nabla_j \nabla_i p^i = K_{ijt} p^t,$$

or

$$\nabla^i \nabla_j p_i = K_{jt} p^t.$$

Taking account of  $K_{ji} = \frac{K}{n} g_{ji}$  we then have

$$\begin{aligned} \int_M (\nabla^j \nabla^i q) (\nabla_j p_i) dV &= -\frac{K}{n} \int_M (\nabla_i q) p^i dV \\ &= \frac{K}{n} \int_M q (\nabla_i p^i) dV \\ &= 0. \end{aligned}$$

Thus from (3.9), by integration we find

$$\begin{aligned} \int_M \left[ (\nabla_j p_i + \nabla_i p_j) (\nabla^j p^i + \nabla^i p^j) \right. \\ \left. + 4 \left( \nabla_j \nabla_i q - \frac{1}{n} \Delta q g_{ji} \right) \left( \nabla^j \nabla^i q - \frac{1}{n} \Delta q g^{ji} \right) \right] dV = 0, \end{aligned}$$

from which

$$(3.10) \quad \nabla_j p_i + \nabla_i p_j = 0,$$

$$(3.11) \quad \nabla_j \nabla_i q = \frac{1}{n} \Delta q g_{ji},$$

showing that  $p^h$  is a Killing vector field and  $q^h$  a gradient conformal Killing vector field.

**Remark 3.1.** Theorem 3.1 generalizes Theorem B, (1).

**Remark 3.2.** We can see in the following way the fact that a sphere admits a gradient conformal Killing vector field. Let

$$(3.12) \quad X^A = X^A(x^h), \quad \sum X^A X^A = r^2$$

be the equations of  $n$ -dimensional sphere of radius  $r$  in an  $(n+1)$ -dimensional Euclidean space, where  $A = 1, \dots, n+1$ .

The equations of Gauss and those of Weingarten of the sphere are, respectively,

$$(3.13) \quad \nabla_j B_i^A = \frac{1}{r} g_{ji} N^A,$$

and



$$(3.14) \quad \nabla_j N^A = -\frac{1}{r} B_j^A,$$

where  $B_i^A = \nabla_i X^A$  and  $N^A$  are components of the unit normal to the sphere.

Considering a parallel vector field  $B_i^A u^i + \alpha N^A$  in the Euclidean space along the sphere, we have

$$\nabla_j (B_i^A u^i + \alpha N^A) = 0,$$

from which

$$\frac{1}{r} u_j N^A + B_i^A \nabla_j u^i + (\nabla_j \alpha) N^A - \frac{\alpha}{r} B_j^A = 0,$$

and consequently

$$\nabla_j u^i = \frac{\alpha}{r} \delta_j^i, \quad \Delta_j \alpha = -\frac{1}{r} u_j,$$

thus giving

$$\nabla_j \nabla_i \alpha = -\frac{1}{r} \nabla_j u_i,$$

that is,

$$\nabla_j \nabla_i \alpha = -\frac{1}{r^2} \alpha g_{ji}.$$

#### 4. Generalizations of Theorem E

We introduce here the notations :

$$(4.1) \quad f = G_{ji} G^{ji}, \quad g = Z_{kjih} Z^{kjih}.$$

**Theorem 4.1.** *If a compact orientable  $M$  of dimension  $n > 2$  with  $K = \text{const.}$  admits an infinitesimal nonhomothetic conformal transformation  $v^h$  such that*

$$(4.2) \quad \mathcal{L} \left\{ \sum_{k=0}^l \alpha_k \left( -\frac{n-1}{K} \right)^k \Delta^k(\mathcal{L}f) + \sum_{k=0}^m \beta_k \left( -\frac{n-1}{K} \right)^k \Delta^k(\mathcal{L}g) \right\} \leq 0,$$

*$l$  and  $m$  being nonnegative integers, and  $\alpha_k$  and  $\beta_k$  constants such that the sums  $\sum_{k=0}^l \alpha_k$  and  $\sum_{k=0}^m \beta_k$  are nonnegative and not both zero, then  $M$  is isometric to a sphere.*

We need the following

**Lemma 4.1.** *If a compact orientable  $M$  with  $K = \text{const.}$  admits an infinitesimal conformal transformation  $v^h : \mathcal{L}g_{ji} = 2\rho g_{ji}$ , then we have*

$$\begin{aligned}
 \int_M \rho F dV &= \int_M \left(-\frac{n-1}{K}\right) \rho \Delta F dV \\
 (4.3) \qquad &= \int_M \left(-\frac{n-1}{K}\right)^2 \rho \Delta^2 F dV \\
 &\qquad \dots\dots\dots \\
 &= \int_M \left(-\frac{n-1}{K}\right)^l \rho \Delta^l F dV
 \end{aligned}$$

for any function  $F$  and any nonnegative integer  $l$ .

