

INTEGRABILITY CRITERIA FOR SYSTEMS OF NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction

In [5], we showed the existence of local solutions of analytic linear partial differential equations; in this paper, we generalize our previous result to an arbitrary analytic (nonlinear) system of partial differential equations and prove the Cartan-Kähler theorem for such a system formulated in terms of the jet bundles introduced by Ehresmann. The reader is strongly advised to read the proof of the linear Cartan-Kähler theorem (Theorem 4.1) of [5] as an introduction to the present paper, although this paper is essentially independent of [5].

We consider partial differential equations on fibered manifolds and do not define nonlinear equations in terms of differential operators as one can for linear equations (see [5]). In the nonlinear case this point of view would be too restrictive (see §7) and so we let arbitrary fibered submanifolds of jet bundles be partial differential equations, since many equations occurring in differential geometry are of this type (see E. Cartan [3]).

To generalize the methods of [5], where the jet bundles considered are vector bundles, we define affine bundles and analyse the structure of affine bundles which the jet bundles possess (§§3 and 5). If every solution of order k of a partial differential equation R_k of order k can be extended to a solution of order $k + 1$, and if the solutions of order $k + 1$ satisfy a regularity condition, the solutions of order $k + 1$ form an equation R_{k+1} of order $k + 1$, the prolongation of R_k . This prolongation R_{k+1} is actually an affine bundle over the equation R_k whose affine bundle structure is induced by the affine bundle structure of a jet bundle. Under these conditions on R_k , this affine bundle structure of R_{k+1} over R_k , together with the affine bundle structure of certain jet bundles (see Proposition 5.3), permits us to define the *curvature* κ of the partial differential equation R_k . The curvature κ is the obstruction to extending a solution of order $k + 1$ to a solution of order $k + 2$. Our definition of κ reduces for homogeneous linear equations to the definition of κ given by Quillen [7], which is equivalent, with a change in sign, to the one Bott

originally gave in [1], as was shown by Quillen [7]. In [5], we used still another definition of the curvature of a homogeneous linear equation.

We define the *symbol* of a partial differential equation and associate to it, following Spencer [8], certain cohomology groups $H^{k+l,j}$ whose vanishing, for $l \geq 0$, is equivalent to Cartan's notion of involutiveness, as was shown by Serre. A crucial property of the curvature κ is the fact that it takes its values in a family of vector spaces induced by the Spencer cohomology group $H^{k,2}$ (see Proposition 8.4). *This permits us to determine, as in [5], the integrability conditions of the equation, which depend only on the formal solutions up to order $k + 1$ and the second Spencer cohomology groups $H^{k+l,2}$.* Our formal existence theorems (Theorem 8.1 and Theorem 8.2), together with Spencer's estimate (see L. Ehrenpreis, V.W. Guillemin, and S. Sternberg [4] and W. J. Sweeney [10]) lead to the existence of analytic solutions for analytic equations and to generalizations of the Cartan-Kähler theorem and the Cartan-Kuranishi theorem (see M. Kuranishi [6]).

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2. Fibered manifolds

Let X be a differentiable manifold. (By differentiable, we shall always mean differentiable of class C^∞ .) We do not require that the dimensions of the different components of a differentiable manifold be the same. We shall denote by $T = T(X)$ the tangent bundle of X and by $T^* = T^*(X)$ the cotangent bundle of X . By $S^k T^*$, $\wedge^l T^*$, $\otimes^m T^*$, we shall mean the k -th symmetric product of T^* , the l -th exterior product of T^* and the tensor product of m copies of T^* respectively.

Definition 2.1. A *fibered manifold* $\pi: E \rightarrow X$ is a differentiable manifold E together with a differentiable surjective map π (projection) of E onto X of maximal rank such that the dimension of E at $e \in E$ is greater than or equal to the dimension of X at $\pi(e) \in X$.

We shall usually write simply E to denote a fibered manifold $\pi: E \rightarrow X$. If $\pi: E \rightarrow X$ is a fibered manifold over X , we shall denote by $E_x = \pi^{-1}(x)$ the fiber of E at $x \in X$, by \mathcal{E} the sheaf of germs of differentiable sections of E and by \mathcal{E}_x the stalk of \mathcal{E} at $x \in X$. If s is a section of E over an open set $U \subset X$ and $x \in X$, then s_x is the germ of s at x . If Y is a submanifold of X , we denote by $E|Y$ the restriction of E to Y , that is, the fibered manifold

$\pi^{-1}(Y)$ over Y . Let $T(E)$ denote the tangent bundle of E and $F(E)$ the sub-bundle of $T(E)$ consisting of all tangent vectors tangent to the fibers, that is, we have the exact sequence of vector bundles over E

$$0 \longrightarrow F(E) \longrightarrow T(E) \xrightarrow{\pi_*} \pi^{-1}T(X) \longrightarrow 0$$

where $\pi^{-1}T(X)$ is the vector bundle over E induced by π from $T(X)$. By O_X we denote X considered as a fibered manifold over X with projection the identity map of X .

Example. If Y is any differentiable manifold, then $pr_1: X \times Y \rightarrow X$, where pr_1 is the projection onto the first factor, is a fibered manifold over X called the trivial fibered manifold over X with fiber Y . A section of this fibered manifold over an open set U is simply the graph of a differentiable map from U to Y .

Remark. Any fiber bundle and in particular any vector bundle is a fibered manifold over X .

If $\xi: W \rightarrow Y$ is a vector bundle over a differentiable manifold Y , then 0 is the zero section of W over Y and $0(y)$ is the zero vector of W_y . If Z is another differentiable manifold, $f_1: Z \rightarrow X$, $f_2: Z \rightarrow Y$ are differentiable maps and if $\xi_1: W_1 \rightarrow X$, $\xi_2: W_2 \rightarrow Y$ are vector bundles, we shall denote by $W_1 \otimes_Z W_2$ the vector bundle $f_1^{-1}W_1 \otimes f_2^{-1}W_2$.

Throughout this paper, $\pi: E \rightarrow X$, $\pi': E' \rightarrow X$, $\pi'': E'' \rightarrow X$ denote fibered manifolds over X .

Definition 2.2. A morphism of fibered manifolds over X from $\pi: E \rightarrow X$ to $\pi': E' \rightarrow X$ is a differentiable map $\varphi: E \rightarrow E'$ such that $\pi' \circ \varphi = \pi$. We denote by φ_x the map from E_x to E'_x , by φ the sheaf map from \mathcal{E} to \mathcal{E}' and by φ_* the map from $T(E)$ to $T(E')$ or from $F(E)$ to $F(E')$ induced by φ .

Definition 2.3. We say that the sequence of fibered manifolds over X

$$E \xrightarrow{\varphi} E' \xrightarrow{\varphi'} E''$$

is exact if there exists a section s'' of E'' over X such that:

- (i) the sequence of sets

$$E \xrightarrow{\varphi} E' \xrightarrow[s'' \circ \pi']{\varphi'} E''$$

is exact;

- (ii) the sequence of vector spaces

$$F_e(E) \xrightarrow{\varphi_*} F_{\varphi(e)}(E') \xrightarrow{\varphi'_*} F_{\varphi'(\varphi(e))}(E'')$$

is exact, for every $e \in E$.

We say that a morphism $\varphi: E \rightarrow E'$ is a monomorphism if it is an imbedding of manifolds, and is an epimorphism if $\varphi: E \rightarrow E'$ is a fibered manifold. If φ is both a monomorphism and an epimorphism, it is called an isomorphism. We say that the sequence

$$O_X \longrightarrow E \xrightarrow{\varphi} E'$$

is exact if φ is a monomorphism and that the sequence

$$E \xrightarrow{\varphi} E' \longrightarrow O_X$$

is exact if φ is an epimorphism. Note that these definitions are compatible with Definition 2.3.

The fibered product $E \times_X E'$ of E and E' over X consisting of all pairs $(e, e') \in E \times E'$ satisfying $\pi(e) = \pi'(e')$ is a differentiable manifold in a natural way and a fibered manifold over X . The projections $pr_1: E \times_X E' \rightarrow E$, $pr_2: E \times_X E' \rightarrow E'$ sending $(e, e') \in E \times_X E'$ into e, e' respectively are epimorphisms of fibered manifolds over X .

Definition 2.4. A fibered manifold $\pi': E' \rightarrow X$ is said to be a fibered submanifold of a fibered manifold $\pi: E \rightarrow X$ if E' is a subset of E and if the inclusion of E' into E is a monomorphism of fibered manifolds over X .

Definition 2.5. If $\varphi: E \rightarrow E'$ is a morphism of fibered manifolds and s' is a section of E' over X , then $\text{Ker}_{s', \varphi}$, the *kernel* of φ with respect to s' , is the subset of E consisting of all $e \in E$ satisfying $\varphi(e) = s'(\pi(e))$.

Proposition 2.1. If $\varphi: E \rightarrow E'$ is a morphism of fibered manifolds of locally constant rank, then the image of φ , denoted by $\text{Im } \varphi$, is a fibered submanifold of E' . If s' is a section of E' over X satisfying $s'(X) \subset \varphi(E)$, then $\text{Ker}_{s', \varphi}$ is a fibered submanifold of E and the sequence

$$O_X \longrightarrow \text{Ker}_{s', \varphi} \xrightarrow{i} E \xrightarrow{\varphi} \text{Im } \varphi \longrightarrow O_X$$

is exact, where i is the inclusion of $\text{Ker}_{s', \varphi}$ into E .

The above proposition and the following lemma follow from the implicit function theorem in a straightforward way and their proofs will be omitted.

Lemma 2.1. If $\pi': E' \rightarrow X$ is a fibered submanifold of $\pi: E \rightarrow X$, then, for each point $e \in E'$, there exist an open neighborhood V of e in E , a fibered manifold $\zeta: Z \rightarrow U$ over $U = \pi V$, a section z of Z over U and a morphism $\varphi: V \rightarrow Z$ of fibered manifolds over U of constant rank such that $E' \cap V = \text{Ker}_z \varphi$.

The statement of the following proposition is due to B. Mazur:

Proposition 2.2. Let $\pi': E' \rightarrow X$ be a fibered submanifold of $\pi: E \rightarrow X$. Then there exist an open fibered submanifold $\pi'': E'' \rightarrow X$ of $\pi: E \rightarrow X$ containing E' , a fibered manifold $\zeta: Z \rightarrow X$, a section z of Z over X and a

morphism $\varphi: E'' \rightarrow Z$ of fibered manifolds over X of locally constant rank such that $E' = \text{Ker}_z\varphi$ if and only if the normal bundle of the imbedding of E' into E is isomorphic to a vector bundle induced by π' from a vector bundle over X .

Proof. If the morphism φ satisfying the above conditions exists, we may assume that $Z = \text{Im } \varphi$ by Proposition 2.1. The normal bundle N of the imbedding of E' into E is isomorphic to $\varphi^{-1}N_z$, if N_z denotes the normal bundle of the imbedding of $z(X)$ into Z . Since $E' = \text{Ker}_z\varphi$, the vector bundle N is isomorphic to $\pi'^{-1}(z^{-1}N_z)$. Because z is a section of Z over X , the bundle N satisfies the desired condition. Conversely, assume that there exists a vector bundle $\zeta: Z \rightarrow X$ such that N is isomorphic to $\pi'^{-1}Z$. Because N is isomorphic to the quotient of $F(E) | E'$ by its sub-bundle $F(E')$, by the proof of the tubular neighborhood theorem, there exist an open fibered submanifold $\pi'': E'' \rightarrow X$ of E containing E' and an isomorphism $\psi: E'' \rightarrow N$ of fibered manifolds over X such that the diagram

$$\begin{array}{ccc}
 E'' & \xrightarrow{\varphi} & N \\
 \uparrow & \nearrow 0 & \\
 E' & &
 \end{array}$$

commutes, where 0 is the zero section of N . The natural epimorphism pr_2 of fibered manifolds over X from $\pi'^{-1}Z = E' \times_X Z$ to Z induces an epimorphism θ of fibered manifolds over X from N to Z . Define $\varphi: E'' \rightarrow Z$ to be the composition $\theta \circ \psi$ and z to be the zero section of Z ; clearly φ has the required properties.

3. Affine bundles

We begin by recalling certain properties of affine spaces. Let k be a field.

By an *affine space* A over k modeled on a vector space T over k , we mean a homogeneous space of the additive group of T on which T acts freely. We denote by $t + a$ the image of $a \in A$ under the action of $t \in T$, and if $a, b \in A$, by $a - b$ the unique element of T such that $(a - b) + b = a$. For each $a \in A$, we have a natural identification of A with T sending $x \in A$ into $x - a \in T$ which induces on A the structure of a vector space over k with origin a . A non-empty subset V of A is an affine subspace of A if V is a subspace of A considered as a vector space with origin a point of V ; the set D of all elements $x - y$ of T , with x, y belonging to V , is a linear subspace of T and V is an affine space modeled on D in a canonical way. If A' is an affine space modeled on a vector space T' over k , by a morphism $\varphi: A \rightarrow A'$ of affine spaces over k , we mean a map preserving barycenters. For any such morphism, there exists a unique linear map $\psi: T \rightarrow T'$ such that

$$\varphi(t + x) = \psi(t) + \varphi(x)$$

for all $x \in A$, $t \in T$, which we call the linear map associated to φ (see N. Bourbaki [2]).

If $\varphi: A \rightarrow A'$ is a morphism of affine spaces whose associated linear map is $\psi: T \rightarrow T'$, we define an equivalence relation on A' : we say that $a' \in A'$ is equivalent to $b' \in A'$ if and only if $a' - b' \in \psi(T)$. We denote the set of equivalence classes by $\text{Coker } \varphi$ and by ρ the natural projection of A' onto $\text{Coker } \varphi$. Note that $\rho\varphi(a) = \rho\varphi(b)$, if $a, b \in A$, so that $\rho\varphi(A)$ is a distinguished point of $\text{Coker } \varphi$. It is easily verified that $\text{Coker } \varphi$ is an affine space modeled on the cokernel of ψ , $\text{Coker } \psi$, and that ρ is a morphism of affine spaces whose associated linear map is the natural projection of T' onto $\text{Coker } \psi$. Since $\text{Coker } \varphi$ has a distinguished element $\rho\varphi(A)$, it can be identified in a natural way with $\text{Coker } \psi$. We call $\text{Coker } \varphi$ the *cokernel* of φ .

If $i: V \rightarrow A$ is the inclusion of an affine subspace V of A into A , the cokernel of i will be called the quotient A/V of A by V .

If $a' \in A'$, we denote by $\text{Ker}_{a'}\varphi$ the inverse image under φ of a' . If $\text{Ker}_{a'}\varphi$ is non-empty, it is easily seen that it is an affine subspace of A modeled on the kernel of ψ , $\text{Ker } \psi$.

Definition 3.1. An affine bundle A over X modeled on a vector bundle $\xi: W \rightarrow X$ is a fibered manifold $\pi: A \rightarrow X$ together with a morphism of fibered manifolds over X

$$W \times_X A \rightarrow A$$

sending $(w, a) \in W \times_X A$ into $w + a \in A$, such that, for each $x \in X$, the fiber A_x is an affine space modeled on the vector space W_x under the action of W_x on A_x sending $(w, a) \in W_x \times A_x$ into $w + a \in A_x$.

