J. DIFFERENTIAL GEOMETRY
53 (1999) 1-13

INTEGRAL INVARIANTS OF 3-MANIFOLDS. II

RAOUL BOTT & ALBERTO S. CATTANEO

Abstract

This note is a sequel to our earlier paper of the same title [4] and describes invariants of rational homology 3-spheres associated to acyclic orthogonal local systems. Our work is in the spirit of the Axelrod–Singer papers [1], generalizes some of their results, and furnishes a new setting for the purely topological implications of their work.

1. Introduction

This note is an addendum to our earlier paper of the same title [4]. Our aim here will be to construct invariants for framed 3-dimensional homology spheres (M, f), associated to an acyclic orthogonal local system E on M.

Like in our earlier note, we follow the guidelines of the Axelrod–Singer paper [1] on the asymptotics of the Chern–Simons theory, and we have again put aside the physics inspired aspects of the subject, concentrating our efforts on the construction of potential configuration-space integral invariants of (M, f). More precisely we are seeking invariants that depend on the diffeomorphism type of M and the *homotopy class* of the framing f.

For simplicity we assume throughout that M is a connected, oriented 3-dimensional homology sphere so that—up to conjugacy—local systems over M are classified by representations of $\pi_1(M; p)$ where p is some fixed point in M.

Received December 30, 1997. We thankfully acknowledge support from NSF for R.B. and INFN (grant No. 5565/95) and from D.O.E. (grant No. DE-FG02-94ER25228, Amendment No. A003) for A.S.C.

Our invariants are associated to local systems E which are induced by an *orthogonal* representation ρ_E of $\pi_1(M;p)$ on some $\mathbb{R}^m$, and we call such systems orthogonal. Furthermore, a local system E is called acyclic if $H^*(M;E)=0$.

With this understood, our principal observation is given by the following

Theorem 1.1. *An orthogonal and acyclic local system E over M gives rise to a purely combinatorial graph cohomology $\mathcal{G}_E$, and if $\Gamma \in \mathcal{G}_E$ is a connected trivalent cocycle of $\mathcal{G}_E$, then Γ determines a numerical invariant $I_\Gamma(M,f)$ of (M,f). This invariant has the structural form:*

$$I_\Gamma(M,f) = A_\Gamma(M) + \phi(\Gamma)\,\mathrm{CS}(M,f), \tag{1.1}$$

where $A_\Gamma(M)$ denotes a sum of configuration-space integrals specified by Γ and a fixed—but arbitrary—Riemannian structure g on M, $\phi(\Gamma)$ is a number universally associated to Γ, and $\mathrm{CS}(M,f)$ stands for the Chern–Simons integral of M relative to f and the Levi-Civita connection of g.

Combined with the invariants described in [4], where we treated the trivial coefficient system $\mathbb{R}$—which is not acyclic—, one is therefore in the possession of a large number of integral invariants of (M,f), and it would be very interesting to understand their relation to the finite type invariants described in [5] (see also [3]) and whether in their totality they are in any sense exhaustive.

The proof of Theorem 1.1, as well as the precise definition of $\mathcal{G}_E$, will be brought in sections 2 and 3, and runs pretty well along the lines of our earlier paper [4]. In fact the acyclicity of E allows for a simplification of the initial step in our procedure, and we will explain this phenomenon here and now.

Recall that the compactified configuration space $C_2(M)$ is a manifold with boundary isomorphic to $M\times M$ with its diagonal, Δ, blown up. Thus $\partial C_2(M)=S$ is isomorphic to the unit sphere bundle of the tangent bundle of M.

This situation therefore gives rise to the diagram below consisting of sections of the exact sequences associated to the pair $(C_2(M),S)$ and its image $(M\times M,\Delta)$ under the natural projection π of $C_2(M)$ to $M\times M$.

$$\begin{array}{ccccccccc}
\longrightarrow & H^2(C_2(M)) & \longrightarrow & H^2(S) & \xrightarrow{\delta} & H^3(C_2(M),S) & \longrightarrow & H^3(C_2(M)) & \longrightarrow \\
& \pi^*\big\uparrow & & \pi^*\big\uparrow & & \pi^*\big\uparrow\approx & & \pi^*\big\uparrow & \\
\longrightarrow & H^2(M\times M) & \longrightarrow & H^2(\Delta) & \xrightarrow{\delta} & H^3(M\times M,\Delta) & \longrightarrow & H^3(M\times M) & \longrightarrow
\end{array} \tag{1.2}$$