*Proof.* Remembering

$$\Delta \rho = -\frac{K}{n-1} \rho \quad (K > 0),$$

that is,

$$\rho = -\frac{n-1}{K} \Delta \rho,$$

we have, for any scalar field  $F$ ,

$$\int_M \rho F dV = \int_M \left(-\frac{n-1}{K}\right) (\Delta \rho) F dV,$$

that is,

$$\int_M \rho F dV = \int_M \left(-\frac{n-1}{K}\right) \rho \Delta F dV.$$

Repeating the same process, we hence obtain (4.3).

*Proof of the theorem.* We have, from (1.16) and Lemma 4.1,

$$\begin{aligned}
 \frac{n-2}{2} \int_M G_{ji} \rho^j \rho^i dV &= \int_M \rho^2 f dV + \frac{1}{4} \int_M \rho \mathcal{L} f dV \\
 &= \int_M \rho^2 f dV + \frac{1}{4} \int_M \left(-\frac{n-1}{K}\right) \rho \Delta \mathcal{L} f dV \\
 &\qquad \dots\dots\dots \\
 &= \int_M \rho^2 f dV + \frac{1}{4} \int_M \left(-\frac{n-1}{K}\right)^l \rho \Delta^l \mathcal{L} f dV.
 \end{aligned}$$

We also have, from (1.17) and Lemma 4.1,

$$\begin{aligned} 2 \int_M G_{ji} \rho^j \rho^i dV &= \int_M \rho^2 g dV + \frac{1}{4} \int_M \rho \mathcal{L} g dV \\ &= \int_M \rho^2 g dV + \frac{1}{4} \int_M \left( -\frac{n-1}{K} \right) \rho \Delta \mathcal{L} g dV \\ &\quad \dots\dots\dots \\ &= \int_M \rho^2 g dV + \frac{1}{4} \int_M \left( -\frac{n-1}{K} \right)^m \rho \Delta^m \mathcal{L} g dV. \end{aligned}$$

From these equations, we have

$$\begin{aligned} &\left\{ \frac{n-2}{2} (\alpha_0 + \alpha_1 + \dots + \alpha_l) + 2(\beta_0 + \beta_1 + \dots + \beta_m) \right\} \int_M G_{ji} \rho^j \rho^i dV \\ &= \int_M \rho^2 \{ (\alpha_0 + \alpha_1 + \dots + \alpha_l) f + (\beta_0 + \beta_1 + \dots + \beta_m) g \} dV \\ &\quad + \frac{1}{4} \int_M \rho \left\{ \alpha_0 \mathcal{L} f + \alpha_1 \left( -\frac{n-1}{K} \right) \Delta \mathcal{L} f + \dots + \alpha_l \left( -\frac{n-1}{K} \right)^l \Delta^l \mathcal{L} f \right. \\ &\quad \quad \left. + \beta_0 \mathcal{L} g + \beta_1 \left( -\frac{n-1}{K} \right) \Delta \mathcal{L} g + \dots \right. \\ &\quad \quad \left. + \beta_m \left( -\frac{n-1}{K} \right)^m \Delta^m \mathcal{L} g \right\} dV, \end{aligned}$$

and consequently, by Lemma 2.1,

$$\begin{aligned} &\left\{ \frac{n-2}{2} (\alpha_0 + \alpha_1 + \dots + \alpha_l) + 2(\beta_0 + \beta_1 + \dots + \beta_m) \right\} \int_M G_{ji} \rho^j \rho^i dV \\ &= \int_M \rho^2 \{ (\alpha_0 + \alpha_1 + \dots + \alpha_l) f + (\beta_0 + \beta_1 + \dots + \beta_m) g \} dV \\ &\quad - \frac{1}{4n} \int_M \mathcal{L} \left\{ \alpha_0 \mathcal{L} f + \alpha_1 \left( -\frac{n-1}{K} \right) \Delta \mathcal{L} f + \dots \right. \\ &\quad \quad \left. + \alpha_l \left( -\frac{n-1}{K} \right)^l \Delta^l \mathcal{L} f + \beta_0 \mathcal{L} g \right. \\ &\quad \quad \left. + \beta_1 \left( -\frac{n-1}{K} \right) \Delta \mathcal{L} g + \dots \right. \\ &\quad \quad \left. + \beta_m \left( -\frac{n-1}{K} \right)^m \Delta^m \mathcal{L} g \right\} dV. \end{aligned}$$

Thus, if the conditions of the theorem are satisfied, we have

$$\int_M G_{ji} \rho^j \rho^i dV \geq 0,$$

and consequently, by Theorem D,  $M$  is isometric to a sphere.

**Theorem 4.2.** *Suppose that a compact orientable  $M$  of dimension  $n > 2$  with  $K = \text{const.}$  satisfies*

$$(4.4) \quad \alpha_0 f - \alpha_1 \Delta f + \beta_0 g - \beta_1 \Delta g = \text{const.},$$

where  $\alpha_0, \alpha_1, \beta_0, \beta_1$  are nonnegative constants not all zero such that, if  $n > 6$ ,

$$(4.5) \quad \frac{8K}{n-1} \alpha_1 \geq (n-6)\alpha_0 \geq 0, \quad \frac{8K}{n-1} \beta_1 \geq (n-6)\beta_0 \geq 0.$$

If  $M$  admits an infinitesimal nonhomothetic conformal transformation  $v^h$ :  $\mathcal{L}g_{ji} = 2\rho g_{ji}$ ,  $\rho \neq \text{constant}$ , then  $M$  is isometric to a sphere.