If $a \in A_x$, the map $w \mapsto w + a$ of W_x into A_x is a diffeomorphism. If $b \in A_x$, the map of A_x into W_x sending a into $a - b$ is also a diffeomorphism.

If s is a section of A over an open set $U \subset X$, the map of $A|U$ into $W|U$ sending a into $a - s(\pi(a))$ is an isomorphism of fibered manifolds over U . Therefore A is locally trivial, that is, given $x \in X$, there exist a neighborhood V of x and an isomorphism $\varphi: A|V \rightarrow V \times A_x$ of fibered manifolds over V .

Note that O_X is an affine bundle over X modeled on the zero-dimensional vector bundle over X .

Definition 3.2. If $\pi_1: E_1 \rightarrow E$, $\pi'_1: E'_1 \rightarrow E'$ are fibered manifolds, then a differentiable map $\varphi_1: E_1 \rightarrow E'_1$ is said to be a morphism of fibered manifolds from E_1 to E'_1 over a morphism $\varphi: E \rightarrow E'$ of fibered manifolds over X if $\pi'_1 \circ \varphi_1 = \varphi \circ \pi_1$. If E_1, E'_1 are vector (respectively affine) bundles over E, E' respectively and if the map from $(E_1)_e$ to $(E'_1)_{\varphi(e)}$ induced by φ_1 is a linear map of vector spaces (respectively morphism of affine spaces) for each $e \in E$, then we say that φ_1 is a morphism of vector (respectively affine) bundles from E_1 to

E'_1 over the morphism φ . A morphism $\varphi_1: E_1 \rightarrow E'_1$ of affine bundles over a morphism $\varphi: E \rightarrow E'$ of fibered manifolds over X is said to be a monomorphism of affine bundles if φ_1 is a monomorphism of fibered manifolds over X . If $E = E'$ and φ is the identity map of E , then we say that a morphism of fibered manifolds from E_1 to E'_1 over φ is a morphism of fibered manifolds over E .

Proposition 3.1. *If $\alpha: A \rightarrow E$, $\alpha': A' \rightarrow E'$ are affine bundles over E, E' modeled on vector bundles $\xi: W \rightarrow E$, $\xi': W' \rightarrow E'$ respectively and if $\varphi: A \rightarrow A'$ is a morphism of affine bundles over a morphism $\tilde{\varphi}: E \rightarrow E'$, there exists a unique morphism of vector bundles $\psi: W \rightarrow W'$ over $\tilde{\varphi}$ such that*

$$\varphi(w + a) = \psi(w) + \varphi(a),$$

for all $a \in A$, $w \in W$, with $\alpha(a) = \xi(w)$.

We say that ψ is the morphism of vector bundles associated to φ . Given a morphism $\psi: W \rightarrow W'$ of vector bundles over a morphism $\tilde{\varphi}: E \rightarrow E'$ and sections s, s' of A, A' over E, E' respectively, the map

$$a \longmapsto \psi(a - s(\alpha(a))) + s'(\tilde{\varphi}(\alpha(a)))$$

is a morphism of affine bundles from A to A' over $\tilde{\varphi}$.

The following proposition is easily verified:

Proposition 3.2. *Let A, A', A'' be affine bundles over X modeled on vector bundles W, W', W'' over X . Then the sequence of morphisms of affine bundles over X*

$$A \xrightarrow{\varphi} A' \xrightarrow{\varphi'} A''$$

is exact if and only if the associated sequence of morphisms of vector bundles over X

$$W \xrightarrow{\psi} W' \xrightarrow{\psi'} W''$$

is exact.

Definition 3.3. A locally trivial fibered submanifold $\pi': A' \rightarrow X$ of an affine bundle $\pi: A \rightarrow X$ is said to be an affine sub-bundle of A if A'_x is an affine subspace of A_x , for each $x \in X$.

An affine sub-bundle $\pi': A' \rightarrow X$ of an affine bundle $\pi: A \rightarrow X$ modeled on a vector bundle $\xi: W \rightarrow X$ is an affine bundle modeled on a sub-bundle W' of W . Indeed, for each $x \in X$, the subset A'_x of A_x is an affine space modeled on the subspace W'_x of W_x consisting of all elements $a' - b'$, with $a', b' \in A'_x$. Since A' is locally trivial,

$$W' = \bigcup_{x \in X} W'_x$$

is a sub-bundle of W on which A' is modeled.

If $\pi': A' \rightarrow X$ is a fibered submanifold of $\pi: A \rightarrow X$, if A'_x is an affine subspace of A_x for each $x \in X$, and if

$$W' = \bigcup_{x \in X} W'_x$$

is a sub-bundle of W , then A' is an affine sub-bundle of A .

Proposition 3.3. *Let $\pi: A \rightarrow X$, $\pi': A' \rightarrow X$ be affine bundles over X modeled on vector bundles $\xi: W \rightarrow X$, $\xi': W' \rightarrow X$. Then a morphism $\varphi: A \rightarrow A'$ of affine bundles over X is of locally constant rank if and only if the linear map $\phi_x: W_x \rightarrow W'_x$, with $x \in X$, induced by the morphism of vector bundles associated to φ , has locally constant rank.*

The proof of this proposition will be left to the reader.

If $\varphi: A \rightarrow A'$ is a morphism of affine bundles over X , then for $x \in X$, define $(\text{Coker } \varphi)_x = \text{Coker } \varphi_x$, and let

$$\text{Coker } \varphi = \bigcup_{x \in X} (\text{Coker } \varphi)_x .$$

If A, A' are modeled on vector bundles W, W' over X , we can canonically identify the affine space $\text{Coker } \varphi_x$ with the vector space on which it is modeled, $\text{Coker } \phi_x$, where ϕ is the morphism of vector bundles associated to φ , and hence also $\text{Coker } \varphi$ with $\text{Coker } \phi$. If φ has locally constant rank, then so does ϕ by Proposition 3.3, so that $\text{Coker } \phi$ is a vector bundle over X ; we can give $\text{Coker } \varphi$ the structure of fibered manifold over X induced from the structure of fibered manifold of $\text{Coker } \phi$ via the above identification. Then $\text{Coker } \varphi$ becomes an affine bundle modeled on $\text{Coker } \phi$. If ρ_x is the natural projection of A'_x onto $(\text{Coker } \varphi)_x$, we obtain a map $\rho: A' \rightarrow \text{Coker } \varphi$ which is easily seen to be a morphism of affine bundles whose associated morphism of vector bundles is the natural projection $\sigma: W' \rightarrow \text{Coker } \phi$. Furthermore,

$$A \xrightarrow{\varphi} A' \xrightarrow{\rho} \text{Coker } \varphi \longrightarrow O_X$$

is an exact sequence of affine bundles over X by Proposition 3.2, since

$$W \xrightarrow{\phi} W' \xrightarrow{\sigma} \text{Coker } \phi \longrightarrow 0$$

is an exact sequence of vector bundles over X . Therefore, we have proved

Proposition 3.4. *If $\varphi: A \rightarrow A'$ is a morphism of affine bundles over X of locally constant rank, then there exist an affine bundle $\text{Coker } \varphi$, the cokernel of φ , and a morphism of affine bundles $\rho: A \rightarrow \text{Coker } \varphi$ such that the sequence*

$$A \xrightarrow{\varphi} A' \xrightarrow{\rho} \text{Coker } \varphi \longrightarrow O_X$$

is exact. Moreover, $\text{Coker } \varphi$ can be identified in a natural way with the cokernel of the morphism of vector bundles associated to φ .

The above proposition can be applied to the inclusion i of an affine sub-

bundle A' of A into A , which is a monomorphism of affine bundles; the cokernel of i will be called the *quotient* A/A' of A by A' .

Proposition 3.5. *Let $\alpha: A \rightarrow E$, $\alpha': A' \rightarrow E'$ be affine bundles over E, E' modeled on vector bundles $\xi: W \rightarrow E$, $\xi': W' \rightarrow E'$ and let φ be a morphism of affine bundles, over a morphism $\tilde{\varphi}: E \rightarrow E'$ of locally constant rank, whose associated morphism of vector bundles is a map $\psi: W \rightarrow W'$. Let s' be a section of A' over X satisfying $s'(X) \subset \varphi(A)$; setting $K = \text{Ker}_{s',\varphi}$, $\bar{K} = \text{Ker}_{\alpha' \circ s', \tilde{\varphi}}$, assume that $\alpha: K \rightarrow \bar{K}$ is surjective. If the restriction of ψ to $W|_{\bar{K}}$ has locally constant rank, then K is an affine sub-bundle of $A|_{\bar{K}}$ over \bar{K} modeled on $(\text{Ker } \psi)|_{\bar{K}}$ and the restriction of φ to $A|_{\bar{K}}$ has locally constant rank.*

Proof. Since $s'(X) \subset \varphi(A)$, we have $(\alpha' \circ s')(X) \subset \tilde{\varphi}(E)$ and Proposition 2.1 applies; hence \bar{K} is a fibered submanifold of $\pi: E \rightarrow X$. Since φ is a morphism of affine bundles over $\tilde{\varphi}$, for each $e \in \bar{K}$, the map $\varphi_e: A_e \rightarrow A'_{\tilde{\varphi}(e)}$ is a morphism of affine spaces. Because $s'(\pi(e))$ belongs to $A'_{\tilde{\varphi}(e)}$ and because $\alpha: K \rightarrow \bar{K}$ is surjective, $\text{Ker}_{s'(\pi(e)),\varphi_e}$ is an affine subspace of A_e modeled on $\text{Ker } \varphi_e$. Since $(\text{Ker } \psi)|_{\bar{K}}$ is a sub-bundle of $W|_{\bar{K}}$, to show that K is an affine sub-bundle of $A|_{\bar{K}}$, it is sufficient to show that for every $e \in \bar{K}$, there exists a section of $A|_{\bar{K}}$ over a neighborhood of $e \in \bar{K}$ whose image lies in K . We first note that the vector bundle $\text{Im}(\psi|_{(W|_{\bar{K}})})$ is the subset of W' consisting of all elements of W' of the form $\varphi(a) - s'(\pi(a))$, where $a \in A|_{\bar{K}}$. Indeed, given $w \in W|_{\bar{K}}$, there exists $a \in K$, with $\alpha(a) = \xi(w)$; we have $\varphi(w + a) = \psi(w) + s'(\pi(a))$ and so $\psi(w) = \varphi(w + a) - s'(\pi(a))$. Given $a \in A|_{\bar{K}}$, choose $a' \in K$ with $\alpha(a') = \alpha(a)$; then

$$\varphi(a) - s'(\pi(a)) = \varphi(a) - \varphi(a') = \psi(a - a').$$

Now choose a morphism of vector bundles $\sigma: \tilde{\varphi}^{-1}\{\text{Im}(\psi|_{(W|_{\bar{K}})})\} \rightarrow W|_{\bar{K}}$ over \bar{K} such that $\psi \circ \sigma = \text{id}$. Given an element $e \in \bar{K}$, choose a section s of $A|_{\bar{K}}$ over a neighborhood V of $e \in \bar{K}$. Clearly $\varphi \circ s$ and $s' \circ \pi$ are mappings of V into A' satisfying $\alpha' \circ (\varphi \circ s) = \alpha' \circ (s' \circ \pi)$. Therefore these maps induce a map $s' \circ \pi - \varphi \circ s$ from V to W' such that $\xi'(s' \circ \pi - \varphi \circ s) = \tilde{\varphi}$. By the preceding remarks, the section

$$v \mapsto \sigma(v, (s' \circ \pi - \varphi \circ s)(v))$$

of W over V is well-defined and will denoted by $\sigma(s' \circ \pi - \varphi \circ s)$; the section

$$\bar{s} = \sigma(s' \circ \pi - \varphi \circ s) + s$$

of A over V is also well-defined. We have

$$\varphi \circ \bar{s} = \psi(\sigma(s' \circ \pi - \varphi \circ s)) + \varphi \circ s = (s' \circ \pi - \varphi \circ s) + \varphi \circ s = s' \circ \pi.$$

Hence the image of the section \bar{s} lies in K and $\alpha: K \rightarrow \bar{K}$ is an affine bundle. Moreover, for $a \in A|_V$

$$\varphi(a) = \psi(a - \bar{s}(\alpha(a))) + s'(\pi(\alpha(a)))$$

which implies that $\varphi | (A | \bar{K})$ has locally constant rank.

Proposition 3.6. *If $\pi: A \rightarrow X$ is an affine bundle over X modeled on a vector bundle $\xi: W \rightarrow X$, and if $\pi^{-1}W$ denotes the vector bundle over A induced from W by π , there is a natural isomorphism μ of vector bundles over A from $\pi^{-1}W$ to $F(A)$.*

Proof. An element $w \in W_x$, with $x \in X$, determines a one-parameter family of diffeomorphisms $\mu_t(w)$ of A_x , by setting $\mu_t(w)(b) = tw + b$, for all $b \in A_x$, $t \in \mathbb{R}$. Then

$$\tilde{w} = \frac{d}{dt} \mu_t(w) |_{t=0}$$

is a vector field on A_x , and since A_x is a submanifold of A , for each $b \in A_x$, $\tilde{w}(b)$ is an element of $F_b(A)$. Consider the map μ sending $(a, w) \in \pi^{-1}W$, with $a \in A_x$, $w \in W_x$, for some $x \in X$, into $\tilde{w}(a) \in F(A)$. Then μ is clearly a monomorphism of vector bundles over A , since if $\tilde{w}(a) = 0$, then $\mu_t(w)(a) = a$ for some $t \neq 0$ and $w = 0(x)$. Because the dimensions of these two vector bundles are the same, μ is actually an isomorphism.

Proposition 3.7 (Naturality of μ). *If $\alpha: A \rightarrow E$, $\alpha': A' \rightarrow E'$ are affine bundles over E, E' modeled on vector bundles $\xi: W \rightarrow E$, $\xi': W' \rightarrow E'$ and if $\varphi: A \rightarrow A'$ is a morphism of affine bundles over a morphism $\tilde{\varphi}: E \rightarrow E'$, whose associated morphism of vector bundles is a map $\psi: W \rightarrow W'$, then the diagram of vector bundles over A*

$$(3.1) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & \alpha^{-1}W & \xrightarrow{\mu} & F(A) & \xrightarrow{\alpha_*} & \alpha^{-1}F(E) & \longrightarrow & 0 \\ & & \downarrow \psi & & \downarrow \varphi_* & & \downarrow \tilde{\varphi}_* & & \\ 0 & \longrightarrow & \alpha'^{-1}W' & \xrightarrow{\mu'} & F(A') & \xrightarrow{\alpha'_*} & \alpha'^{-1}F(E') & \longrightarrow & 0 \end{array}$$

is commutative and has exact rows, where $\psi: \alpha^{-1}W \rightarrow \alpha'^{-1}W'$, $\tilde{\varphi}_*: \alpha^{-1}F(E) \rightarrow \alpha'^{-1}F(E')$ are morphisms of vector bundles over $\tilde{\varphi}$ induced by $\psi: W \rightarrow W'$, $\tilde{\varphi}_*: F(E) \rightarrow F(E')$ respectively.