The vertical isomorphism in the third column of (1.2) follows from excision near the blow-up of Δ. Note also that this diagram is acted upon by the involution T which exchanges the factors in $M \times M$, and each of the sequences therefore splits canonically into a $+$ and $-$ part corresponding to the ± 1 eigenvalues of T. In the bottom row $H^*(\Delta)$ is clearly invariant under T so that the antisymmetric part of (1.2) reduces to

(1.3)

$$\begin{array}{ccccccc} H^2_-(C_2(M)) & \longrightarrow & H^2_-(S) & \xrightarrow{\delta} & H^3_-(C_2(M), S) & & \\ & & & & \pi^* \Big\uparrow \approx & & \\ & & 0 & \longrightarrow & H^3_-(M \times M, \Delta) & \xrightarrow{\approx} & H^3_-(M \times M) \end{array}$$

Now in [4] we showed that the form η given by half the Euler form of the tangent bundle along the fiber in the fibering $S \to \Delta$ generates $H^2(S; \mathbb{R})$ as a module over $H^*(\Delta; \mathbb{R})$ and that this η is antisymmetric: $T^*\eta = -\eta$. In short $[\eta]$ generates $H^2_-(S; \mathbb{R})$.

So far our discussion involved the constant coefficient system $\mathbb{R}$. But the sequences (1.2) and (1.3) as well as the action of T remain valid for a general local system E on M, provided we use the local system $F = \pi_1^{-1}E \otimes \pi_2^{-1}E$ on $M \times M$, and $\pi^{-1}F$ on $C_2(M)$.

This understood, assume now that E is orthogonal and acyclic.

The orthogonality gives rise to an arrow

$$I : \mathbb{R} \to E \otimes E, \tag{1.4}$$

defined by sending 1 to $\sum_i e_i \otimes e_i$, where $\{e_i\}$ is any orthonormal frame in E. We may therefore also apply I to η to obtain a *closed* form $I(\eta) \in \Omega^2(S; E \otimes E)$.

Next observe that the Künneth formula implies that

$$H^*(M \times M; \pi_1^{-1}E \otimes \pi_2^{-1}E) = H^*(M; E) \otimes H^*(M; E).$$

Hence, under our acyclicity assumption, all the terms on the right of δ in (1.3) vanish! It follows immediately that the class of $I(\eta) \in \Omega^2(S; E \otimes E)$ is in the image of a class $[\hat{\eta}] \in H^2(C_2(M); \pi^{-1}F)$.

Actually we need the following slight refinement of this assertion:

Lemma 1.2. *Under the orthogonality and acyclicity assumptions there exists a form $\hat{\eta} \in \Omega^2(C_2(M); \pi^{-1}F)$ with the following properties:*

1. *The restriction of* $\hat{\eta}$ *to* S *is* $I(\eta)$*:* $i_\partial^*\hat{\eta} = I(\eta)$.

2. $\hat{\eta}$ *is closed under* $d_{\pi^{-1}F}$*:* $d_{\pi^{-1}F}\hat{\eta} = 0$.

3. $\hat{\eta}$ *is antisymmetric:* $T^*\hat{\eta} = -\hat{\eta}$.

The construction of $\hat{\eta}$ proceeds precisely along the guidelines given in [4].

Let U be a tubular neighborhood of Δ in $M \times M$, and let $p : U \to \Delta$ be a projection which fibers U over Δ into discs on which T acts linearly as the antipodal map, and such that ∂U can be identified with S. Then $\widetilde{U} = \pi^{-1}U$ has the structure of $S \times [0,1]$ and hence fibers over S with the unit interval as fiber. We write $\sigma : \widetilde{U} \to S$ for the projection onto S in this fibering, and note that T acts on $S \times [0,1]$ by the antipodal map on S crossed with the identity on $[0,1]$. Now choose a smooth function χ on $[0,1]$ which is identically $+1$ near 0 and identically 0 near $+1$, and write χ also for its pullback to $\widetilde{U}$. It follows that the form