To prove the theorem, we need the following

**Lemma 4.2.** *For a conformal Killing vector  $v^h$  in  $M$ , that is, for a vector field  $v^h$  satisfying*

$$\mathcal{L}g_{ji} = \nabla_j v_i + \nabla_i v_j = 2\rho g_{ji},$$

we have

$$(4.6) \quad \Delta(\mathcal{L}F) = \mathcal{L}(\Delta F) + 2\rho \Delta F - (n-2)\rho^i \nabla_i F$$

for any scalar field  $F$ .

*Proof.* Since  $v^h$  is a conformal Killing vector field, we have

$$(4.7) \quad g^{ji} \nabla_j \nabla_i v^h + K_i^h v^i + (n-2)\rho^h = 0,$$

(see [6] for example). We also have, for an arbitrary scalar field  $F$ ,

$$(4.8) \quad g^{ji} \nabla_j \nabla_i \nabla_h F - K_h^i \nabla_i F = \nabla_h(\Delta F).$$

Thus we have

$$\begin{aligned} \Delta(\mathcal{L}F) &= g^{ji} \nabla_j \nabla_i (v^h \nabla_h F) \\ &= (g^{ji} \nabla_j \nabla_i v^h) \nabla_h F + (\nabla^j v^i + \nabla^i v^j) \nabla_j \nabla_i F \\ &\quad + v^h g^{ji} \nabla_j \nabla_i \nabla_h F, \end{aligned}$$

and consequently, by using (4.7) and (4.8),

$$\begin{aligned} \Delta(\mathcal{L}F) &= -K_{ji} v^i \nabla^j F - (n-2)\rho^h \nabla_h F \\ &\quad + 2\rho \Delta F + K_{ji} v^j \nabla^i F + v^h \nabla_h(\Delta F), \end{aligned}$$

that is,

$$\Delta(\mathcal{L}F) = \mathcal{L}\Delta F + 2\rho\Delta F - (n - 2)\rho^h\nabla_h F .$$

**Lemma 4.3.** For any scalar field  $F$  and a scalar field  $\rho$  satisfying  $\Delta\rho = k\rho$ ,  $k$  being a constant, in a compact orientable  $M$  we have

$$(4.9) \quad \int_M \rho\rho^h\nabla_h F dV = -\frac{1}{2} \int_M \rho^2(\Delta F) dV ,$$

$$(4.10) \quad \int_M \rho^2(\Delta F) dV = 2k \int_M \rho^2 F dV + 2 \int_M \rho_i\rho^i F dV .$$

*Proof.* Integral formula (4.9) follows from

$$\nabla^h(\rho^2\nabla_h F) = 2\rho\rho^h\nabla_h F + \rho^2\Delta F$$

by integration. On the other hand, we have

$$\begin{aligned} \int_M \rho^2(\Delta F) dV &= \int_M (\Delta\rho^2) F dV \\ &= 2 \int_M (\rho\Delta\rho + \rho_i\rho^i) F dV \\ &= 2k \int_M \rho^2 F dV + 2 \int_M \rho_i\rho^i F dV , \end{aligned}$$

which proves (4.10).

*Proof of the theorem.* From (1.16), (4.3), (4.6) and (4.9), we find

$$\begin{aligned} &\frac{n-2}{2} \int_M G_{ji}\rho^j\rho^i dV \\ &= \int_M \rho^2 f dV + \frac{1}{4} \int_M \rho \mathcal{L}f dV \\ &= \int_M \rho^2 f dV + \frac{1}{4} \int_M \left(-\frac{n-1}{K}\right) \rho \Delta \mathcal{L}f dV \\ &= \int_M \rho^2 f dV + \frac{1}{4} \int_M \left(-\frac{n-1}{K}\right) \rho [\mathcal{L}\Delta f + 2\rho\Delta f - (n-2)\rho^i\nabla_i f] dV \\ &= \int_M \rho^2 f dV + \frac{1}{4} \int_M \rho \mathcal{L} \left(-\frac{n-1}{K} \Delta f\right) dV \\ &\quad + \frac{1}{4} \int_M \left[-\frac{2(n-1)}{K} \rho^2 \Delta f - \frac{(n-1)(n-2)}{2K} \rho^2 \Delta f\right] dV \\ &= \int_M \rho^2 f dV + \frac{1}{4} \int_M \rho \mathcal{L} \left(-\frac{n-1}{K} \Delta f\right) dV - \frac{(n-1)(n+2)}{8K} \int_M \rho^2 \Delta f dV . \end{aligned}$$