Proof. The exactness of the rows of (3.1) follows from Proposition 3.6. If $e \in E$, $w \in W_e$, we have

$$\varphi \circ (\mu_t(w)(a)) = \varphi(tw + a) = t\psi(w) + \varphi(a) = \mu'_t(\psi(w))(\varphi(a))$$

for all $a \in A_e$. The proposition is an easy consequence of this identity.

4. Jet bundles as fibered manifolds

Let s be a section of the fibered manifold E over a neighborhood of $x \in X$. The equivalence class of all sections of E agreeing with s to order k at x is called the k -jet of s at x and is denoted by $j_k(s)(x)$. The point $x \in X$ is called the source of the jet $j_k(s)(x)$, and $s(x)$ its target. Let $J_k(E)_x$ denote the set of all k -jets at x of sections of E and let

$$J_k(E) = \bigcup_{x \in X} J_k(E)_x .$$

We call $J_k(E)$ the bundle of k -jets of E . For $k \geq l$, we denote by $\pi_l : J_k(E) \rightarrow J_l(E)$ the map sending a k -jet into the l -jet it determines. In particular, the map $\pi_0 : J_k(E) \rightarrow E$ sends a k -jet into its target, so we shall identify $J_0(E)$ with E .

Proposition 4.1. *There exists a natural differentiable manifold structure on $J_k(E)$. In fact, $J_k(E)$ is a fibered manifold over X whose projection $\pi : J_k(E) \rightarrow X$ sends a k -jet into its source; moreover $\pi_l : J_k(E) \rightarrow J_l(E)$ is an epimorphism of fibered manifolds over X , for $k \geq l$.*

If s is a section of E over an open set $U \subset X$, the map $y \mapsto j_k(s)(y)$ of U into $J_k(E)$ is a section $j_k(s)$ of $J_k(E)$ over U , and the map $s \mapsto j_k(s)$ induces a morphism of sheaves $j_k : \mathcal{E} \rightarrow \mathcal{J}_k(\mathcal{E})$.

Note that the fibered manifold $J_k(O_X)$ is isomorphic to O_X and will be identified with O_X , and that the fibered manifold $J_k(E \times_X E')$ is naturally isomorphic to $J_k(E) \times_X J_k(E')$. If E is a vector bundle, then $J_k(E)$ has the structure of a vector bundle over X . If U is an open subset of X , then the restriction $J_k(E)|_U$ of $J_k(E)$ to U is isomorphic to the bundle $J_k(E|_U)$ of k -jets of $E|_U$. Furthermore, if E is the trivial fibered manifold over X with fiber a differentiable manifold Y , then $J_k(E)$ is diffeomorphic to the manifold of k -jets of (local) differentiable maps of X into Y in the sense of Ehresmann.

Let $\varphi : J_k(E) \rightarrow E'$ be a morphism of fibered manifolds over X . A sheaf morphism of the form $\varphi \circ j_k : \mathcal{E} \rightarrow \mathcal{E}'$ is called a differential operator from E to E' of order k . The l -th prolongation $p_l(\varphi) : J_{k+l}(E) \rightarrow J_l(E')$ of φ is defined to be the unique morphism of fibered manifolds over X such that the diagram

$$\begin{array}{ccc}
 \mathcal{J}_{k+l}(\mathcal{E}) & \xrightarrow{p_l(\varphi)} & \mathcal{J}_l(\mathcal{E}') \\
 \uparrow j_{k+l} & & \uparrow j_l \\
 \mathcal{E} & \xrightarrow{\varphi \circ j_k} & \mathcal{E}'
 \end{array}$$

commutes. In particular, the l -th prolongation of the identity map id_k of $J_k(E)$

is a monomorphism of fibered manifolds $p_i(\text{id}_k): J_{k+i}(E) \rightarrow J_i(J_k(E))$. We shall identify $J_{k+i}(E)$ with its image in $J_i(J_k(E))$ under the natural monomorphism $p_i(\text{id}_k)$. The map φ also induces a morphism of fibered manifolds $J_i(\varphi): J_i(J_k(E)) \rightarrow J_i(E')$ such that the diagram

$$\begin{array}{ccc}
 \mathcal{F}_i(\mathcal{F}_k(\mathcal{E})) & \xrightarrow{\mathcal{F}_i(\varphi)} & \mathcal{F}_i(\mathcal{E}') \\
 \uparrow i_i & & \uparrow i_i \\
 \mathcal{F}_k(E) & \xrightarrow{\varphi} & E'
 \end{array}$$

commutes, where $\mathcal{F}_i(\varphi)$ is the map of sheaves induced by $J_i(\varphi)$.

Proposition 4.2. *If $\varphi: J_k(E) \rightarrow E'$, $\varphi': J_i(E') \rightarrow E''$ are morphisms of fibered manifolds over X , then*

$$p_m(\varphi' \circ p_i(\varphi)) = p_m(\varphi') \circ p_{i+m}(\varphi), \quad \text{for } m \geq 0.$$

The proposition follows directly from the definitions of the maps involved.

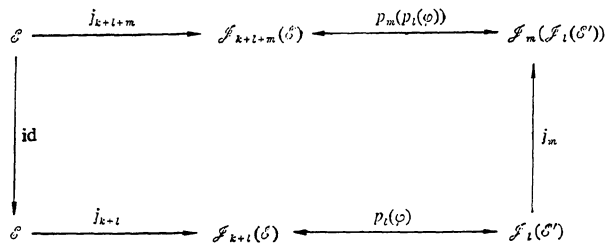
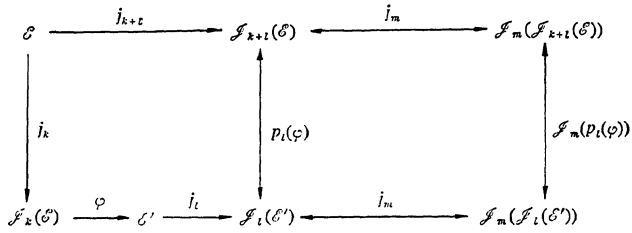
Proposition 4.3. *If $\varphi: J_k(E) \rightarrow E'$ is a morphism of fibered manifolds, then the diagram*

$$\begin{array}{ccc}
 J_{k+i+m}(E) & \xrightarrow{p_{i+m}(\varphi)} & J_{i+m}(E') \\
 \downarrow p_m(\text{id}_{k+i}) & \searrow p_m(p_i(\varphi)) & \downarrow p_m(\text{id}_i) \\
 J_m(J_{k+i}(E)) & \xrightarrow{J_m(p_i(\varphi))} & J_m(J_i(E'))
 \end{array}$$

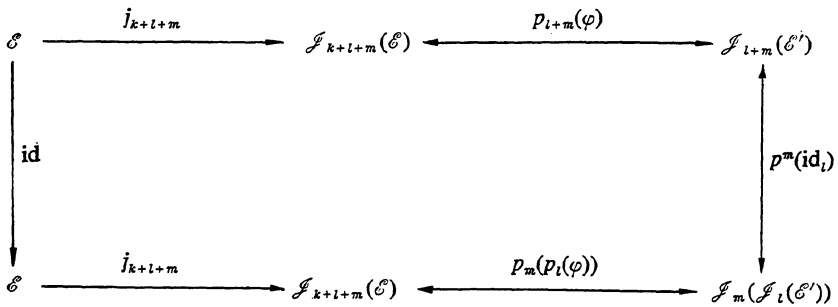
commutes.

Proof. It follows directly from the definitions of the maps involved that the following diagrams commute:

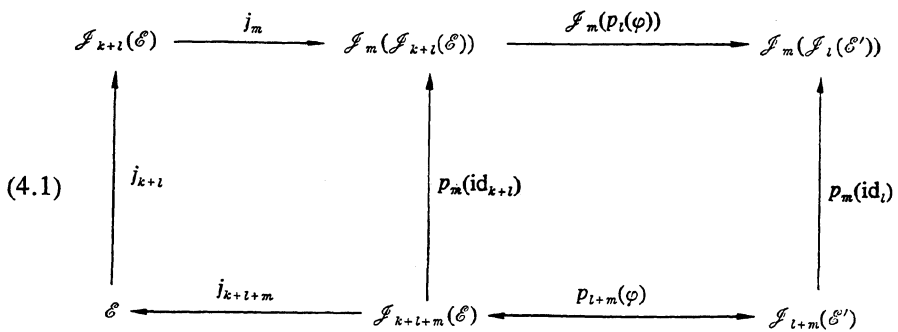
$$\begin{array}{ccccccc}
 \mathcal{E} & \xrightarrow{i_{k+i+m}} & \mathcal{F}_{k+i+m}(\mathcal{E}) & \xrightarrow{p_{i+m}(\varphi)} & \mathcal{F}_{i+m}(\mathcal{E}') & \xrightarrow{p^m(\text{id}_i)} & \mathcal{F}_m(\mathcal{F}_i(\mathcal{E}')) \\
 \downarrow \text{id} & & & & \uparrow i_{i+m} & & \uparrow i_m \\
 \mathcal{E} & \xrightarrow{i_k} & \mathcal{F}_k(\mathcal{E}) & \xrightarrow{\varphi} & \mathcal{E}' & \xrightarrow{i_i} & \mathcal{F}_i(\mathcal{E}')
 \end{array}$$



Therefore the diagram



and the outer diagram of



also commute. However the first square of (4.1) commutes by the definition of the map $p_m(\text{id}_{k+l})$; hence

$$\begin{aligned} p_m(p_l(\varphi)) \circ j_{k+l+m} &= p_m(\text{id}_l) \circ p_{l+m}(\varphi) \circ j_{k+l+m} \\ &= \mathcal{J}_m(p_l(\varphi)) \circ p_m(\text{id}_{k+l}) \circ j_{k+l+m} \end{aligned}$$

as maps from \mathcal{E} to $\mathcal{J}_m(\mathcal{J}_l(\mathcal{E}'))$ and so the proposition is established.

If F is an open fibered submanifold of $J_k(E)$ and $\varphi : F \rightarrow E'$ is a morphism of fibered manifolds over X , then one can define the l -th prolongation $p_l(\varphi)$ as a morphism of fibered manifolds from the open fibered submanifold $\pi_k^{-1}F$ of $J_{k+l}(E)$ to $J_l(E')$ as follows. It is the unique morphism such that

$$p_l(\varphi)(j_{k+l}(s)(x)) = j_l(\varphi \circ j_k(s))(x)$$

for all sections s of E over a neighborhood of $x \in X$ satisfying $j_k(s)(x) \in F$. Note that $p_l(\varphi)$ is well-defined, since, for any such section s of E , it follows, from the fact that F is an open subset of $J_k(E)$, that $j_k(s)(y)$ belongs to F for all y in a neighborhood of x . It is easily seen that the analogues of Propositions 4.2, 4.3 hold for such φ .

Proposition 4.4. *Let $\pi_1 : E_1 \rightarrow E$, $\pi'_1 : E'_1 \rightarrow E$ be fibered manifolds over E and let $\varphi : E_1 \rightarrow E'_1$ be a morphism of fibered manifolds over E of locally constant rank. Then $p_k(\varphi) : J_k(E_1) \rightarrow J_k(E'_1)$ is a morphism over $J_k(E)$ of locally constant rank. If φ is a monomorphism (respectively an epimorphism) over E , then $p_k(\varphi)$ is a monomorphism (respectively an epimorphism) over $J_k(E)$. If s'_1 is a section of E'_1 over E satisfying $s'_1(E) \subset \varphi(E_1)$, then $p_k(s'_1)(J_k(E)) \subset p_k(\varphi)(J_k(E_1))$ and the sequence*

$$O_{J_k(E)} \longrightarrow J_k(\text{Ker}_{s'_1} \varphi) \xrightarrow{p_k(i)} J_k(E_1) \xrightarrow{p_k(\varphi)} J_k(E'_1)$$

is exact and $p_k(\varphi) \circ p_k(i) = p_k(s'_1) \circ p_k(\pi_1)$, where i is the inclusion of the fibered manifold $\text{Ker}_{s'_1} \varphi$ over E into E_1 .

Proof. The proposition is a consequence of Propositions 4.1 and 2.1.

If we set $E = O_X$ in the above proposition, then $p_k(s'_1) = j_k(s'_1)$ and we obtain

Corollary 4.1. *If $\varphi : E \rightarrow E'$ is a morphism of fibered manifolds of locally constant rank, then so is $p_k(\varphi) : J_k(E) \rightarrow J_k(E')$. If φ is a monomorphism (respectively an epimorphism) so is $p_k(\varphi)$. If s' is a section of E' over X satisfying $s'(X) \subset \varphi(E)$, then $j_k(s')(X) \subset p_k(\varphi)(J_k(E))$ and the sequence*

$$O_X \longrightarrow J_k(\text{Ker}_{s'} \varphi) \xrightarrow{p_k(i)} J_k(E) \xrightarrow{p_k(\varphi)} J_k(E')$$

is exact and $p_k(\varphi) \circ p_k(i) = j_k(s') \circ \pi$, where i is the inclusion of the fibered manifold $\text{Ker}_{s'} \varphi$ into E .

Proposition 4.5. *Let $\pi' : E' \rightarrow X$ be a fibered submanifold of $\pi : E \rightarrow X$ and let $\rho : M \rightarrow E$ be a fibered manifold. Then, for $k \geq 0$, considering $J_k(E')$ as a subset of $J_k(E)$, we have*

$$J_k(\rho^{-1}(E')) = (J_k(\rho))^{-1}J_k(E')$$

as submanifolds of $J_k(M)$.

Proof. This proposition follows from the implicit function theorem and Proposition 4.1.

5. Jet bundles as affine bundles

Let \mathcal{O}_x denote the ring of germs f_x at $x \in X$ of real-valued differentiable functions f defined on a neighborhood of x and let \mathcal{M}_x denote its unique maximal ideal.

Lemma 5.1. *Let s be a section of E over a neighborhood U of $x_0 \in X$ and let f be any real-valued function on U , with $f_{x_0} \in \mathcal{M}_{x_0}^k$, where $k \geq 1$. Suppose that \tilde{s} is any deformation of the section s , which is a map $\tilde{s} : U \times (-\varepsilon, \varepsilon) \rightarrow E$ satisfying the following conditions:*

- (i) $\tilde{s}(x, 0) = s(x)$, for all $x \in U$;
- (ii) $\pi \circ \tilde{s} = pr_1$, where pr_1 is the projection of $U \times (-\varepsilon, \varepsilon)$ onto U .

Then the k -jet at x_0 of the section

$$(5.1) \quad x \mapsto \tilde{s}(x, f(x))$$

of E over some neighborhood of x_0 depends only on $j_k(s)(x_0)$, on the class of f_{x_0} in $\mathcal{M}_{x_0}^k/\mathcal{M}_{x_0}^{k+1}$, and on the tangent vector $\left. \frac{d\tilde{s}(x_0, t)}{dt} \right|_{t=0} \in F_{s(x_0)}(E)$ to the deformation \tilde{s} at x_0 .