$$\widetilde{\eta} = \sigma^* I(\eta)\,\chi$$

on $\widetilde{U}$ extends by 0 to a form on all of $C_2(M)$ with values in $\pi^{-1}F$. It is also clear that $\widetilde{\eta}$ restricts to $I(\eta)$ on S, so that $d_{\pi^{-1}F}\widetilde{\eta}$ represents $\delta(I(\eta))$ in the upper sequence. On the other hand $d_{\pi^{-1}F}\widetilde{\eta}$ vanishes identically near S and so may be considered an antisymmetric form on $M \times M$. But then, by the acyclicity assumption, there must exist an antisymmetric form α on $M \times M$ such that

$$d_F\alpha = d_{\pi^{-1}F}\widetilde{\eta}.$$

Now $\hat{\eta} = \widetilde{\eta} - \pi^*\alpha$ has all the desired properties.

2. The Θ-invariant

Using the closed form $\hat{\eta}$ defined in the previous section, see Lemma 1.2, we can define an invariant for the framed 3-dimensional homology sphere (M, f).

First we notice that $\hat{\eta}^3$ is a 6-form on the 6-dimensional space $C_2(M)$ which takes values in $\pi_1^{-1}E^{\otimes 3} \otimes \pi_2^{-1}E^{\otimes 3}$. If we associate to each vertex i ($i = 1, 2$) a homomorphism

$$\rho_i : \mathbb{R} \to E \otimes E \otimes E \tag{2.1}$$

which is equivariant as a module over $\pi_1(M)$, then we obtain the *closed* real-valued 6-form $\langle \rho_1\rho_2\,,\,\hat{\eta}^3\rangle$. Here $\langle\cdot\,,\,\cdot\rangle$ denotes the scalar product over E and its extensions to $E^{\otimes 3}$ and to $\pi_1^{-1}E^{\otimes 3}\otimes\pi_2^{-1}E^{\otimes 3}$.

Remark 2.1. The existence of such homomorphisms depends on E. In some cases, the only possible choice will be the trivial one: $\rho = 0$.

If the vector space spanned by these homomorphisms has dimension greater than one, then one can choose ρ_1 and ρ_2 linearly independent.

Example 2.2. A particular case, considered in [1], occurs when E is the adjoint representation of a compact Lie group G. Then a natural choice for ρ is obtained by using the structure constants f_{abc} relative to a left- and right-invariant inner product on the Lie algebra of G; namely,

$$\rho(x) = x\sum_{abc} f_{abc}\, e_a\otimes e_b\otimes e_c.$$

The equivariance under the full group G ensures the equivariance under the action of $\pi_1(M)$. Notice that the antisymmetry of the structure constants implies that this homomorphism is completely antisymmetric.

Note that if E denotes a representation of such a Lie group G, the equivariant homomorphisms are dual to the projections to the trivial representations in $E\otimes E\otimes E$. Again, the equivariance under the full group G ensures the equivariance under the action of $\pi_1(M)$.

Example 2.3. If E_j denotes the irreducible representation of spin $j\in\mathbb{Z}/2$ of $SU(2)$, then the Clebsch–Gordan formula,

$$E_j\otimes E_k = \bigoplus_{l=|j-k|}^{j+k} E_l,$$

implies that

$$E_j\otimes E_j\otimes E_j = \bigoplus_{k=j-\lfloor j\rfloor}^{j}(2k+1)E_k \oplus \bigoplus_{k=j+1}^{3j}(3j-k+1)E_k.$$

So $E_j^{\otimes 3}$ contains no trivial representations if j is a half-integer and one trivial representation if j is an integer. In the case $j = 1$ we recover the choice of example 2.2. Notice that this projection is obtained by selecting the representation of spin j in $E_j\otimes E_j$, then by tensoring by the last copy of E_j, and finally by projecting on the trivial representation. Therefore, all these projections (and the corresponding homomorphisms) are completely antisymmetric.