Thus, we have

$$\begin{aligned} \frac{n-2}{2} \int_M G_{ji} \rho^j \rho^i dV &= \int_M \rho^2 f dV + \frac{1}{4} \int_M \rho \mathcal{L} f dV, \\ \frac{n-2}{2} \int_M G_{ji} \rho^j \rho^i dV &= \int_M \rho^2 f dV + \frac{1}{4} \int_M \rho \mathcal{L} \left( -\frac{n-1}{K} \Delta f \right) dV \\ &\quad - \frac{(n-1)(n+2)}{8K} \int_M \rho^2 \Delta f dV. \end{aligned}$$

Similarly, we find

$$\begin{aligned} 2 \int_M G_{ji} \rho^j \rho^i dV &= \int_M \rho^2 g dV + \frac{1}{4} \int_M \rho \mathcal{L} g dV, \\ 2 \int_M G_{ji} \rho^j \rho^i dV &= \int_M \rho^2 g dV + \frac{1}{4} \int_M \rho \mathcal{L} \left( -\frac{n-1}{K} \Delta g \right) dV \\ &\quad - \frac{(n-1)(n+2)}{8K} \int_M \rho^2 \Delta g dV. \end{aligned}$$

From the above four equations, we obtain

$$\begin{aligned} &\left\{ \frac{n-2}{2} (a+a') + 2(b+b') \right\} \int_M G_{ji} \rho^j \rho^i dV \\ &= \int_M \rho^2 [(a+a')f + (b+b')g] dV \\ (4.11) \quad &+ \frac{1}{4} \int_M \rho \left[ \mathcal{L} \left( af - \frac{n-1}{K} a' \Delta f + bg - \frac{n-1}{K} b' \Delta g \right) \right] dV \\ &- \frac{(n-1)(n+2)}{8K} \int_M \rho^2 (a' \Delta f + b' \Delta g) dV, \end{aligned}$$

$a, a', b, b'$  being nonnegative constants. Now we choose  $a, a', b, b'$  in such a way that we have

$$(4.12) \quad \alpha_0 = a, \quad \alpha_1 = \frac{n-1}{K} a', \quad \beta_0 = b, \quad \beta_1 = \frac{n-1}{K} b'.$$

Then we have, from (4.4),

$$(4.13) \quad af - \frac{n-1}{K} a' \Delta f + bg - \frac{n-1}{K} b' \Delta g = c \text{ (const.)}$$

and

$$a' \Delta f + b' \Delta g = \frac{K}{n-1} (af + bg) - \frac{Kc}{n-1},$$

and consequently, from (4.11),

$$\begin{aligned} & \left\{ \frac{n-2}{2} (a + a') + 2(b + b') \right\} \int_M G_{ji} \rho^j \rho^i dV \\ &= \int_M \rho^2 \left[ (a + a')f + (b + b')g \right. \\ & \quad \left. - \frac{(n-1)(n+2)}{8K} \left\{ \frac{K}{n-1} (af + bg) - \frac{Kc}{n-1} \right\} \right] dV, \end{aligned}$$

that is,

$$\begin{aligned} (4.14) \quad & \left\{ \frac{n-2}{2} (a + a') + 2(b + b') \right\} \int_M G_{ji} \rho^j \rho^i dV \\ &= \frac{1}{8} \int_M \rho^2 \left[ \{8a' - (n-6)a\}f + \{8b' - (n-6)b\}g + (n+2)c \right] dV. \end{aligned}$$

Now, constants

$$8a' - (n-6)a, \quad 8b' - (n-6)b$$

are both nonnegative for  $n \leq 6$ . Since

$$\begin{aligned} 8a' - (n-6)a &= \frac{8K}{n-1} \alpha_1 - (n-6)\alpha_0, \\ 8b' - (n-6)b &= \frac{8K}{n-1} \beta_1 - (n-6)\beta_0, \end{aligned}$$

they are nonnegative also for  $n \geq 6$  by virtue of the assumption.

Moreover, we have, from (4.13),

$$a \int_M f dV + b \int_M g dV = c \int_M dV,$$

and consequently  $c$  is nonnegative.

Thus we have, from (4.14),

$$\int_M G_{ji} \rho^j \rho^i dV \geq 0,$$

and consequently, by Theorem D,  $M$  is isometric to a sphere.

**Theorem 4.3.** *If a compact orientable  $M$  of dimension  $n > 2$  with  $K = \text{const.}$  admits an infinitesimal nonhomothetic conformal transformation  $v^h$  such that*

$$(4.15) \quad \mathcal{L}\mathcal{L}(\alpha_0 f + \alpha_1 \Delta f + \beta_0 g + \beta_1 \Delta g) \leq 0,$$

$\alpha_0, \alpha_1, \beta_0, \beta_1$  being constants not all zero such that

$$(4.16) \quad \frac{4(n-1)}{K} \alpha_0 \geq (n+6)\alpha_1 \geq 0, \quad \frac{4(n-1)}{K} \beta_0 \geq (n+6)\beta_1 \geq 0,$$

then  $M$  is isometric to a sphere.