We leave the proof of this lemma to the reader. If $v_0 = \left. \frac{d\tilde{s}(x_0, t)}{dt} \right|_{t=0}$ and \bar{f}_{x_0} is the class of f_{x_0} in $\mathcal{M}_{x_0}^k/\mathcal{M}_{x_0}^{k+1}$, we denote the k -jet at x_0 of the section (5.1) of E by

$$(\bar{f}_{x_0}, v_0) + j_k(s)(x_0).$$

From Lemma 5.1, it follows immediately that the $(k - 1)$ -jet at x_0 of the section (5.1) of E is $j_{k-1}(s)(x_0)$ and that

$$(0, v_0) + j_k(s)(x_0) = j_k(s)(x_0).$$

Remark 5.1. For each $\lambda \in \mathbf{R}$, we have a deformation \tilde{s}_λ of s by setting

$$\tilde{s}_\lambda(x, t) = \tilde{s}(x, \lambda t), \quad \text{for } |\lambda t| < \varepsilon, \quad x \in U,$$

such that

$$\tilde{s}(x, \lambda f(x)) = \tilde{s}_i(x, f(x))$$

for x in some neighborhood of x_0 . Hence

$$(5.2) \quad (\lambda \bar{f}_{x_0}, v_0) + j_k(s)(x_0) = (\bar{f}_{x_0}, \lambda v_0) + j_k(s)(x_0).$$

Remark 5.2. If g is another real-valued function on U ,

$$(5.3) \quad (\bar{g}_{x_0} + \bar{f}_{x_0}, v_0) + j_k(s)(x_0) = (\bar{g}_{x_0}, v_0) + ((\bar{f}_{x_0}, v_0) + j_k(s)(x_0)).$$

Indeed, let

$$\tilde{\tilde{s}}(x, t) = \tilde{s}(x, f(x) + t)$$

for x in some neighborhood of x_0 and t in some neighborhood of $0 \in \mathbf{R}$. It is clear that $\tilde{\tilde{s}}$ is a deformation of the section (5.1). The tangent vector to the deformation $\tilde{\tilde{s}}$ at x_0 is v_0 because $f(x_0) = 0$. Hence the right-hand side of (5.3) is the k -jet at x_0 of the section

$$x \mapsto \tilde{\tilde{s}}(x, g(x)) = \tilde{s}(x, (g + f)(x))$$

of E over some neighborhood of x_0 , which is precisely the left-hand side of (5.3).

Remark 5.3. If $v_0, w_0 \in F_{s(x_0)}(E)$, let v, w denote vector fields tangent to the fibers of E defined on some neighborhood \bar{U} of $s(x_0)$ such that $v(s(x_0)) = v_0, w(s(x_0)) = w_0$ and $[v, w] \equiv 0$ on \bar{U} . Then, denoting by $\exp tv, \exp tw, \exp t(w + v)$, the one-parameter family of (local) diffeomorphisms of E generated by $v, w, w + v$ respectively, we have

$$\exp t(w + v) = (\exp tw) \circ (\exp tv)$$

for all sufficiently small t . By applying Lemma 5.1 to the deformation \tilde{s} of s defined by

$$\tilde{s}(x, t) = \exp t(w + v)(s(x)) = \exp tw(\exp tv(s(x))),$$

we obtain the identity

$$(5.4) \quad (\bar{f}_{x_0}, w_0 + v_0) + j_k(s)(x_0) = (\bar{f}_{x_0}, w_0) + ((\bar{f}_{x_0}, v_0) + j_k(s)(x_0))$$

for all $v_0, w_0 \in F_{s(x_0)}(E)$.

Remark 5.4. If $\varphi: E \rightarrow E'$ is a morphism of fibered manifolds over X , then $\varphi \circ \tilde{s}$ is a deformation of the section $\varphi \circ s$ of E' over U . It is clear that

$$(5.5) \quad p_k(\varphi)((\bar{f}_{x_0}, v_0) + j_k(s)(x_0)) = (\bar{f}_{x_0}, \varphi_* v_0) + j_k(\varphi \circ s)(x_0)$$

because $\left. \frac{d(\varphi \circ \tilde{\gamma})(x_0, t)}{dt} \right|_{t=0} = \varphi_* v_0$ and the left-hand side of (5.5) is the k -jet at x_0 of the section

$$x \mapsto \varphi(\tilde{\gamma}(x, f(x))) = (\varphi \circ \tilde{\gamma})(x, f(x))$$

of E' over some neighborhood of x_0 .

We denote by π_0 and π the natural projections of the vector bundle $S^k T^* \otimes_E F(E)$ onto E and X respectively. If $\varphi: E \rightarrow E'$ is a morphism of fibered manifolds over X , then $\varphi_*: F(E) \rightarrow F(E')$ induces a morphism of vector bundles from $S^k T^* \otimes_E F(E)$ to $S^k T^* \otimes_{E'} F(E')$ over φ which we shall also denote by φ_* .

Proposition 5.1. *There is a natural morphism of fibered manifolds over $J_{k-1}(E)$, for $k \geq 1$,*

$$S^k T^* \otimes_E F(E) \times_E J_k(E) \rightarrow J_k(E)$$

sending (a, p) into $a + p$ which induces on $J_k(E)$ the structure of an affine bundle over $J_{k-1}(E)$. In fact, if $\varphi: E \rightarrow E'$ is a morphism of fibered manifolds over X the following diagram is commutative:

$$\begin{array}{ccc} S^k T^* \otimes_E F(E) \times_E J_k(E) & \xrightarrow{\quad\quad\quad} & J_k(E) \\ \downarrow \varphi_* \times p_k(\varphi) & & \downarrow p_k(\varphi) \\ S^k T^* \otimes_{E'} F(E') \times_{E'} J_k(E') & \xleftarrow{\quad\quad\quad} & J_k(E') \end{array}$$

Proof. Since $\mathcal{M}_{x_0}^k / \mathcal{M}_{x_0}^{k+1}$ can be canonically identified with $S^k T_{x_0}^*$, for $x_0 \in X$, Lemma 5.1 together with Remarks 5.1, 5.2 and 5.3 defines a mapping, for each $e \in E$, from $S^k T_{\pi(e)}^* \otimes F_e(E) \times J_k(E)_e$ to $J_k(E)_e$ sending $(\tilde{f} \otimes v, p)$ into $\tilde{f} \otimes v + p = (\tilde{f}, v) + p$. This mapping determines a morphism of fibered manifolds over $J_{k-1}(E)$ from $S^k T^* \otimes_E F(E) \times_E J_k(E)$ to $J_k(E)$ sending (a, p) into $a + p$, since the induced action of $S^k T^* \otimes_E F(E)$ on $J_k(E)$ preserves the fibers of $J_k(E)$ over $J_{k-1}(E)$. It is easily seen that this morphism endows $J_k(E)$ with the structure of an affine bundle over $J_{k-1}(E)$ modeled on the vector bundle $S^k T^* \otimes_{J_{k-1}(E)} F(E)$. The naturality of this morphism follows from Remark 5.4.

If $p_1, p_2 \in J_k(E)$ satisfy $\pi_{k-1}(p_1) = \pi_{k-1}(p_2)$, we denote by $p_1 - p_2$ the unique element of $S^k T^* \otimes_E F(E)$ such that

$$(p_1 - p_2) + p_2 = p_1 .$$

From Proposition 5.1 and Proposition 3.7, we obtain

Proposition 5.2. *The sequence of vector bundles over $J_k(E)$*

$$0 \rightarrow S^k T^* \otimes_{J_k(E)} F(E) \xrightarrow{\mu} F(J_k(E)) \xrightarrow{\pi_{k-1}^{-1}} \pi_{k-1}^{-1} F(J_{k-1}(E)) \rightarrow 0$$

is exact.

We shall identify $S^k T^* \otimes_{J_k(E)} F(E)$ with its image in $F(J_k(E))$ under the map μ .

Proposition 5.3. *Let $\xi: W \rightarrow E$ be a vector bundle over E . Then:*

- (i) $J_k(\xi): J_k(W) \rightarrow J_k(E)$ is a vector bundle.
- (ii) There is a natural morphism

$$(5.6) \quad S^k T^* \otimes_E W \times_E J_k(W) \rightarrow J_k(W)$$

of fibered manifolds over $Y = J_k(E) \times_{J_{k-1}(E)} J_{k-1}(W)$, sending (a, q) into $a + q$, induced by the morphism

$$S^k T^* \otimes_W F(W) \times_W J_k(W) \rightarrow J_k(W)$$

of Proposition 5.1.

- (iii) There is a morphism of vector bundles over $J_k(E)$

$$\varepsilon: S^k T^* \otimes_{J_k(E)} W \rightarrow J_k(W)$$

induced by the morphism (5.6).

- (iv) The sequence of vector bundles over $J_k(E)$

$$(5.7) \quad 0 \rightarrow S^k T^* \otimes_{J_k(E)} W \xrightarrow{\varepsilon} J_k(W) \xrightarrow{\eta^{k-1}} \pi_{k-1}^{-1} J_{k-1}(W) \rightarrow 0$$

is exact, where $\eta_l: J_k(W) \rightarrow J_l(W)$, with $l \leq k$, and $\eta = \pi \circ \xi: W \rightarrow X$ are the natural projections.

(v) The morphism (5.6) induces on $J_k(W)$ the structure of an affine bundle over Y modeled on the vector bundle $S^k T^* \otimes_Y W$.

Proof. (i) If s_1, s_2 are sections of W over $U \subset X$, satisfying $\xi \circ s_1 = \xi \circ s_2$, and f is a real-valued function defined on U , then $s_1 + s_2: x \mapsto s_1(x) + s_2(x)$, $f \cdot s_1: x \mapsto f(x)s_1(x)$ are well-defined sections of W over U satisfying $\xi \circ (s_1 + s_2) = \xi \circ s_1 = \xi \circ (f \cdot s_1)$. Then $J_k(W)$ becomes a vector bundle over $J_k(E)$ by setting

$$\begin{aligned} j_k(s_1)(x) + j_k(s_2)(x) &= j_k(s_1 + s_2)(x) \\ \lambda j_k(s_1)(x) &= j_k(\lambda \cdot s_1)(x), \end{aligned}$$

if $\lambda \in \mathbf{R}$, $x \in U$. The zero section of this vector bundle is precisely $J_k(0)$, if 0 denotes the zero section of the vector bundle W over E .

(ii) The inclusion μ of $\xi^{-1}W$ into $F(W)$ determines a map $W \times_E J_k(W)$ into $F(W) \times_W J_k(W)$ sending (w, q) into $(\mu(\eta_0(q), w), q)$ and hence also a map

from $S^k T^* \otimes_E W \times_E J_k(W)$ to $S^k T^* \otimes_W F(W) \times_W J_k(W)$. We define the map (5.6) to be the composition of this map from $S^k T^* \otimes_E W \times_E J_k(W)$ to $S^k T^* \otimes_W F(W) \times_W J_k(W)$ and of the morphism of Proposition 5.1 from $S^k T^* \otimes_W F(W) \times_W J_k(W)$ to $J_k(W)$.

The morphism (5.6) can be computed explicitly in terms of the vector bundle structure on $J_k(W)$ as follows:

Lemma 5.2. *Let p be an element of $J_k(E)$, with $\pi_0(p) = e \in E$ and $\pi(p) = x_0 \in X$, and let $w \in W$ satisfy $\xi(w) = e$. Let f be any real-valued function defined on a neighborhood U of x_0 , with $f_{x_0} \in \mathcal{M}_{x_0}^k$, and s, u be any sections of W over U satisfying $j_k(\xi \circ s)(x_0) = j_k(\xi \circ u)(x_0) = p$ and $s(x_0) = w$. Then*

$$(5.8) \quad \bar{f}_{x_0} \otimes w + j_k(u)(x_0) = j_k(f \cdot s + u)(x_0) ,$$

where \bar{f}_{x_0} is the class of f_{x_0} in $\mathcal{M}_{x_0}^k / \mathcal{M}_{x_0}^{k+1}$.

Proof. The map $\tilde{u}: U \times \mathbf{R} \rightarrow W$ defined by

$$\tilde{u}(x, t) = u(x) + ts(x) , \quad x \in U, t \in \mathbf{R},$$

is a deformation of u , and the tangent vector at x_0 to this deformation is clearly $\mu(u(x_0), w) \in F_{u(x_0)}(W)$. Now both sides of equation (5.8) are by definition the k -jet at x_0 of the section

$$x \mapsto \tilde{u}(x, f(x)) = u(x) + f(x)s(x)$$

of W over U .

We now return to the proof of Proposition 5.3.

From Lemma 5.2, it follows directly that the map (5.6) is a morphism of fibered manifolds over Y .

Let $\xi: S^k T^* \otimes_E W \rightarrow E$ be the natural projection induced by $\xi: W \rightarrow E$. From Lemma 5.2, we obtain

Corollary 5.1. *For all $a \in S^k T^* \otimes_E W, q_1, q_2 \in J_k(W)$ satisfying $J_k(\xi)(q_1) = J_k(\xi)(q_2) = p \in J_k(E)$ and $\xi(a) = \pi_0(p)$, we have the formulas:*

$$(5.9) \quad \begin{aligned} a + (q_1 + q_2) &= (a + q_1) + q_2 \\ \lambda a + \lambda q_1 &= \lambda(a + q_1) , \end{aligned} \quad \text{for all } \lambda \in \mathbf{R} .$$

(iii) Define the morphism ε of fibered manifolds over $J_k(E)$ by

$$\varepsilon(p, a) = a + J_k(0)(p) ,$$

for all $p \in J_k(E), a \in S^k T^* \otimes_E W$, satisfying $\pi_0(p) = \xi(a)$. From (5.9), it follows that ε is a morphism of vector bundles over $J_k(E)$.

(iv) Because $\mu: \xi^{-1}W \rightarrow F(W)$ is injective, Proposition 5.1 implies that ε is injective and is therefore a monomorphism of vector bundles over $J_k(E)$.

Moreover, it is clear that $\eta_{k-1} \circ \varepsilon = 0$. Since η_{k-1} is an epimorphism, counting the dimensions of the vector bundles $S^k T^* \otimes_{J_k(E)} W$, $J_k(W)$ and $\pi_{k-1}^{-1} J_{k-1}(W)$ over $J_k(E)$, we conclude that the sequence (5.7) is exact.

If $q \in J_k(W)$ satisfies $J_k(\xi)(q) = p \in J_k(E)$ and $\eta_{k-1}(q) = J_{k-1}(0)(p)$, we denote by $\varepsilon^{-1}q$ the unique element of $S^k T^* \otimes_E W$ such that

$$\varepsilon(p, \varepsilon^{-1}q) = q ,$$

which exists and is unique by the exactness of (5.7).

(v) To verify that the morphism (5.6) induces on $J_k(W)$ the structure of an affine bundle over Y , it is sufficient to prove

Lemma 5.3. *Given $q_1, q_2 \in J_k(W)$ satisfying $J_k(\xi)(q_1) = J_k(\xi)(q_2) = p \in J_k(E)$ and $\eta_{k-1}(q_1) = \eta_{k-1}(q_2)$, there exists a unique element $a \in S^k T^* \otimes_E W$ such that*

$$a + q_2 = q_1 .$$

In fact, a is the unique element $\varepsilon^{-1}(q_1 - q_2)$ of $S^k T^ \otimes_E W$ which satisfies*

$$(5.10) \quad \varepsilon(p, \varepsilon^{-1}(q_1 - q_2)) = q_1 - q_2 ,$$

where $q_1 - q_2$ is the element of $J_k(W)$ determined by the vector bundle structure of $J_k(E)$.