Example 2.4. With the notation of the previous example, consider $E = E_{1/2} \oplus E_1$. It turns out that $E^{\otimes 3}$ contains three copies of the trivial representation: the first is obtained by choosing the trivial representation in $E_1^{\otimes 3}$; the second by choosing the trivial representation in $E_{1/2} \otimes E_{1/2} \otimes E_1$, and the other two by cyclic rotations of the second. Notice that the second projection is obtained by selecting the representation of spin 1 in $E_{1/2} \otimes E_{1/2}$, then by tensoring by E_1, and finally by projecting on the trivial representation. Therefore, this projection is symmetric with respect to the exchange of the spin-(1/2) components.

Integrating the closed form we have obtained over $C_2(M)$ yields the number

$$A_{(\Theta,\rho_1,\rho_2)} \doteq \int_{C_2(M)} \langle \rho_1 \rho_2 \, , \, \hat{\eta}^3 \rangle \, , \tag{2.2}$$

which is our first potential invariant. We recall, see [4], that the definition of η relies on the choice of a metric on M and of a compatible connection; moreover, the construction of $\hat{\eta}$ requires the choice of a function χ and of a 2-form α as explained after Lemma 1.2. An invariant must be independent of all these choices. Actually, we have the following

Theorem 2.5. *Given a section f of the orthonormal frame bundle, the combination*

$$I_{(\Theta,\rho_1,\rho_2)}(M, f) = A_{(\Theta,\rho_1,\rho_2)}(M) - \frac{\langle \rho_1 \, , \, \rho_2 \rangle}{4} \, \mathrm{CS}(M, f), \tag{2.3}$$

is independent of all the choices involved (except for the framing). Here

$$\begin{aligned} \mathrm{CS}(M, f) &= -\frac{1}{8\pi^2} \int_M f^* \, \mathrm{Tr} \left(\theta \, d\theta + \frac{2}{3} \, \theta^3 \right) \\ &= \frac{1}{4\pi^2} \int_M f^* \left(\theta^i d\theta_i - \frac{1}{3} \, \epsilon_{ijk} \theta^i \theta^j \theta^k \right), \end{aligned} \tag{2.4}$$

is the Chern–Simons integral of the same metric connection used to define η.

Thus, $I_{(\Theta,\rho_1,\rho_2)}(M, f)$ is an invariant for the framed rational homology sphere (M, f).

Remark 2.6. In the case discussed in example 2.2, we have

$$\langle \rho_1 \, , \, \rho_2 \rangle = \sum_{abc} f_{abc} f_{abc} = -c_v \, \dim G,$$

where c_v is the Casimir of the adjoint representation of G.

Proof. As in [4], we introduce the unit interval I as a parameter space, and recall that, as shown there, letting θ vary on I corresponds to defining on $S \times I$ a form—which we still denote by η—given by half the Euler form of the tangent bundle along the fiber in the fibering $S \times I \to \Delta \times I$.

Then all the arguments contained in section 1 are still true if we multiply by I each space involved (say, M, $M \times M$, Δ, $C_2(M)$ and S), since $H^n(I) = \delta_{n0}\,\mathbb{R}$. In particular, we have a form—which we keep denoting by $\hat\eta$—which satisfies the properties of Lemma 1.2 with $C_2(M)$ replaced by $C_2(M)\times I$. (To be precise, by π now we mean the projection $C_2(M) \times I \to M \times M \times I$.)

If we denote by σ the projection $C_2(M) \times I \to I$, then

$$A_{(\Theta,\rho_1,\rho_2),\tau} = \sigma_* \left\langle \rho_1\rho_2\,,\,\hat\eta^3 \right\rangle$$

is a function depending on the parameter $\tau \in I$, in whose variations we are interested.

To do so, we recall that, given two spaces M_1 and M_2 and projections $\pi_i : M_1 \times M_2 \to M_i$, Stokes' theorem can be rewritten as

$$d\pi_{i*}\omega = \pi_{i*}d\omega - (-1)^{\deg \pi^\partial_{i*}\omega}\,\pi^\partial_{i*}\omega, \tag{2.5}$$

where π^∂_{1*} denotes integration along the boundary of M_2 and vice versa. (Notice that the signs in (2.5) are correct if integration acts from the right.)