To prove this theorem, we need the following

**Lemma 4.4.** *If a compact orientable  $M$  of dimension  $n > 2$  with  $K = \text{const.}$  admits an infinitesimal nonhomothetic conformal transformation  $v^h$ , then*

$$(4.17) \quad \begin{aligned} \frac{n-2}{2} \int_M G_{ji} \rho^j \rho^i dV &= \frac{n+6}{4} \int_M \rho^2 f dV - \frac{n-1}{4K} \int_M \rho \mathcal{L} \Delta f dV \\ &\quad - \frac{(n-1)(n+2)}{4K} \int_M \rho_i \rho^i f dV, \end{aligned}$$

$$(4.18) \quad \begin{aligned} 2 \int_M G_{ji} \rho^j \rho^i dV &= \frac{n+6}{4} \int_M \rho^2 g dV - \frac{n-1}{4K} \int_M \rho \mathcal{L} \Delta g dV \\ &\quad - \frac{(n-1)(n+2)}{4K} \int_M \rho_i \rho^i g dV. \end{aligned}$$

*Proof.* From (1.16), we have

$$\frac{n-2}{2} \int_M G_{ji} \rho^j \rho^i dV = \int_M \rho^2 f dV + \frac{1}{4} \int_M \rho \mathcal{L} f dV.$$

Substituting  $\rho = -\frac{n-1}{K} \Delta \rho$  into the last term of the second member of this equation, we find

$$\begin{aligned} \frac{n-2}{2} \int_M G_{ji} \rho^j \rho^i dV &= \int_M \rho^2 f dV - \frac{n-1}{4K} \int_M (\Delta \rho) \mathcal{L} f dV \\ &= \int_M \rho^2 f dV - \frac{n-1}{4K} \int_M \rho \Delta (\mathcal{L} f) dV, \end{aligned}$$

and consequently, by (4.6),



$$\begin{aligned} \frac{n-2}{2} \int_M G_{ji} \rho^j \rho^i dV &= \int_M \rho^2 f dV \\ &\quad - \frac{n-1}{4K} \int_M \rho \{ \mathcal{L} \Delta f + 2\rho \Delta f - (n-2) \rho^i \nabla_i f \} dV. \end{aligned}$$

Thus by (4.9) and (4.10) with  $k = -\frac{K}{n-1}$ , we find

$$\begin{aligned} \frac{n-2}{2} \int_M G_{ji} \rho^j \rho^i dV &= \int_M \rho^2 f dV - \frac{n-1}{4K} \int_M \left( \rho \mathcal{L} \Delta f + \frac{n+2}{2} \rho^2 \Delta f \right) dV \\ &= \int_M \rho^2 f dV - \frac{n-1}{4K} \int_M \rho \mathcal{L} \Delta f dV \\ &\quad - \frac{(n-1)(n+2)}{8K} \int_M \left( -\frac{2K}{n-1} \rho^2 f + 2\rho_i \rho^i f \right) dV \\ &= \frac{n+6}{4} \int_M \rho^2 f dV - \frac{n-1}{4K} \int_M \rho \mathcal{L} \Delta f dV \\ &\quad - \frac{(n-1)(n+2)}{4K} \int_M \rho_i \rho^i f dV. \end{aligned}$$

We can similarly prove (4.18).

*Proof of the theorem.* We first write down (1.16), (4.17), (1.17) and (4.18):

$$\begin{aligned} \frac{n-2}{2} \int_M G_{ji} \rho^j \rho^i dV &= \int_M \rho^2 f dV + \frac{1}{4} \int_M \rho \mathcal{L} f dV, \\ \frac{n-2}{2} \int_M G_{ji} \rho^j \rho^i dV &= \frac{n+6}{4} \int_M \rho^2 f dV - \frac{n-1}{4K} \int_M \rho \mathcal{L} \Delta f dV \\ &\quad - \frac{(n-1)(n+2)}{4K} \int_M \rho_i \rho^i f dV, \\ 2 \int_M G_{ji} \rho^j \rho^i dV &= \int_M \rho^2 g dV + \frac{1}{4} \int_M \rho \mathcal{L} g dV, \\ 2 \int_M G_{ji} \rho^j \rho^i dV &= \frac{n+6}{4} \int_M \rho^2 g dV - \frac{n-1}{4K} \int_M \rho \mathcal{L} \Delta g dV \\ &\quad - \frac{(n-1)(n+2)}{4K} \int_M \rho_i \rho^i g dV, \end{aligned}$$

from which we obtain

$$\begin{aligned}
& \left\{ \frac{n-2}{2}(a-a') + 2(b-b') \right\} \int_M G_{ji} \rho^j \rho^i dV \\
&= \frac{1}{4} \{4a - (n+6)a'\} \int_M \rho^2 f dV + \frac{1}{4} \{4b - (n+6)b'\} \int_M \rho^2 g dV \\
&+ \frac{1}{4} \int_M \rho \mathcal{L} \left( af + \frac{n-1}{K} a' \Delta f + bg + \frac{n-1}{K} b' \Delta g \right) dV \\
&+ \frac{(n-1)(n+2)}{4K} \int_M \rho_i \rho^i (a'f + b'g) dV,
\end{aligned}$$

or by Lemma 2.1,

$$\begin{aligned}
& \left\{ \frac{n-2}{2}(a-a') + 2(b-b') \right\} \int_M G_{ji} \rho^j \rho^i dV \\
(4.19) \quad &= \frac{1}{4} \{4a - (n+6)a'\} \int_M \rho^2 f dV + \frac{1}{4} \{4b - (n+6)b'\} \int_M \rho^2 g dV \\
&- \frac{1}{4n} \int_M \mathcal{L} \mathcal{L} \left( af + \frac{n-1}{K} a' \Delta f + bg + \frac{n-1}{K} b' \Delta g \right) dV \\
&+ \frac{(n-1)(n+2)}{4K} \int_M \rho_i \rho^i (a'f + b'g) dV,
\end{aligned}$$