Proof. Equation (5.10) is equivalent to

$$(5.11) \quad \varepsilon^{-1}(q_1 - q_2) + q_2 = q_1 .$$

Indeed, we can rewrite (5.10) as

$$\varepsilon^{-1}(q_1 - q_2) + J_k(0)(p) = q_1 - q_2 .$$

The above equation is equivalent to

$$(\varepsilon^{-1}(q_1 - q_2) + J_k(0)(p)) + q_2 = q_1 ,$$

which, by Corollary 5.1, is the same as

$$\varepsilon^{-1}(q_1 - q_2) + (J_k(0)(p) + q_2) = q_1$$

or equation (5.11).

If we set $E = O_X$ in the above proposition, we obtain the exact sequence of vector bundles over X

$$0 \rightarrow S^k T^* \otimes W \xrightarrow{\varepsilon} J_k(W) \xrightarrow{\xi_{k-1}} J_{k-1}(W) \rightarrow 0$$

for any vector bundle $\xi: W \rightarrow X$ over X .

Proposition 5.4. *If $\varphi: E \rightarrow E'$ is a morphism of fibered manifolds over X , then, for $k \geq 1$, the map $p_k(\varphi): J_k(E) \rightarrow J_k(E')$ is a morphism of affine*

bundles over $p_{k-1}(\varphi)$ whose associated morphism of vector bundles is $\varphi_*: S^k T^* \otimes_{J_k(E)} F(E) \rightarrow S^k T^* \otimes_{J_k(E')} F(E')$.

This proposition is a direct consequence of Proposition 5.1.

Proposition 5.5. For $l \geq 1$, the map $p_l(\text{id}_k): J_{k+l}(E) \rightarrow J_l(J_k(E))$ is a monomorphism of affine bundles over $p_{l-1}(\text{id}_k): J_{k+l-1}(E) \rightarrow J_{l-1}(J_k(E))$.

Proof. The diagram

$$\begin{array}{ccc}
 J_{k+l}(E) & \xrightarrow{p_l(\text{id}_k)} & J_l(J_k(E)) \\
 \uparrow \pi_{k+l-1} & & \uparrow \pi_{l-1} \\
 J_{k+l-1}(E) & \xrightarrow{p_{l-1}(\text{id}_k)} & J_{l-1}(J_k(E))
 \end{array}$$

is easily seen to commute, so that $p_l(\text{id}_k)$ is a morphism of fibered manifolds over $p_{l-1}(\text{id}_k)$ which is easily seen to be a monomorphism of affine bundles.

Define a monomorphism $\Delta_{l,k}: S^{k+l} T^* \rightarrow S^l T^* \otimes S^k T^*$ of vector bundles over X as follows: let $\Delta_{l,k}$ be the composition of the natural inclusion of $S^{k+l} T^*$ into $\otimes^{k+l} T^* = \otimes^l T^* \otimes \otimes^k T^*$ and the map from $\otimes^l T^* \otimes \otimes^k T^*$ to $S^l T^* \otimes S^k T^*$ induced by the natural projections of $\otimes^l T^*$, $\otimes^k T^*$ onto $S^l T^*$, $S^k T^*$. If $l = 1$, we set $\delta = \delta_k = \Delta_{1,k}$.

By Proposition 4.3, the following diagram commutes:

$$\begin{array}{ccccc}
 J_{k+l}(E) & \xleftarrow{p_l(\text{id}_k)} & J_l(J_k(E)) & \xleftarrow{J_l(\pi_{k-1})} & J_l(J_{k-1}(E)) \\
 \uparrow \pi_{k+l-1} & & \uparrow \pi_{l-1} & \nearrow p_l(\text{id}_{k-1}) & \uparrow \pi_{l-1} \\
 J_{k+l-1}(E) & \xleftarrow{p_{l-1}(\text{id}_k)} & J_{l-1}(J_k(E)) & \xleftarrow{J_{l-1}(\pi_{k-1})} & J_{l-1}(J_{k-1}(E))
 \end{array}$$

By Propositions 5.4. and 5.5, the composition $J_l(\pi_{k-1}) \circ p_l(\text{id}_k)$ is a morphism of affine bundles over $J_{l-1}(\pi_{k-1}) \circ p_{l-1}(\text{id}_k)$ whose associated morphism of vector bundles is the composition

$$\begin{aligned}
 S^{k+l} T^* \otimes_{J_{k+l-1}(E)} F(E) &\rightarrow S^l T^* \otimes_{J_{l-1}(J_k(E))} F(J_k(E)) \\
 &\xrightarrow{\pi_{k-1}^*} S^l T^* \otimes_{J_{l-1}(J_{k-1}(E))} F(J_{k-1}(E))
 \end{aligned}$$

where the map from $S^{k+l} T^* \otimes_{J_{k+l-1}(E)} F(E)$ to $S^l T^* \otimes_{J_{l-1}(J_k(E))} F(J_k(E))$ is the morphism of vector bundles associated to $p_l(\text{id}_k)$. Since the morphism $J_l(\pi_{k-1}) \circ p_l(\text{id}_k)$ factors through $J_{k+l-1}(E)$, its associated morphism of vector bundles is the zero morphism over $J_{l-1}(\pi_{k-1}) \circ p_{l-1}(\text{id}_k)$. Therefore, by

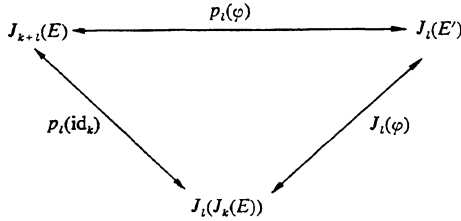
Proposition 5.2, the morphism of vector bundles associated to $p_l(\text{id}_k)$ is determined by a morphism of vector bundles from $S^{k+l}T^* \otimes_{J_{k+l-1}(E)} F(E)$ to $S^l T^* \otimes_{J_{l-1}(J_k(E))} F(E)$ over $p_{l-1}(\text{id}_k)$, which is easily seen to be precisely the morphism induced by $\Delta_{l,k}$.

Proposition 5.6. *If $\varphi: J_k(E) \rightarrow E'$ is a morphism of fibered manifolds over X , then, for $l \geq 1$, the map $p_l(\varphi): J_{k+l}(E) \rightarrow J_l(E')$ is a morphism of affine bundles over $p_{l-1}(\varphi)$, whose associated morphism of vector bundles is induced by the map*

$$\varphi_* \circ \Delta_{l,k}: S^{k+l}T^* \otimes_{J_k(E)} F(E) \rightarrow S^l T^* \otimes_{E'} F(E')$$

over φ .

Proof. The diagram



commutes by Proposition 4.3. Hence by Propositions 5.4 and 5.5, $p_l(\varphi)$ is a morphism of affine bundles whose associated morphism of vector bundles is the map

$$\varphi_* \circ \Delta_{l,k}: S^{k+l}T^* \otimes_{J_{k+l-1}(E)} F(E) \rightarrow S^l T^* \otimes_{J_{l-1}(E')} F(E')$$

over $p_{l-1}(\varphi)$.

If φ is a morphism of an open fibered submanifold F of $J_k(E)$ into E' , analogues of Propositions 5.4, 5.6 hold for φ .

We now conclude this section by giving an alternate description of the affine bundle structure which $J_k(E)$ possesses.

Let p be an element of $J_k(E)$ and let s be a section of E over a neighborhood of $x = \pi(p)$ such that $j_k(s)(x) = p$; set $q = \pi_{k-1}(p)$. Then the linear map $j_{k-1}(s)_*: T_x \rightarrow Tq(J_{k-1}(E))$ depends only on p and will be denoted simply by p_* . In fact, p_* determines p uniquely; indeed, if p' is an element of $J_k(E)$ such that $\pi_{k-1}(p') = q$ and $p'_* = p_*$ as maps from T_x to $Tq(J_{k-1}(E))$, then $p' = p$. In general, if $p' \in J_k(E)$ satisfies $\pi_{k-1}(p') = q$, then $\pi_{k-1} p_* = \pi_{k-1} p'_*$ and so $p'_* - p_*$ belongs to $T^* \otimes S^{k-1}T^* \otimes_{J_{k-1}(E)} F(E)$ by Proposition 5.2. The map from $J_k(E)q$ to $(T^* \otimes S^{k-1}T^* \otimes_{J_{k-1}(E)} F(E))q$ sending p' into $p'_* - p_*$ is injective. Actually $p'_* - p_*$ belongs to the subspace $(\delta(S^k T^*) \otimes_{J_{k-1}(E)} F(E))q$ of $(T^* \otimes S^{k-1}T^* \otimes_{J_{k-1}(E)} F(E))q$ and every element of this subspace is of the form $p'_* - p_*$, for some $p' \in J_k(E)q$. Since δ is injective, we can identify $S^k T^*$

with its image in $T^* \otimes S^{k-1}T^*$ under the map δ . Hence p determines a bijective map from $J_k(E)q$ to $(S^kT^* \otimes_{J_{k-1}(E)} F(E))q$. It is easily seen that this map endows $J_k(E)q$ with the structure of an affine space modeled on the vector space $(S^kT^* \otimes_{J_{k-1}(E)} F(E))q$. This gives rise to a morphism of fibered manifolds over $J_{k-1}(E)$

$$S^kT^* \otimes_E F(E) \times_E J_k(E) \rightarrow J_k(E)$$

sending (a, p) into the unique element p' of $J_k(E)$ such that $p'_* = \delta(\pi_{k-1}(p), a) + p_*$. It is readily verified that this morphism is precisely the morphism of Proposition 5.1.

6. The Spencer cohomology

We shall henceforth assume that all the components of the differentiable manifold X have the same dimension n .

We have a morphism of vector bundles over X

$$\delta : S^kT^* \longrightarrow T^* \otimes S^{k-1}T^*$$

defined in §5. We extend δ to a vector bundle morphism

$$\delta : \wedge^j T^* \otimes S^k T^* \longrightarrow \wedge^{j+1} T^* \otimes S^{k-1} T^*$$

sending $\omega \otimes u$ into $(-1)^j \omega \wedge \delta u$, where $\omega \in \wedge^j T^*$, $u \in S^k T^*$. Then we have the complex

$$(6.1) \quad 0 \longrightarrow S^k T^* \xrightarrow{\delta} T^* \otimes S^{k-1} T^* \xrightarrow{\delta} \wedge^2 T^* \otimes S^{k-2} T^* \xrightarrow{\delta} \dots \\ \longrightarrow \wedge^n T^* \otimes S^{k-n} T^* \longrightarrow 0$$

(i.e., $\delta^2 = 0$), where $S^l T^* = 0$ for $l < 0$. We call δ the formal differentiation operator and we now state the formal Poincaré lemma:

Lemma 6.1 (see D. C. Spencer [8], D. G. Quillen [7], or S. Sternberg [9]). *The sequences (6.1) are exact for $k \geq 1$.*

Let Y be a differentiable manifold and let $f: Y \rightarrow X$ be a differentiable map. Let W be a vector bundle over Y . Suppose that g_k is a family of subspaces of $S^k T^* \otimes_Y W$ over Y , where $k \geq 1$. Define the l -th prolongation g_{k+l} of g_k to be the kernel of the composition ϕ_l :

$$S^{k+l} T^* \otimes_Y W \xrightarrow{\Delta_{l,k}} S^l T^* \otimes S^k T^* \otimes_Y W \xrightarrow{\phi} S^l T^* \otimes_Y ((S^k T^* \otimes_Y W)/g_k)$$

where the map ϕ is induced by the natural projection of $S^k T^* \otimes_Y W$ onto $(S^k T^* \otimes_Y W)/g_k$. We set $g_{k-l} = S^{k-l} T^* \otimes_Y W$ for $l < 0$.

Since the diagram

$$(6.2) \quad \begin{array}{ccc} S^{k+1}T^* & \xrightarrow{J_{l,k}} & S^l T^* \otimes S^k T^* \\ \downarrow \delta & & \downarrow \delta \\ T^* \otimes S^{k+l-1} T^* & \xrightarrow{J_{l-1,k}} & T^* \otimes S^{l-1} T^* \otimes S^k T^* \end{array}$$

commutes, it is easily seen that $\delta(g_{k+l}) \subset T^* \otimes_Y g_{k+l-1}$ and hence δ induces a map

$$\delta : \wedge^j T^* \otimes_Y g_{k+l} \longrightarrow \wedge^{j+1} T^* \otimes_Y g_{k+l-1} .$$

Definition 6.1. The *Spencer cohomology* of g_k is the cohomology of the sequences

$$(6.3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & g_m & \xrightarrow{\delta} & T^* \otimes_Y g_{m-1} & \xrightarrow{\delta} & \wedge^2 T^* \otimes_Y g_{m-2} \longrightarrow \dots \\ & & & & \longrightarrow & \wedge^{m-k} T^* \otimes_Y g_k & \xrightarrow{\delta} & \wedge^{m-k+1} T^* \otimes S^{k-1} T^* \otimes_Y W \end{array}$$

where $m \geq k$. We denote by $H^{m-j,j} = H^{m-j,j}(g_k)$ the cohomology of the sequence (6.3) at $\wedge^j T^* \otimes_Y g_{m-j}$. We say that g_k is involutive if the sequences (6.3) are exact and that g_k is r -acyclic if $H^{m,j} = 0$ for $m \geq k, 0 \leq j \leq r$.

Lemma 6.2. *The sequences*

$$0 \longrightarrow g_{k+l} \xrightarrow{\delta} T^* \otimes_Y g_{k+l-1} \xrightarrow{\delta} \wedge^2 T^* \otimes_Y g_{k+l-2}$$

are exact for $l \geq 1$, that is, g_k is 1-acyclic.

Proof. The commutativity of diagram (6.2) implies the commutativity of the following diagram, whose rows are exact, where $l \geq 1$:

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & g_{k+l} & \longrightarrow & S^{k+l} T^* \otimes_Y W & \xrightarrow{\psi_l} & S^l T^* \otimes_Y ((S^k T^* \otimes_Y W)/g_k) \\ & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\ 0 & \longrightarrow & T^* \otimes_Y g_{k+l-1} & \longrightarrow & T^* \otimes S^{k+l-1} T^* \otimes_Y W & \xrightarrow{\psi_{l-1}} & T^* \otimes S^{l-1} T^* \otimes_Y ((S^k T^* \otimes_Y W)/g_k) \\ & & \downarrow \delta & & \downarrow \delta & & \\ 0 & \longrightarrow & \wedge^2 T^* \otimes_Y g_{k+l-2} & \longrightarrow & \wedge^2 T^* \otimes S^{k+l-2} T^* \otimes_Y W & & \end{array}$$

By Lemma 6.1, the last two columns of this diagram are also exact and hence so is the first.