Since $\left\langle \rho_1\rho_2\,,\,\hat\eta^3 \right\rangle$ is a closed form, we simply have

$$dA_{(\Theta,\rho_1,\rho_2),\tau} = \sigma^\partial_* \left\langle \rho_1\rho_2\,,\,\hat\eta^3 \right\rangle = \langle \rho_1\,,\,\rho_2 \rangle\,\sigma^\partial_*\eta^3;$$

the last identity following from property 1 in Lemma 1.2.

Now we recall that in [4] (see Lemma 3.15 there) we proved that

$$\pi^\partial_*\eta^3 = \frac{1}{4}\,p_1,$$

where π^∂ is the projection $S \times I \to \Delta \times I$, and p_1 is the first Pontrjagin form on $\Delta \times I = M \times I$.

Denoting by σ_M the projection $M \times I \to I$, we finally get

$$dA_{(\Theta,\rho_1,\rho_2),\tau} = \frac{\langle \rho_1\,,\,\rho_2\rangle}{4}\,\sigma_{M*}p_1,$$

from which the theorem follows. q.e.d.

3. The higher invariants

Using the natural projections $\pi_{ij} : C_n(M) \to C_2(M)$ we can pull back the form $\hat{\eta}$ defined in section 1. We will write

$$\hat{\eta}_{ij} = \pi_{ij}^* \hat{\eta},$$

and by property 3 of Lemma 1.2 we have

$$\hat{\eta}_{ij} = -\hat{\eta}_{ji}.$$

These forms on $C_n(M)$ allow for writing other invariants of the 3-dimensional homology sphere M associated to cocycles in an appropriate graph cohomology (depending on the bundle E).

Definition 3.1. We call a decorated graph a graph with oriented and numbered edges and numbered vertices (by convention we start the enumeration by 1). We require edges always to connect distinct vertices. If two vertices are connected by exactly one edge, we call that edge *regular.*

The edge numbering induces a numbering of the v_i half-edges at each vertex i, corresponding to which we attach a homomorphism

$$\rho_i : \mathbb{R} \to E^{\otimes v_i},$$

which is equivariant as a module over $\pi_1(M)$.

Denoting by V the number of vertices and by E the number of edges, we grade the collection of decorated graphs by

$$\begin{aligned} \operatorname{ord}\Gamma &= E - V, \\ \deg\Gamma &= 2E - 3V. \end{aligned} \tag{3.1}$$

Remark 3.2. Compared with the decorated graphs we introduced in [4], this definition adds two further decorations: the numbering of the edges and the equivariant homomorphisms attached to the vertices.

Remark 3.3. A trivalent diagram has degree zero, and its order is given by $m = V/2 = E/3$.

We thank S. Garoufalidis for pointing out that our choice of the words "order" and "degree" is a bit unfortunate, for people working with finite type invariants call m the degree (instead of the order) of a trivalent graph.

However, we prefer to stick to our old notation [4] since the term degree is consistent with the cohomology defined by the coboundary operator δ (see Proposition 3.4).

Denoting by $v(\Gamma)$ the set of vertices and by $e(\Gamma)$ the ordered set of oriented edges in Γ, we can associate to the 3-dimensional homology sphere M and to the *trivalent* decorated graph Γ the number

$$A_\Gamma(M) \doteq \int_{C_n(M)} \left\langle \prod_{i\in v(\Gamma)} \rho_i \, , \prod_{(ij)\in e(\Gamma)} \hat{\eta}_{ij} \right\rangle , \tag{3.2}$$

where $n = 2 \operatorname{ord} \Gamma$ is the number of vertices, and (ij) denotes the edge connecting the vertex i to the vertex j.

Next we give the collection of decorated graphs the structure of an algebra over $\mathbb{Q}$ (the product simply being the disjoint union of graphs). We will denote this algebra by $\mathcal{G}_E^0$ and will extend (3.2) by linearity.

In view of the definition of $A_\Gamma(M)$, we introduce the following equivalence relation on $\mathcal{G}_E^0$: if two decorated graphs Γ and Γ' differ only by a permutation of order p in the vertex numbering and by l edge-orientation reversals, we set

$$\Gamma = (-1)^{(p+l)}\, \Gamma'. \tag{3.3}$$

Notice that to equivalent graphs we associate the same number $A_\Gamma(M)$. We will denote by $\mathcal{G}_E$ the algebra of graphs modulo the above equivalence relation.