$a, a', b, b'$  being constants. Now we choose these constants so as to have

$$(4.20) \quad \alpha_0 = a, \quad \alpha_1 = \frac{n-1}{K} a', \quad \beta_0 = b, \quad \beta_1 = \frac{n-1}{K} b'.$$

Then from (4.16) we find

$$\begin{aligned}
4a - (n+6)a' &\geq 0, & a' &\geq 0, \\
4b - (n+6)b' &\geq 0, & b' &\geq 0,
\end{aligned}$$

and

$$\begin{aligned}
4(a-a') &\geq (n+2)a' \geq 0, \\
4(b-b') &\geq (n+2)b' \geq 0,
\end{aligned}$$

and consequently

$$\frac{n-2}{2} (a-a') + 2(b-b') \geq 0,$$

the equality sign occurring when and only when  $a = a' = b = b' = 0$ , that is,  $\alpha_0 = \alpha_1 = \beta_0 = \beta_1 = 0$ .

Thus from the assumption and (4.19), we have

$$\int_M G_{ji} \rho^j \rho^i dV \geq 0,$$

and consequently, by theorem D,  $M$  is isometric to a sphere.

**Remark 4.1.** If

$$\mathcal{L}(\alpha_0 f + \alpha_1 \Delta f + \beta_0 g + \beta_1 \Delta g) = \text{constant},$$

then (4.15) is automatically satisfied. But if  $\mathcal{L}h = \text{constant}$  for a scalar field  $h$  in a compact space, the constant must be zero, because  $h$  attains an extreme value at a certain point of the space at which  $\mathcal{L}h = v^i \nabla_i h = 0$ . The same remark applies to Theorems E, 4.1, 5.1, 6.1, 6.2 and 6.4.

### 5. A theorem similar to that of Hsiung

To obtain Theorem F, Hsiung [2] used the tensor

$$aZ_{kjih} + bg_{kh}G_{ji},$$

but we would like to use here the tensor

$$(5.1) \quad W_{kjih} = aZ_{kjih} + \frac{b}{n-2} (g_{kh}G_{ji} - g_{jh}G_{ki} + G_{kh}g_{ji} - G_{jh}g_{ki}),$$

$a$  and  $b$  being constants.

It is easily seen that

$$(5.2) \quad W_{kjih}g^{kh} = (a + b)G_{ji},$$

and that, when  $a + b = 0$ ,

$$(5.3) \quad W_{kjih} = aC_{kjih},$$

where

$$(5.4) \quad C_{kjih} = K_{kjih} - \frac{1}{n-2} (g_{kh}K_{ji} - g_{jh}K_{ki} + K_{kh}g_{ji} - K_{jh}g_{ki}) + \frac{K}{(n-1)(n-2)} (g_{kh}g_{ji} - g_{jh}g_{ki})$$

is the covariant Weyl conformal curvature tensor.

In general, we have

$$(5.5) \quad W_{kjih}W^{kjih} = a^2Z_{kjih}Z^{kjih} + \frac{4(2a+b)b}{n-2}G_{ji}G^{ji},$$

and for the case  $a + b = 0$  we have

$$(5.6) \quad W_{kjih}W^{kjih} = a^2C_{kjih}C^{kjih}.$$

Using the tensor  $W_{kjih}$  defined above we can obtain

**Theorem 5.1.** *Suppose that a compact orientable  $M$  of dimension  $n > 2$  with  $K = \text{const.}$  admits an infinitesimal nonhomothetic conformal transformation  $v^h$ . If*

$$(5.7) \quad \mathcal{L}\mathcal{L}(W_{kjih}W^{kjih}) \leq 0,$$

or equivalently,

$$(5.8) \quad (n-2)a^2\mathcal{L}\mathcal{L}(Z_{kjih}Z^{kjih}) + 4(2a+b)b\mathcal{L}\mathcal{L}(G_{ji}G^{ji}) \leq 0,$$

$a$  and  $b$  being constants such that  $a + b \neq 0$ ,  $M$  is isometric to a sphere.