Lemma 6.3. *The m -th prolongation $g_{(k+l)+m}$ of the family of subspaces: $g_{k+l} \subset S^{k+l} T^* \otimes_Y W$ is the same as the $(l+m)$ -th prolongation of g_k .*

Proof. We proceed by induction on m . For $m = 0$, the lemma is trivial. Assume that $m \geq 1$ and that $g_{(k+l)+q} = g_{k+l+q}$ for $q < m$. By Lemma 6.2, the diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & g_{k+l+m} & \xrightarrow{\delta} & T^* \otimes_Y g_{k+l+m-1} & \xrightarrow{\delta} & \wedge^2 T^* \otimes_Y g_{k+l+m-2} \\
 & & & & \downarrow & & \downarrow \\
 0 & \longrightarrow & (g_{k+l})_{+m} & \xrightarrow{\delta} & T^* \otimes_Y g_{(k+l)+m-1} & \xrightarrow{\delta} & \wedge^2 T^* \otimes_Y g_{(k+l)+m-2}
 \end{array}$$

is exact and commutative, so that $g_{(k+l)+m} = g_{k+l+m}$.

We now state the δ -Poincaré lemma (see D. G. Quillen [7] or S. Sternberg [9]).

Lemma 6.4. *If the dimension of W_y is independent of $y \in Y$, there exists an integer $k_0 > k$ depending only on n, k and the dimension of $W_y, y \in Y$, such that g_{k_0} is involutive.*

Hence, by Lemma 6.3, the Spencer cohomology groups $H^{m,j}(g_k)$ vanish for $m \geq k_0$.

Lemma 6.5. *Let W' be a vector bundle over Y and let g_k be the kernel of a morphism of vector bundles $\psi: S^k T^* \otimes_Y W \rightarrow W'$ over Y . If g_k is 2-acyclic and if g_{k+1} is a vector bundle over Y , then g_{k+l} is a vector bundle over Y for $l \geq 1$.*

Proof. The exactness of the sequence

$$T^* \otimes_Y g_{k+1} \xrightarrow{\delta} \wedge^2 T^* \otimes_Y g_k \xrightarrow{\delta} \wedge^3 T^* \otimes S^{k-1} T^* \otimes_Y W$$

implies the exactness of the sequence

$$\begin{aligned}
 0 \longrightarrow \delta(T^* \otimes_Y g_{k+1}) &\longrightarrow \wedge^2 T^* \otimes S^k T^* \otimes_Y W \\
 &\xrightarrow{\psi \oplus \delta} (\wedge^2 T^* \otimes_Y W') \oplus (\wedge^3 T^* \otimes S^{k-1} T^* \otimes_Y W)
 \end{aligned}$$

where $\psi \oplus \delta$ is the morphism of vector bundles over Y sending u into $\psi(u) \oplus \delta u$. Hence the function $y \mapsto \dim(\delta(T^* \otimes_Y g_{k+1}))_y$ on Y is upper semi-continuous. Since g_{k+2} is the kernel of the composition

$$S^{k+2} T^* \otimes_Y W \xrightarrow{A_{2,k}} S^2 T^* \otimes S^k T^* \otimes_Y W \xrightarrow{\psi} S^2 T^* \otimes_Y W',$$

the function $y \mapsto \dim(g_{k+2})_y$ on Y is also upper semi-continuous. The sequence

$$(6.4) \quad 0 \longrightarrow g_{k+2} \xrightarrow{\delta} T^* \otimes_Y g_{k+1} \xrightarrow{\delta} \delta(T^* \otimes_Y g_{k+1}) \longrightarrow 0$$

is exact by Lemma 6.2. Therefore, taking the Euler-Poincaré characteristic of (6.4), we see that the function $y \mapsto \dim(g_{k+2})_y + \dim(\delta(T^* \otimes_Y g_{k+1}))_y$ on Y is locally constant, because g_{k+1} is a vector bundle over Y . Hence g_{k+2} and $\delta(T^* \otimes_Y g_{k+1})$ are both vector bundles. Consider the exact sequence, for $l \geq 0$,

$$0 \rightarrow g_{k+l+3} \xrightarrow{\delta} T^* \otimes_Y g_{k+l+2} \xrightarrow{\delta} \wedge^2 T^* \otimes_Y g_{k+l+1} \xrightarrow{\delta} \wedge^3 T^* \otimes S^{k+l} T^* \otimes_Y W .$$

Lemma 3.3 of [5] implies that g_{k+l+3} is a vector bundle whenever g_{k+l+2} and g_{k+l+1} are vector bundles, completing the proof.

7. Differential equations

Definition 7.1. A partial differential equation R_k of order k on E is a fibered submanifold of $\pi : J_k(E) \rightarrow X$. A solution of R_k is a section s of E over an open set $U \subset X$ such that $j_k(s)(x) \in R_k$ for all $x \in U$. The l -th prolongation of R_k is the subset

$$R_{k+l} = J_l(R_k) \cap J_{k+l}(E)$$

of $J_{k+l}(E)$, where $J_l(R_k)$ is considered as a subset of $J_l(J_k(E))$. The symbol of R_k is the family of subspaces

$$g_k = F(R_k) \cap \{S^k T^* \otimes_{R_k} F(E)\}$$

of the vector bundle $S^k T^* \otimes_{R_k} F(E)$. Note that g_k is the kernel of the morphism of vector bundles over R_k

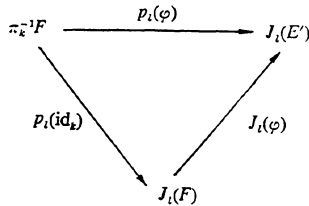
$$S^k T^* \otimes_{R_k} F(E) \rightarrow (F(J_k(E)) | R_k) / F(R_k) .$$

Note that $\pi_{k+l}(R_{k+l+1}) \subset R_{k+l}$. Let g_{k+l} be the l -th prolongation of g_k ; then g_{k+l} is a sub-family of vector spaces of the vector bundle $S^{k+l} T^* \otimes_{R_k} F(E)$.

If F is an open fibered submanifold of $J_k(E)$ over X , by Proposition 2.1, any morphism $\varphi : F \rightarrow E'$ of fibered manifolds of locally constant rank and a section s' of E' over X satisfying $s'(X) \subset \varphi(F)$ determine a partial differential equation $R_k = \text{Ker}_{s',\varphi}$. For such an equation, we have

$$R_{k+l} = \text{Ker}_{j_l(s'), p_l(\varphi)} .$$

In fact, the morphism $p_l(\text{id}_k) : J_{k+l}(E) \rightarrow J_l(J_k(E))$ induces a morphism $p_l(\text{id}_k)$ from the open fibered submanifold $\pi_k^{-1}F$ of $J_{k+l}(E)$ into $J_l(F)$; the diagram



commutes by Proposition 4.3 and so by Corollary 4.1, we conclude that

$$R_{k+l} = (\text{Ker}_{j_l(s')} J_l(\varphi)) \cap J_{k+l}(E) = \text{Ker}_{j_l(s')} p_l(\varphi) .$$

Moreover by Proposition 2.1, for $p \in R_k$ the vector space $(g_k)_p$ is the kernel of the map $\varphi_* : (S^k T^* \otimes_{R_k} F(E))_p \rightarrow F_{\varphi(p)}(E')$.

Conversely, given an equation $R_k \subset J_k(E)$ of order k on E , such fibered manifolds F, E' , morphism φ and section s' of E' such that $R_k = \text{Ker}_{s'} \varphi$ exist if and only if the normal bundle of the imbedding of R_k into $J_k(E)$ is isomorphic to a vector bundle induced by π from a vector bundle over X , by Proposition 2.2. Hence the condition that a fibered submanifold R_k of $J_k(E)$ be the kernel of a morphism $\varphi : J_k(E) \rightarrow E'$ of locally constant rank imposes a topological restriction on R_k . For this reason, we have given the above definition of a partial differential equation rather than defining an equation as a kernel of such a morphism φ . In the linear case, both points of view coincide (see [5]).

Definition 7.2. We say that a partial differential equation $R_k \subset J_k(E)$ of order k on E is formally integrable if, for $l \geq 0$, g_{k+l+1} is a vector bundle over R_k , and the map $\pi_{k+l} : R_{k+l+1} \rightarrow R_{k+l}$ is surjective.

Proposition 7.1. Let $R_k \subset J_k(E)$ be a partial differential equation of order k on E . Then the following statements are equivalent :

- (i) g_{k+1} is a vector bundle over R_k and the map $\pi_k : R_{k+1} \rightarrow R_k$ is surjective.
- (ii) $\pi_k : R_{k+1} \rightarrow R_k$ is a fibered submanifold of $\pi_k : J_{k+1}(E) | R_k \rightarrow R_k$.
- (iii) $\pi_k : R_{k+1} \rightarrow R_k$ is an affine sub-bundle of $\pi_k : J_{k+1}(E) | R_k \rightarrow R_k$ modeled on the vector bundle g_{k+1} .

Moreover, if any one of these assertions is satisfied, the l -th prolongation $R_{(k+1)+l}$ of the equation R_{k+1} is the same as the $(l + 1)$ -th prolongation of the equation R_k .

Proof. By Lemma 2.1, for each $p \in R_k$, there are an open neighborhood V of p in $J_k(E)$, a fibered manifold E' over $U = \pi V$, a section s' of E' over U and a morphism $\varphi : V \rightarrow E'$ of fibered manifolds of constant rank such that $R_k \cap V = \text{Ker}_{s'} \varphi$. Then, by Proposition 5.6, $p_l(\varphi) : J_{k+l}(E) | V \rightarrow J_l(E')$ is a morphism of affine bundles over $p_{l-1}(\varphi) : J_{k+l-1}(E) | V \rightarrow J_{l-1}(E')$ whose associated morphism of vector bundles is induced by the map $\varphi_* \circ \Delta_{l,k}$ over φ . Since φ has constant rank, Proposition 4.3 and Corollary 4.1 imply, by a previous argument, that $R_{k+l} \cap \pi_k^{-1}(V) = \text{Ker}_{j_l(s')} p_l(\varphi)$; Proposition 2.1 implies that $(g_{k+l})_p$ is the kernel of $\varphi_* \circ \Delta_{l,k} : (S^{k+l} T^* \otimes_{R_k} F(E))_p \rightarrow (S^l T^* \otimes_{E'} F(E'))_{\varphi(p)}$, if $p \in R_k \cap V$.

We first show that (i) implies (iii). Since g_{k+1} is a vector bundle over R_k , by Proposition 3.3, it follows that $p_1(\varphi) | \pi_k^{-1}(R_k \cap V)$ has locally constant rank and that (iii) holds.

Clearly (iii) implies (ii). Finally, let us prove that (ii) implies (i). To verify that g_{k+1} is a vector bundle over R_k , it suffices to show that we have an exact sequence

$$(7.1) \quad 0 \longrightarrow (g_{k+1})p \longrightarrow F_q(R_{k+1}) \xrightarrow{\pi_k^*} Fp(R_k) \longrightarrow 0$$

for all $q \in R_{k+1}$, $p \in R_k$, with $\pi_k(q) = p$. The map

$$S^{k+1}T^* \otimes_E F(E) \times_E J_{k+1}(E) \rightarrow J_{k+1}(E)$$

of Proposition 5.1 induces a map

$$g_{k+1} \times_{R_k} R_{k+1} \rightarrow J_{k+1}(E)$$

whose image is R_{k+1} . Indeed, the image of this map is contained in $J_1(R_k)$, since $g_{k+1} = (S^{k+1}T^* \otimes_{R_k} F(E)) \cap (T^* \otimes_{R_k} F(R_k))$ and $p_1(\text{id}_k) : J_{k+1}(E) \rightarrow J_1(J_k(E))$ is a monomorphism of affine bundles over $J_k(E)$. Given $q_1, q_2 \in R_{k+1}$, with $\pi_k(q_1) = \pi_k(q_2)$, by Proposition 5.1 there exist elements $a \in T^* \otimes_{R_k} F(R_k)$, $a' \in S^{k+1}T^* \otimes_E F(E)$ such that $a + q_1 = q_2$, $a' + q_1 = q_2$ as elements of $J_1(R_k)$ and $J_{k+1}(E)$ respectively. Since $p_1(\text{id}_k) : J_{k+1}(E) \rightarrow J_1(J_k(E))$ and the mapping of $J_1(R_k)$ into $J_1(J_k(E))$ are both monomorphisms of affine bundles, we have $a = a' \in g_{k+1}$. We have thus shown that, for all $p \in R_k$, the fiber $(R_{k+1})_p$ is an affine subspace of $(J_{k+1}(E))_p$ modeled on $(g_{k+1})_p$. By Proposition 3.7, we have a natural inclusion of $(g_{k+1})p$ into $F_q(R_{k+1})$ for all $q \in R_{k+1}$, with $\pi_k(q) = p$, determining the sequence (7.1). Since $F_q(R_{k+1}) \subset F_q(J_1(R_k)) \cap F_q(J_{k+1}(E))$, this sequence is exact.

We now assume that (i), (ii) or (iii) holds. To prove the remaining part of the proposition, it suffices to show that

$$R_{k+l+1} \cap \pi_k^{-1}(V) = R_{(k+1)+l} \cap \pi_k^{-1}(V) ,$$

where $\pi_k^{-1}(V)$ denotes, as throughout the rest of this proof, the inverse image in $J_{k+l+1}(E)$ of the open set $V \subset J_k(E)$ under the map π_k . We set $V' = R_k \cap V$ and $\rho = \pi_k : J_{k+1}(E) \rightarrow J_k(E)$. First note that V' is a fibered submanifold of V over U and that, since $\rho : R_{k+1} \rightarrow R_k$ is a fibered manifold, $R_{k+1} \cap \rho^{-1}(V') = R_{k+1} \cap \rho^{-1}(V)$ is a fibered manifold over U . The sequences of fibered manifolds over U

$$\begin{aligned} O_U &\longrightarrow V' \longrightarrow V \xrightarrow{\varphi} E' \\ O_U &\longrightarrow R_{k+1} \cap \rho^{-1}(V) \longrightarrow \rho^{-1}(V') \xrightarrow{p_1(\varphi)} J_1(E') \end{aligned}$$

are exact, since φ and the restriction of $p_1(\varphi)$ to $\rho^{-1}(V')$ have locally constant rank. Hence, by Corollary 4.1, the sequences of fibered manifolds over U

$$(7.2) \quad O_U \longrightarrow J_l(V') \longrightarrow J_l(V) \xrightarrow{J_l(\varphi)} J_l(E')$$

$$(7.3) \quad O_U \longrightarrow J_l(R_{k+1} \cap \rho^{-1}(V)) \longrightarrow J_l(\rho^{-1}(V')) \xrightarrow{J_l(p_1(\varphi))} J_l(J_1(E'))$$

are also exact. Since $\rho^{-1}(V)$ is open in $J_{k+1}(E)$, we have

$$\begin{aligned} R_{(k+1)+l} \cap \pi_k^{-1}(V) &= J_l(R_{k+1}) \cap \pi_k^{-1}(V) \\ &= J_l(R_{k+1} \cap \rho^{-1}(V)) \cap \pi_k^{-1}(V) . \end{aligned}$$