Then we introduce an operator δ on $\mathcal{G}_E^0$ that acts by contracting a regular edge one at a time in Γ, followed by a consistent renumbering of edges and vertices. To the contraction of the regular edge connecting the vertex i to the vertex j we associate a sign $\sigma(i,j)$ defined by

$$\sigma(i,j) = \begin{cases} (-1)^j & \text{if } j > i, \\ (-1)^{i+1} & \text{if } j < i. \end{cases} \tag{3.4}$$

Assuming that this edge corresponds to the kth of the v_i half-edges at i and to the lth of the v_j half-edges at j, we attach to the vertex obtained after contraction the equivariant homomorphism

$$\widetilde{\rho}_i : \mathbb{R} \to E^{\otimes(v_i+v_j-2)}$$

defined by

$$\widetilde{\rho}_i = m_{k,v_i+l}(\rho_i \otimes \rho_j),$$

where m_{rs} denotes the scalar product between the rth and the sth terms in the tensor product.

Notice that the homomorphism attached to a vertex after contracting two different (regular) edges starting from it does not depend on the order of the contractions. Therefore, the argument we gave in [4] is enough to prove the following

Proposition 3.4. *The operator δ descends to $\mathcal{G}_E$ and satisfies $\delta^2 = 0$ there. Moreover, if we denote by $\mathcal{G}_{E;n,t}$ the (equivalence classes of) decorated graphs of order n and degree t, we have*

$$\delta : \mathcal{G}_{E;n,t} \to \mathcal{G}_{E;n,t+1}.$$

We call a cocycle an element of the kernel of δ in $\mathcal{G}_E$. Notice that the action of δ can be restricted to the algebra of (equivalence classes of) decorated connected graphs. Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. As in the proof of Theorem 2.5 we introduce the unit interval I as a parameter space, and define $\hat{\eta}$ as a form on $C_n(M) \times I$. Denoting by σ the projection $C_n(M) \times I \to C_n(M)$, we define

$$A_{\Gamma,\tau}(M) = \sigma_* \left\langle \prod_{i \in v(\Gamma)} \rho_i \, , \prod_{(ij) \in e(\Gamma)} \hat{\eta}_{ij} \right\rangle ,$$

and consider its change as τ varies on I.

Since the integrand form is closed, equation (2.5) implies that $dA_{\Gamma,\tau}(M)$ is given by boundary contributions only.

But on the boundary $\hat{\eta}$ reduces to $I(\eta)$—see Lemma 1.2—so we can use essentially the same arguments as in the trivial coefficient case [4]. Therefore, we will only give a brief sketch of the proof here and refer to [4] for further details.

First recall that the face in $\partial C_n(M)$ corresponding to the collapse of q points has the structure of a fibering over C_{n-q+1}, with the $(3q-4)$-dimensional fiber isomorphic to $C_q(\mathbb{R})$ modulo translations and scalings.

If we denote by e the number of edges connecting two collapsing vertices, we see that the vertical form-degree is given by $2e$. Moreover, since we are considering trivalent graphs, we have the relation $2e + e_0 = 3q$, where e_0 denotes the number of edges connecting a collapsing vertex with a noncollapsing one. Therefore, the push-forward along the fiber of the form associated to the edges connecting collapsing vertices yields a form of degree $4 - e_0$ if $e_0 \leq 4$, and zero otherwise.

By a theorem due to Axelrod and Singer [1]—which we have recast in [4] in a form suitable to our construction—this form must be the pullback of a multiple of a characteristic form on $M \times I$, namely, the constant function or the first Pontrjagin form p_1.

The former case corresponds to $e_0 = 4$, and it can be shown that the only case when the integral does not vanish is when this is given by the collapse of just two vertices. These boundary terms are then taken care of by the requirement that Γ be a cocycle.

The latter case corresponds to $e_0 = 0$, that is, to the case where all points collapse since we assume the diagrams to be connected. These terms are then taken care of by the correction $\phi(\Gamma)\,\mathrm{CS}$. q.e.d.