To prove this theorem, we need the following

**Lemma 5.1.** *For an infinitesimal conformal transformation  $v^h$  in  $M$ :  $\mathcal{L}g_{ji} = 2\rho g_{ji}$ , we have*

$$(5.9) \quad \begin{aligned} \mathcal{L}W_{kjih} &= 2a\rho Z_{kjih} + \frac{2b\rho}{n-2}(g_{kh}G_{ji} - g_{jh}G_{ki} + G_{kh}g_{ji} - G_{jh}g_{ki}) \\ &- (a+b)(g_{kh}\nabla_j\rho_i - g_{jh}\nabla_k\rho_i + \nabla_k\rho_h g_{ji} - \nabla_j\rho_h g_{ki}) \\ &+ \frac{2(a+b)}{n}\Delta\rho(g_{kh}g_{ji} - g_{jh}g_{ki}). \end{aligned}$$

*Proof.* This follows from (1.10), (1.11) and

$$(5.10) \quad \mathcal{L}Z_{kjih} = \mathcal{L}(Z_{kji}{}^t g_{ih}) = (\mathcal{L}Z_{kji}{}^t)g_{ih} + 2\rho Z_{kjih}.$$

**Lemma 5.2.** *For an infinitesimal conformal transformation  $v^h$  in  $M$ :  $\mathcal{L}g_{ji} = 2\rho g_{ji}$ , we have*

$$(5.11) \quad (\mathcal{L}W_{kjih})W^{kjih} = 2\rho W_{kjih}W^{kjih} - 4(a+b)^2G_{ji}\nabla^j\rho^i.$$

*Proof.* This follows from (5.5) and (5.9).

**Lemma 5.3.** *For an infinitesimal conformal transformation  $v^h$  in  $M$ , we have*

$$(5.12) \quad \mathcal{L}(W_{kjih}W^{kjih}) = -4\rho W_{kjih}W^{kjih} - 8(a+b)^2G_{ji}\nabla^j\rho^i.$$

*Proof.* This follows from (5.11) and

$$\mathcal{L}(W_{kjih}W^{kjih}) = 2(\mathcal{L}W_{kjih})W^{kjih} - 8\rho W_{kjih}W^{kjih}.$$

**Lemma 5.4.** For an infinitesimal conformal transformation  $v^h$  in  $M$  with  $K = \text{constant}$ , we have

$$(5.13) \quad \begin{aligned} &8(a + b)^2 \nabla^j (G_{ji} \rho \rho^i) \\ &= 8(a + b)^2 G_{ji} \rho^j \rho^i - 4\rho^2 W_{kjih} W^{kjih} - \rho \mathcal{L}(W_{kjih} W^{kjih}). \end{aligned}$$

*Proof.* This follows from  $\nabla^j G_{ji} = 0$  and (5.12).

**Lemma 5.5.** If a compact orientable  $M$  of dimension  $n > 2$  with  $K = \text{const.}$  admits an infinitesimal nonhomothetic conformal transformation  $v^h$ , then we have

$$(5.14) \quad \begin{aligned} &8(a + b)^2 \int_M G_{ji} \rho^j \rho^i dV \\ &= 4 \int_M \rho^2 W_{kjih} W^{kjih} dV + \int_M \rho \mathcal{L}(W_{kjih} W^{kjih}) dV \\ &= 4 \int_M \rho^2 W_{kjih} W^{kjih} dV - \frac{1}{n} \int_M \mathcal{L} \mathcal{L}(W_{kjih} W^{kjih}) dV. \end{aligned}$$

*Proof.* This follows from (5.13) by integrating both sides over  $M$  and using Lemma 2.1.

*Proof of the theorem.* If  $\mathcal{L} \mathcal{L}(W_{kjih} W^{kjih}) \leq 0$ , and  $a + b \neq 0$ , then from (5.14) we have

$$\int_M G_{ji} \rho^j \rho^i dV \geq 0.$$

Thus by Theorem D,  $M$  is isometric to a sphere.

### 6. Characterizations of conformally flat spaces

**Theorem 6.1.** If a compact orientable  $M$  of dimension  $n > 3$  admits an infinitesimal conformal transformation  $v^h: \mathcal{L}g_{ji} = 2\rho g_{ji}$  such that  $\rho$  does not vanish on any  $n$ -dimensional domain and

$$(6.1) \quad \mathcal{L} \mathcal{L} h < 0, \quad h = C_{kjih} C^{kjih},$$

then  $M$  is conformally flat.

*Proof.* Multiplying (5.12), with  $a + b = 0$ , by  $\rho$  and integrating the resulting equation over  $M$ , we find

$$0 = 4 \int_M \rho^2 h dV + \int_M \rho \mathcal{L} h dV,$$

or by Lemma 2.1,

$$(6.2) \quad 0 = 4 \int_M \rho^2 h dV - \frac{1}{n} \int_M \mathcal{L} \mathcal{L} h dV.$$

(6.2) implies

$$\int_M \rho^2 h dV \leq 0,$$

from which  $\rho^2 h = 0$ , or by the assumption of the theorem,  $h = 0$ , that is,  $C_{kjih} = 0$ , which shows that  $M$  is conformally flat.

**Remark 6.1.** If  $\mathcal{L}h = \text{constant}$  in a compact space, we have  $\mathcal{L}h = 0$ . On the other hand, if  $\mathcal{L}h = 0$  in a general Riemannian space, from  $\mathcal{L}h + 4\rho h = 0$  we find  $h = 0$ , which shows that the space is conformally flat.