The exactness of (7.3) and Proposition 4.5 imply that

$$\begin{aligned} R_{(k+1)+l} \cap \pi_k^{-1}(V) &= \{\text{Ker}_{j_l(j_1(s'))} J_l(p_1(\varphi)) \mid J_l(\rho^{-1}(V'))\} \cap \pi_k^{-1}(V) \\ &= \{\text{Ker}_{j_l(j_1(s'))} J_l(p_1(\varphi)) \mid (J_l(\rho))^{-1} J_l(V')\} \cap \pi_k^{-1}(V) . \end{aligned}$$

The diagram

$$\begin{array}{ccc} J_{k+l+1}(E) | V & \xleftarrow{p_{l+1}(\varphi)} & J_{l+1}(E') \\ \uparrow p_l(\text{id}_{k+1}) & & \uparrow p_l(\text{id}_l) \\ J_l(J_{k+1}(E)) | V & \xleftarrow{J_l(p_1(\varphi))} & J_l(J_l(E')) \end{array}$$

commutes by Proposition 4.3; since $p_l(\text{id}_l) \circ j_{l+1}(s') = j_l(j_1(s'))$, we clearly have

$$\begin{aligned} R_{k+l+1} \cap \pi_k^{-1}(V) &= \text{Ker}_{j_{l+1}(s')} p_{l+1}(\varphi) \\ &= \{\text{Ker}_{j_l(j_1(s'))} J_l(p_1(\varphi))\} \cap \pi_k^{-1}(V) . \end{aligned}$$

To complete the proof, we need only to show that

$$\begin{aligned} &\{\text{Ker}_{j_l(j_1(s'))} J_l(p_1(\varphi))\} \cap \pi_k^{-1}(V) \\ &= \{\text{Ker}_{j_l(j_1(s'))} J_l(p_1(\varphi)) \mid (J_l(\rho))^{-1} J_l(V')\} \cap \pi_k^{-1}(V) . \end{aligned}$$

Clearly, the right-hand side is contained in the left-hand side. Let q be an element belonging in the left-hand side; we wish to show that $J_l(\rho)q \in J_l(V')$. The diagram

$$\begin{array}{ccc} J_l(J_{k+1}(E)) | V & \xrightarrow{J_l(p_1(\varphi))} & J_l(J_l(E')) \\ \downarrow J_l(\rho) & & \downarrow J_l(\pi_\delta) \\ J_l(J_k(E)) | V & \xrightarrow{J_l(\varphi)} & J_l(E') \end{array}$$

commutes by Proposition 4.2. Hence, if $x = \pi(q)$.

$$J_i(\varphi)(J_i(\rho)q) = J_i(\pi_0^i j_i(j_1(s')))(x) = j_i(s')(x) .$$

Since V is an open subset of $J_k(E)$, it is easily seen that $J_i(\rho)q \in J_i(V)$; the exactness of (7.2) implies that $J_i(\rho)q$ belongs to $J_i(V')$.

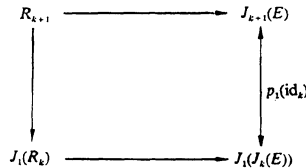
Proposition 7.2. *Let $R_k \subset J_k(E)$ be a partial differential equation of order k on E . If g_{k+m+1} is a vector bundle over R_k and $\pi_{k+m}: R_{k+m+1} \rightarrow R_{k+m}$ is surjective for $0 \leq m \leq l$, then R_{k+m+1} is a submanifold of $J_{k+m+1}(E)$ and $\pi_{k+m}: R_{k+m+1} \rightarrow R_{k+m}$ is an affine sub-bundle of $\pi_{k+m}: J_{k+m+1}(E) | R_{k+m} \rightarrow R_{k+m}$ modeled on the vector bundle $\pi_k^{-1}g_{k+m+1}$ over R_{k+m} induced from g_{k+m+1} by $\pi_k: R_{k+m} \rightarrow R_k$, for $0 \leq m \leq l$. Moreover, the m -th prolongation $R_{(k+l)+m}$ of the equation R_{k+l} is the same as the $(l+m)$ -th prolongation of the equation R_k .*

Proof. We proceed by induction on l . For $l = 1$, the proposition holds by Proposition 7.1. Now assume that the proposition holds for $l - 1$, with $l \geq 2$, and that the hypotheses of the proposition hold; then $R_{(k+l-1)+m} = R_{k+l+m-1}$. Apply Proposition 7.1 to the equation $R_{k+l-1} \subset J_{k+l-1}(E)$ and obtain $R_{k+l+m} = R_{(k+l-1)+m+1} = R_{((k+l-1)+1)+m} = R_{(k+l)+m}$ by our induction hypothesis. By Lemma 6.3 and Proposition 7.1, R_{k+l+1} is an affine bundle over R_{k+l} modeled on the vector bundle $\pi_k^{-1}g_{k+l+1}$.

Corollary 7.1. *If $R_k \subset J_k(E)$ is a formally integrable partial differential equation of order k on E , then, for each $l \geq 0$, R_{k+l} is a submanifold of $J_{k+l}(E)$ and $\pi_{k+l}: R_{k+l+1} \rightarrow R_{k+l}$ is an affine sub-bundle of $\pi_{k+l}: J_{k+l+1}(E) | R_{k+l} \rightarrow R_{k+l}$ modeled on the vector bundle $\pi_k^{-1}g_{k+l+1}$ over R_{k+l} induced from g_{k+l+1} by $\pi_k: R_{k+l} \rightarrow R_k$.*

8. Existence of formal solutions

Let $R_k \subset J_k(E)$ be a partial differential equation of order k on E . Assume that $\pi_k: R_{k+1} \rightarrow R_k$ is surjective and that g_{k+1} is a vector bundle. Then the conclusion of Proposition 7.1 holds. The diagram



commutes; the maps of R_{k+1} into $J_1(J_k(E)) | R_k$ and of $J_1(R_k)$ into $J_1(J_k(E)) | R_k$ are both monomorphisms of affine bundles over R_k . Hence the inclusion of R_{k+1} into $J_1(R_k)$ is also a monomorphism of affine bundles over R_k . We let $\pi^1: C^1 \rightarrow R_k$ be the quotient affine bundle of $J_1(R_k)$ by R_{k+1} , given by Proposition 3.4, and let $\rho: J_1(R_k) \rightarrow C^1$ be the natural epimorphism of affine bundles

over R_k . The affine bundle C^1 is canonically isomorphic to the vector bundle $\{T^* \otimes_{R_k} F(R_k)\} / \delta(g_{k+1})$ over R_k , where δ is the inclusion of g_{k+1} into $T^* \otimes_{R_k} F(R_k)$; we shall identify these two bundles. Then the sequence

$$O_{R_k} \longrightarrow R_{k+1} \xrightarrow{i} J_1(R_k) \xrightarrow{\rho} C^1 \longrightarrow O_{R_k}$$

is an exact sequence of affine bundles over R_k .

We recall that if 0 is the zero section of C^1 over R_k , then $J_l(0) : J_l(R_k) \rightarrow J_l(C^1)$ is the zero section of the vector bundle $J_l(\pi^1) : J_l(C^1) \rightarrow J_l(R_k)$ (see Proposition 5.3).

Proposition 8.1. *The map $p_l(\rho) : J_{l+1}(R_k) \rightarrow J_l(C^1)$ is a morphism of fibered manifolds over $J_l(R_k)$ and*

$$R_{k+l+1} = \text{Ker}_{J_l(0)} p_l(\rho) .$$

Therefore the l -th prolongation of the first order equation $R_{k+1} \subset J_1(R_k)$ on R_k is R_{k+l+1} . Moreover, if \mathcal{S} denotes the sheaf of germs of solutions of R_k , the subsheaf $j_k(\mathcal{S}) \subset \mathcal{R}_k$ is the sheaf of germs of solutions of the first order equation $R_{k+1} \subset J_1(R_k)$ on R_k .

Proof. By Proposition 4.2, the diagram

$$\begin{array}{ccc} J_{l+1}(R_k) & \xleftarrow{p_l(\rho)} & J_l(C^1) \\ \uparrow \pi_l & & \uparrow J_l(\pi^1) \\ J_l(R_k) & \xrightarrow{\text{id}} & J_l(R_k) \end{array}$$

commutes, since ρ is a morphism over R_k . Since ρ has locally constant rank, by Proposition 4.4

$$\text{Ker}_{J_l(0)} J_l(\rho) = J_l(R_{k+1}) .$$

The diagram

$$\begin{array}{ccc} J_{l+1}(R_k) & \xleftarrow{p_l(\rho)} & J_l(C^1) \\ & \swarrow p_l(\text{id}_l) & \nearrow J_l(\rho) \\ & J_l(J_1(R_k)) & \end{array}$$

commutes by Proposition 4.3 so that

$$\begin{aligned} \text{Ker}_{J_l(0)} p_l(\rho) &= J_{l+1}(R_k) \cap \text{Ker}_{J_l(0)} J_l(\rho) \\ &= J_{l+1}(R_k) \cap J_l(R_{k+1}) \\ &= J_{l+1}(R_k) \cap J_l(R_{k+1}) \cap J_{k+l+1}(E) \\ &= R_{k+l+1} \end{aligned}$$

by Proposition 7.1, because $J_{k+l+1}(E) = J_{l+1}(J_k(E)) \cap J_l(J_{k+1}(E))$. The remaining part of the proposition follows from the proof of Proposition 5.2 of [5].

Let $\sigma : T^* \otimes_{R_k} F(R_k) \rightarrow C^1$ denote the natural projection; then σ is the morphism of vector bundles associated to ρ .

Proposition 8.2. *The map $p_l(\rho) : J_{l+1}(R_k) \rightarrow J_l(C^1)$ is a morphism of affine bundles over*

$$\text{id} \times p_{l-1}(\rho) : J_l(R_k) \rightarrow J_l(R_k) \times_{J_{l-1}(R_k)} J_{l-1}(C^1), \quad \text{for } l \geq 1,$$

and its associated morphism of vector bundles is induced by the morphism $\sigma_l = \sigma \circ \Delta_{l,1} : S^{l+1}T^* \otimes_{R_k} F(R_k) \rightarrow S^l T^* \otimes_{R_k} C^1$ over R_k .

Proof. By Propositions 8.1 and 5.6, $p_l(\rho)$ is a morphism of fibered manifolds over $\text{id} \times p_{l-1}(\rho)$ and a morphism of affine bundles over $p_{l-1}(\rho)$ whose associated morphism of vector bundles is induced by the map

$$\rho_* \circ \Delta_{l,1} : S^{l+1}T^* \otimes_{J_1(R_k)} F(J_1(R_k)) \rightarrow S^l T^* \otimes_{C^1} F(C^1)$$

over ρ . By Proposition 5.3, it is clear that $p_l(\rho)$ is a morphism of affine bundles over $\text{id} \times p_{l-1}(\rho)$ whose associated morphism of vector bundles is determined by $\rho_* \circ \Delta_{l,1}$. This vector bundle morphism is induced by $\sigma_l = \sigma \circ \Delta_{l,1}$, because, by Proposition 3.7, the diagram

$$\begin{array}{ccc} T^* \otimes_{J_1(R_k)} F(R_k) & \xrightarrow{\sigma} & (\pi^1)^{-1}C^1 \\ \uparrow \rho'' & & \uparrow \rho^1 \\ F(J_1(R_k)) & \xrightarrow{\rho_*} & F(C^1) \end{array}$$

commutes.

Setting $\sigma_0 = \sigma$, it follows from Lemma 6.3 that g_{k+l} is the kernel of σ_l . Consider the family of vector spaces over R_k

$$C^2 = \{ \wedge^2 T^* \otimes_{R_k} F(R_k) \} / \delta(T^* \otimes_{R_k} g_{k+1})$$

whose projection onto R_k we denote by π^2 ; let σ be the projection of $\wedge^2 T^* \otimes_{R_k} F(R_k)$ onto C^2 . Then C^2 is a vector bundle if and only if g_{k+2} is a vector bundle over R_k , since the sequence

$$0 \longrightarrow g_{k+2} \xrightarrow{\delta} T^* \otimes_{R_k} g_{k+1} \xrightarrow{\delta} \wedge^2 T^* \otimes_{R_k} F(R_k) \xrightarrow{\sigma} C^2 \longrightarrow 0$$

is exact by Lemma 6.2. Let

$$\tau : T^* \otimes_{R_k} C^1 \longrightarrow C^2$$

be the epimorphism of vector bundles over R_k induced by the multiplication map from $T^* \otimes T^*$ to $\wedge^2 T^*$.

Lemma 8.1. *The sequence*

$$(8.1) \quad 0 \longrightarrow g_{k+2} \longrightarrow S^2 T^* \otimes_{R_k} F(R_k) \xrightarrow{\sigma_1} T^* \otimes_{R_k} C^1 \xrightarrow{\tau} C^2 \longrightarrow 0$$

is exact.

Proof. Consider the commutative exact diagram :

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & g_{k+2} & \longrightarrow & S^2 T^* \otimes_{R_k} F(R_k) & \xrightarrow{\sigma_1} & T^* \otimes_{R_k} C^1 \\
 & & \uparrow \delta & & \uparrow \delta & & \downarrow \\
 0 & \longrightarrow & T^* \otimes_{R_k} g_{k+1} & \longrightarrow & T^* \otimes T^* \otimes_{R_k} F(R_k) & \xrightarrow{\sigma_0} & T^* \otimes_{R_k} C^1 \longrightarrow 0 \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow \\
 0 & \longrightarrow & \wedge^2 T^* \otimes_{R_k} F(R_k) & \longrightarrow & \wedge^2 T^* \otimes_{R_k} F(R_k) & \longrightarrow & 0 \\
 & & \downarrow \sigma & & \downarrow & & \\
 & & C^2 & & 0 & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
 \end{array}$$

The map $\sigma \cdot \delta \cdot \sigma_0^{-1} : T^* \otimes_{R_k} C^1 \rightarrow C^2$ is well-defined and induces an isomorphism between the cokernel of $\sigma_1 : S^2 T^* \otimes_{R_k} F(R_k) \rightarrow T^* \otimes_{R_k} C^1$ and C^2 , which is easily seen to be, by the diagram, the map $-\tau$.

Proposition 8.3. *There is a morphism over R_k*

$$\kappa = \kappa(R_k) : R_{k+1} \longrightarrow C^2$$

the curvature of R_k such that the sequence

$$R_{k+2} \xrightarrow{\pi_{k+1}} R_{k+1} \xrightarrow[0 \circ \pi_k]{\kappa} C^2$$

is exact.

Proof. Consider the commutative diagram

$$(8.2) \quad \begin{array}{ccccc} R_{k+2} & \longrightarrow & J_2(R_k) & \xrightarrow{p_1(\rho)} & J_1(C^1) \\ \downarrow \pi_{k+1} & & \downarrow \pi_1 & & \downarrow \eta_0 \\ R_{k+1} & \longrightarrow & J_1(R_k) & \xrightarrow{\rho} & C^1 \end{array}$$

where $\eta_0 : J_1(C^1) \rightarrow C^1$ is the natural projection, and the exact sequence (8.1).