Remark 3.5. It is clear from the above proof that $\phi(\Gamma)$ is linear. Moreover, if Γ is a decorated graph, we can write

$$\phi(\Gamma) = \rho(\Gamma)\,\phi_0(\Gamma),$$

where $\rho(\Gamma)$ is a purely algebraic factor obtained from the homomorphisms ρ_i by associating a scalar product to each edge in Γ, while $\phi_0(\Gamma)$ is given by the boundary integral involving the forms η, and so it is the same as in the trivial coefficient case.

If $\operatorname{ord}\Gamma$ is even, there exists an orientation reversing involution under which the integrand form turns out to be odd (see [1] or [4]). Therefore, in this case, $\phi_0(\Gamma)$ vanishes, and so does $\phi(\Gamma)$.

From the above remark and from Theorems 1.1 and 2.5, we get the following

Corollary 3.6. *If Γ is a connected trivalent cocycle of $\mathcal{G}_E$, and ρ_1 and ρ_2 are equivariant homomorphisms such that $\langle\rho_1\,,\,\rho_2\rangle \neq 0$, then the quantity*

$$J_{\Gamma;\rho_1,\rho_2}(M) = A_\Gamma(M) + \frac{4}{\langle\rho_1\,,\,\rho_2\rangle}\,\phi(\Gamma)\,A_{(\Theta,\rho_1,\rho_2)}(M)$$

is an invariant for the rational homology 3-sphere M. Moreover, if $\operatorname{ord}\Gamma$ *is even, then $\phi(\Gamma) = 0$.*

4. Discussion

The graph cohomology introduced in the previous section is in principle more general than those introduced in [1] and [4] and might give rise to more general invariants.

In the case where E is the adjoint bundle of a Lie group G, we can choose all the equivariant homomorphisms associated to a trivalent graph to be determined by the structure constants, as explained in example 2.2. The cocycles in this case are those studied in [1] and come naturally from perturbative Chern–Simons theory. Antisymmetry of the structure constants implies that it is enough to give a *cyclic order* of the three half-edges at each vertex. The Jacobi identity then implies that the cocycles satisfy the so-called IHX relation (see [2]). It is a non-trivial fact (and we thank S. Garoufalidis for pointing this out) that these cocycles are in one-to-one correspondence with the cocycles of the trivial coefficient case.

If $E_{1/2} \oplus E_1$ (where $E_{1/2}$ and E_1 denote the representation of $SU(2)$ with spin 1/2 and 1) is an orthogonal and acyclic local system, we can choose each homomorphism as the dual of the second projection considered in example 2.4. In this case we can think of the diagram as carrying spin 1/2 over two of the three half-edges at each vertex and spin 1 over the last half-edge. Since each of these homomorphisms is *symmetric* with respect to the exchange of the two spin-1/2 representations, the diagram is symmetric under the exchange of the corrresponding half-edges.

It would be interesting to see if this or more general choices of the bundle E and of the equivariant homomorphisms give rise to new inequivalent cocycles.

Acknowledgements

We again thankfully acknowledge helpful conversations with Scott Axelrod, Robin Forman and Cliff Taubes. We are especially thankful to Stavros Garoufalidis for pointing out the isomorphisms in various graph cohomologies mentioned in section 4. One of these isomorphisms was also explained earlier by Dylan Thurston in his senior thesis at Harvard.

References

[1] S. Axelrod & I. M. Singer, *Chern–Simons perturbation theory,* Proc. XXth DGM Conference, (eds. S. Catto and A. Rocha), World Scientific, Singapore, 1992, 3–45; *Chern–Simons perturbation theory. II,* J. Differential Geom. **39** (1994) 173–213.

[2] D. Bar-Natan, *On the Vassiliev knot invariants,* Topology **34** (1995) 423–472.

[3] D. Bar-Natan, S. Garoufalidis, L. Rozansky & D. P. Thurston, *The Århus invariant of rational homology 3-spheres: A highly nontrivial flat connection on* S^3, q-alg/9706004.

[4] R. Bott & A. S. Cattaneo, *Integral invariants of 3-manifolds,* J. Differential Geom. **48** (1998) 91–133.

[5] T. Q. T. Le, J. Murakami & T. Ohtsuki, *On a universal quantum invariant of 3-manifolds,* q-alg/9512002, to appear in Topology.

HARVARD UNIVERSITY