**Theorem 6.2.** Under the same assumptions as in Theorem 6.1, if  $K = \text{const.}$  and (6.1) is replaced by

$$(6.3) \quad \mathcal{L} \left\{ \sum_{k=0}^l \alpha_k \left( -\frac{n-1}{K} \right)^k \Delta^k(\mathcal{L}h) \right\} \leq 0,$$

$l$  being a nonnegative integer and  $\alpha_k$  constants such that  $\sum_{k=0}^l \alpha_k > 0$ , then  $M$  is conformally flat.

*Proof.* Similarly, as in the proof of Theorem 4.1 we can obtain

$$\begin{aligned} 0 &= 4 \int_M (\alpha_0 + \alpha_1 + \cdots + \alpha_l) \rho^2 h dV \\ &\quad + \int_M \rho \left\{ \alpha_0 \mathcal{L}h + \alpha_1 \left( -\frac{n-1}{K} \right) \Delta(\mathcal{L}h) + \cdots \right. \\ &\quad \left. + \alpha_l \left( -\frac{n-1}{K} \right)^l \Delta^l(\mathcal{L}h) \right\} dV, \end{aligned}$$

or, by virtue of Lemma 2.1,

$$(6.4) \quad \begin{aligned} 0 &= 4 \int_M (\alpha_0 + \alpha_1 + \cdots + \alpha_l) \rho^2 dV \\ &\quad - \frac{1}{n} \int_M \mathcal{L} \left\{ \alpha_0 \mathcal{L}h + \alpha_1 \left( -\frac{n-1}{K} \right) \Delta(\mathcal{L}h) + \cdots \right. \\ &\quad \left. + \alpha_l \left( -\frac{n-1}{K} \right)^l \Delta^l(\mathcal{L}h) \right\} dV, \end{aligned}$$

$\alpha_0, \alpha_1, \dots, \alpha_l$  being constants such that  $\sum_{k=0}^l \alpha_k > 0$ . Thus by (6.3), we have

$$\int_M \rho^2 h dV = 0, \text{ from which } h = 0 \text{ and consequently } C_{kjih} = 0.$$

**Theorem 6.3.** Under the same assumptions as in Theorem 6.1, if  $K = \text{const.}$  and (6.1) is replaced by

$$(6.5) \quad \alpha_0 h - \alpha_1 \Delta h = c \text{ (constant),}$$

$\alpha_0$  and  $\alpha_1$  being positive constants such that

$$(6.6) \quad \frac{8K}{n-1} \alpha_1 > (n-6)\alpha_0 \geq 0, \quad \text{for } n > 6,$$

then  $M$  is conformally flat.

*Proof.* Similarly, as in the proof of Theorem 4.2 we can obtain

$$(6.7) \quad 0 = \int_M \rho^2 [8a' - (n-6)a]h + (n+2)c]dV.$$

Now, the constant  $8a' - (n-6)a$  is positive for  $n \leq 6$ . Since

$$8a' - (n-6)a = \frac{8K}{n-1} \alpha_1 - (n-6)\alpha_0,$$

by (6.6) this constant is also positive for  $n > 6$ .

On the other hand, from (6.5) we have

$$\alpha_0 \int_M h dV = c \int_M dV,$$

which shows that  $c$  is a nonnegative constant.

Thus from (6.7) we see that  $h = 0$  and consequently  $C_{kji h} = 0$ .

**Theorem 6.4.** Under the same assumption as in Theorem 6.1, if  $K = \text{const.}$  and (6.1) is replaced by

$$(6.8) \quad \mathcal{L}\mathcal{L}(\alpha_0 h + \alpha_1 \Delta h) \leq 0,$$

$\alpha_0$  and  $\alpha_1$  being constants such that

$$(6.9) \quad \frac{4(n-1)}{K} \alpha_0 > (n+6)\alpha_1 \geq 0,$$

then  $M$  is conformally flat.

*Proof.* Similarly, as in the proof of Theorem 4.3 we can obtain

$$(6.10) \quad \begin{aligned} 0 = & \{4a - (n+6)a'\} \int_M \rho^2 h dV - \frac{1}{n} \int_M \mathcal{L}\mathcal{L} \left( ah + \frac{n-1}{K} a' \Delta h \right) dV \\ & + \frac{(n-1)(n+2)}{K} \int_M \rho_i \rho^i (a'h) dV, \end{aligned}$$

$a$  and  $a'$  being constants. Now we choose these constants such that

$$(6.11) \quad \alpha_0 = a, \quad \alpha_1 = \frac{n-1}{K} a'.$$

Then from (6.9) we have

$$4a - (n+6)a' = 4\alpha_0 - (n+6) \frac{K}{n-1} \alpha_1 > 0.$$

We also have

$$\mathcal{L} \left( ah + \frac{n-1}{K} a' \Delta h \right) = \mathcal{L}(\alpha_0 h + \alpha_1 \Delta h) = \text{constant}.$$

Thus, from (6.10), we have  $h = 0$  and consequently  $C_{kjih} = 0$ .

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