Let p be an element of R_{k+1} , and q any element of $J_2(R_k)$ satisfying $\pi_1(q) = p$. Then

$$\eta_0(p_1(\rho)q) = \rho(p) = 0(\pi_k(p)) ;$$

hence by Proposition 5.3, there exists a unique element $\varepsilon^{-1}p_1(\rho)q$ of $T^* \otimes_{R_k} C^1$ such that

$$\varepsilon(p, \varepsilon^{-1}p_1(\rho)q) = p_1(\rho)q .$$

We claim that the element $\kappa(R_k)p = \tau\varepsilon^{-1}p_1(\rho)q$ of C^2 depends only on p . Indeed, if q_1 is another element of $J_2(R_k)$ satisfying $\pi_1(q_1) = p$, then by Proposition 8.1,

$$J_1(\pi^1)p_1(\rho)q_1 = J_1(\pi^1)p_1(\rho)q = p .$$

Now $q_1 - q$ belongs to $S^2 T^* \otimes_{R_k} F(R_k)$ and by Proposition 8.2, we have

$$\sigma_1(q_1 - q) + p_1(\rho)q = p_1(\rho)q_1$$

where the left-hand side is defined in terms of the map (5.6), with $W = C^1$, $E = R_k$. Therefore by Lemma 5.3, we obtain

$$\sigma_1(q_1 - q) = \varepsilon^{-1}(p_1(\rho)q_1 - p_1(\rho)q) .$$

Since $\varepsilon : T^* \otimes_{J_1(R_k)} C^1 \rightarrow J_1(C^1)$ is a morphism of vector bundles over $J_1(R_k)$ and since $\eta_0(p_1(\rho)q_1) = \eta_0(p_1(\rho)q) = 0(\pi_k(p))$, we conclude that

$$\sigma_1(q_1 - q) = \varepsilon^{-1}p_1(\rho)q_1 - \varepsilon^{-1}p_1(\rho)q .$$

Because $\tau \circ \sigma_1 = 0$, we obtain

$$\tau \varepsilon^{-1}p_1(\rho)q_1 = \tau \varepsilon^{-1}p_1(\rho)q .$$

Hence $\kappa(R_k) : R_{k+1} \rightarrow C^2$ is a well-defined map. If C^2 is a vector bundle, then $\kappa(R_k)$ is in fact a morphism of fibered manifolds over R_k .

Now, if $q \in R_{k+2}$, then $\pi_{k+1}(q) = p$ belongs to R_{k+1} and $p_1(\rho)q = J_1(0)(p)$ by Proposition 8.1; hence $\varepsilon^{-1}p_1(\rho)q = 0(\pi_k(p))$ by Proposition 5.3, (iv), and so $\kappa(R_k)p = 0$. Conversely, if $p \in R_{k+1}$ satisfies $\kappa(R_k)p = 0$, let q be an element of $J_2(R_k)$ such that $\pi_1(q) = p$; then $\tau \varepsilon^{-1}p_1(\rho)q = 0$. By the exactness of the sequence (8.1), there exists some element $a \in S^2 T^* \otimes_{R_k} F(R_k)$ such that

$$\varepsilon^{-1}p_1(\rho)q = \sigma_1 a$$

or

$$p_1(\rho)q = \varepsilon(p, \sigma_1 a) .$$

Hence by Propositions 8.2 and 5.3 and Corollary 5.1,

$$\begin{aligned} p_1(\rho)((-a) + q) &= \sigma_1(-a) + p_1(\rho)q \\ &= \sigma_1(-a) + (J_1(0)(p) + p_1(\rho)q) \\ &= (\sigma_1(-a) + J_1(0)(p)) + p_1(\rho)q \\ &= \varepsilon(p, \sigma_1(-a)) + \varepsilon(p, \sigma_1 a) \\ &= J_1(0)(p) . \end{aligned}$$

By Proposition 8.1, it follows that the element $(-a) + q$ of $J_2(R_k)$ satisfying $\pi_1((-a) + q) = p$ belongs to R_{k+2} .

Proposition 8.4. *The image of κ lies in the family of subspaces over R_k*

$$H^{k,2} = \frac{\text{Ker} \{ \delta : \wedge^2 T^* \otimes_{R_k} \mathcal{G}_k \rightarrow \wedge^3 T^* \otimes S^{k-1} T^* \otimes_{R_k} F(E) \}}{\delta(T^* \otimes_{R_k} \mathcal{G}_{k+1})}$$

of C^2 .

Proof. Let $C^j(E)$ be the vector bundle

$$\{ \wedge^j T^* \otimes_{J_k(E)} F(J_k(E)) \} / \delta(\wedge^{j-1} T^* \otimes S^{k+1} T^* \otimes_{J_k(E)} F(E))$$

over $J_k(E)$, for $j = 1, 2$. Consider the exact commutative diagram (8.2)

$$\begin{array}{ccccc}
 J_{k+2}(E) & \xleftarrow{p_2(id_k)} & J_2(J_k(E)) & \xleftarrow{p_1(\rho)} & J_1(C^1(E)) \\
 \downarrow \pi_{k+1} & & \downarrow \pi_1 & & \downarrow \eta_0 \\
 J_{k+1}(E) & \xrightarrow{p_1(id_k)} & J_1(J_k(E)) & \xleftarrow{\rho} & C^1(E)
 \end{array}$$

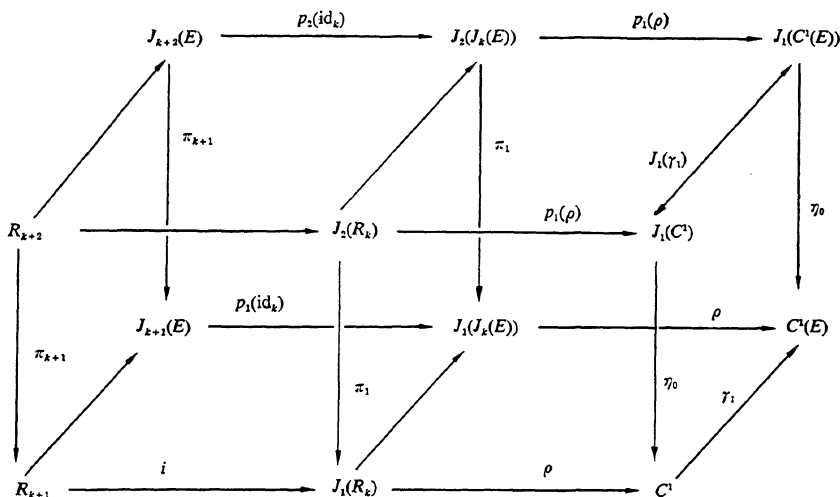
and the exact sequence (8.1)

$$\begin{aligned}
 0 \longrightarrow S^{k+2}T^* \otimes_{J_k(E)} F(E) &\longrightarrow S^2T^* \otimes_{J_k(E)} F(J_k(E)) \\
 &\xrightarrow{\sigma_1} T^* \otimes_{J_k(E)} C^1(E) \xrightarrow{\tau} C^2(E) \longrightarrow 0
 \end{aligned}$$

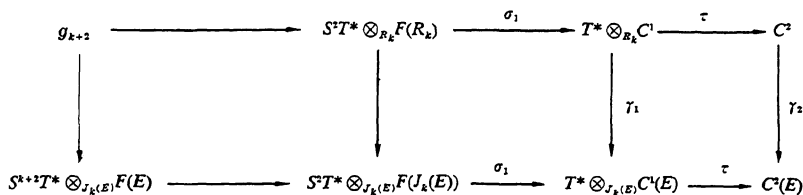
of vector bundles over $J_k(E)$ corresponding to the equation $J_k(E)$ of order k on E . By Proposition 8.3, we obtain a map $\kappa(J_k(E)): J_{k+1}(E) \rightarrow C^2(E)$, which must be the zero map over $J_k(E)$, since $\pi_{k+1}: J_{k+2}(E) \rightarrow J_{k+1}(E)$ is an epimorphism. The exact commutative diagram

$$\begin{array}{ccc}
 \wedge^{j-1}T^* \otimes_{R_k} \mathcal{G}_{k+1} & \longrightarrow & \wedge^{j-1}T^* \otimes S^{k+1}T^* \otimes_{J_k(E)} F(E) \\
 \downarrow \delta & & \downarrow \delta \\
 \wedge^j T^* \otimes_{R_k} F(R_k) & \longrightarrow & \wedge^j T^* \otimes_{J_k(E)} F(J_k(E)) \\
 \downarrow \sigma & & \downarrow \sigma \\
 C^j & \xrightarrow{\gamma_j} & C^j(E) \\
 \downarrow & & \downarrow \\
 0 & & 0
 \end{array}
 \tag{8.3}$$

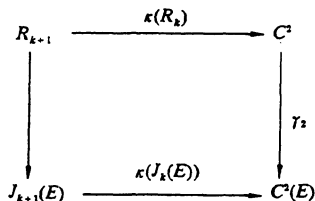
induces a morphism $\gamma_j: C^j \rightarrow C^j(E)$ over the inclusion of R_k into $J_k(E)$ and which is therefore a morphism of fibered manifolds over X , for $j = 1, 2$. The following three-dimensional diagram clearly commutes:



as does the diagram :



Therefore, by Propositions 8.3, 5.3, 5.4 and 3.7 the diagram



commutes, and so by the exactness and commutativity of diagram (8.3) with $j = 2$, the image of $\kappa(R_k)$ must lie in the family of vector spaces

$$\{\wedge^2 T^* \otimes_{R_k} F(R_k)\} \cap \delta(T^* \otimes S^{k+1} T^* \otimes_{R_k} F(E)) / \delta(T^* \otimes_{R_k} S_{k+1})$$

over R_k which is precisely $H^{k,2}$ by Lemma 6.1.

Theorem 8.1. *Let $R_k \subset J_k(E)$ be a partial differential equation of order k*

on E such that $\pi_k : R_{k+1} \rightarrow R_k$ is surjective. If g_k is 2-acyclic, and g_{k+1} is a vector bundle over R_k , then R_k is formally integrable.

Proof. Lemma 6.5 implies that g_{k+l} is a vector bundle for $l \geq 1$. We now proceed by induction on l . Assume that $l \geq 0$ and that $\pi_{k+m} : R_{k+m+1} \rightarrow R_{k+m}$ is a surjective map for $0 \leq m \leq l$. By Proposition 7.2, we can apply Propositions 8.3, 8.4 to the equation $R_{k+l} \subset J_{k+l}(E)$ and we obtain, by Proposition 7.2 and Lemma 6.3, an exact sequence

$$R_{k+l+2} \xrightarrow{\pi_{k+l+1}} R_{k+l+1} \xrightarrow{\kappa(R_{k+l})} \pi_k^{-1}H^{k+l,2}$$

where $\pi_k^{-1}H^{k+l,2}$ is the family of vector spaces over R_{k+l} induced from $H^{k+l,2}$ by $\pi_k : R_{k+l} \rightarrow R_k$. Since g_k is 2-acyclic, it follows that $H^{k+l,2} = 0$ and hence that the map $\pi_{k+l+1} : R_{k+l+2} \rightarrow R_{k+l+1}$ is surjective.

If the dimensions of all components of E are the same, then there exists an integer $k_0 > k$ depending on n, k and the dimension of E such that g_{k_0} is involutive by Lemma 6.4, so by Proposition 7.2 we have clearly also proved

Theorem 8.2. *If the dimensions of all components of E are the same, and $R_k \subset J_k(E)$ is a partial differential equation of order k on E , then there exists an integer $k_0 > k$ depending only on n, k and the dimension of E such that, if g_{k+l+1} is a vector bundle over R_k and $\pi_{k+l} : R_{k+l+1} \rightarrow R_{k+l}$ is surjective, for $0 \leq l \leq k_0 - k$, then R_k is formally integrable.*

9. Existence of analytic solutions

Assume that X is a real analytic manifold and that the fibered manifold $\pi : E \rightarrow X$ is real analytic; we say that a partial differential equation R_k of order k on E is analytic if it is an analytic fibered submanifold of $J_k(E)$.

Theorem 9.1. *Let R_k be an analytic partial differential equation of order k on E which is formally integrable. Then, given $p \in R_{k+l}$, with $\pi(p) = x \in X$, there exists an analytic solution s of the equation R_k over a neighborhood of x such that $j_{k+l}(s)(x) = p$.*

Proof. Consider the diagram, for $l \geq 1$,

$$(9.1) \quad \begin{array}{ccccc} R_{k+l+1} & \longrightarrow & J_{l+1}(R_k) & \xleftarrow{p_l(\rho)} & J_l(C^1) \\ \uparrow \pi_{k+l} & & \uparrow \pi_l & & \downarrow \eta_{l-1} \\ R_{k+l} & \longrightarrow & J_l(R_k) & \xrightarrow{p_{l-1}(\rho)} & J_{l-1}(C^1) \end{array}$$

where η_{l-1} is the natural projection, and the exact sequence

$$(9.2) \quad 0 \longrightarrow g_{k+l+1} \longrightarrow S^{l+1}T^* \otimes_{R_k} F(R_k) \xrightarrow{\sigma_l} S^l T^* \otimes_{R_k} C^1$$

of vector bundles over R_k . Since $\pi_{k+l} : R_{k+l+1} \rightarrow R_{k+l}$ is surjective, by a diagram chase involving (9.1) and (9.2) similar to the one given in Proposition 8.3, it is easily seen that $\sigma_l(S^{l+1}T^* \otimes_{R_k} F(R_k))$ is equal to the set of all elements of the form $\varepsilon^{-1}p_l(\rho)q$, with $q \in J_{l+1}(R_k)$ satisfying $\pi_l(q) \in R_{k+l}$. Hence, restricting our attention to the fibers of all fibered manifolds at $x \in X$, using Spencer's estimate (see L. Ehrenpreis, V. W. Guillemin, and S. Sternberg [4], and W. J. Sweeney [10]) and a diagram chase involving (9.1) and (9.2) similar to the preceding one, one obtains by Proposition 8.1 and Corollary 7.1 the desired analytic solution s over a neighborhood of x .

References

- [1] R. Bott, *Notes on the Spencer resolution*, Mimeographed notes, Harvard University, 1963.
- [2] N. Bourbaki, *Algèbre linéaire*, Hermann, Paris, 1962.
- [3] E. Cartan, *Les systèmes différentiels extérieurs et leurs applications géométriques*, Hermann, Paris, 1945.
- [4] X. Ehrenpreis, V. W. Guillemin & S. Sternberg, *On Spencer's estimate for δ -Poincaré*, Ann. of Math. **82** (1965) 128–138.
- [5] H. Goldschmidt, *Existence theorems for analytic linear partial differential equations*, Ann. of Math. **86** (1967) 246–270.
- [6] M. Kuranishi, *Lectures on exterior differential systems*, Tata Institute of Fundamental Research, Bombay, 1962.
- [7] D. G. Quillen, *Formal properties of over-determined systems of linear partial differential equations*, Ph.D. thesis, Harvard University, 1964.
- [8] D. C. Spencer, *Deformations of structures on manifolds defined by transitive, continuous pseudogroups*, I–II, Ann. of Math. **76** (1962) 306–445.
- [9] S. Sternberg, *Notes on partial differential equations*, Mimeographed notes, Harvard University, to appear.
- [10] W. J. Sweeney, *The δ -Poincaré estimate*, Pacific J. Math. **20** (1967) 559–570.

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