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VIRTUALLY HAKEN DEHN-FILLING

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Abstract

We show that "most" Dehn-fillings of a non-fibered, atoroidal, Haken three-manifold with torus boundary are virtually Haken.

1. Results

Suppose that X is a compact, oriented, three-manifold with boundary a torus T. We will pick a basis of $H_1(T)$ represented by simple loops λ, μ such that $\lambda = 0$ in $H_1(X; \mathbf{Q})$. We call λ a *longitude* and μ a *meridian.* A *slope*, α, on T is the isotopy class of an essential unoriented simple closed curve. The manifold $X(\alpha)$ is the result of Dehn-filling along the slope α. This means that a solid torus is glued along its boundary to T so that a meridian disc of the solid torus is glued onto α. The manifold X is *atoroidal* if every $\mathbf{Z} \times \mathbf{Z}$ subgroup of $\pi_1 X$ is conjugate into $\pi_1 T$. The *distance* between two slopes α, β is $\Delta(\alpha, \beta)$ which is the absolute value of the algebraic intersection number of the homology classes represented by these slopes.

Theorem 1.1. *Suppose that X is a compact, connected, oriented, irreducible, atoroidal three-manifold with boundary a torus T. Suppose that S is a compact, connected, oriented, non-separating, incompressible surface properly embedded in X with non-empty boundary. Suppose that S is not a fiber of a fibration of X over the circle. Let g be the genus*

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of S, and b the number of boundary components of S. Then there is an integer $1 \leq P(X,S) \leq 3g-2+b$ with the following property.

Given an integer $p>0$ let $\pi : \widetilde{X}_p \longrightarrow X$ be the p-fold cyclic cover of X dual to S. Let Z be the closed three-manifold obtained from $\widetilde{X}_p$ by Dehn-filling each component of $\partial\widetilde{X}_p$ along a curve which is not isotopic in $\partial\widetilde{X}_p$ to a lift of a longitude.

Suppose that $p \geq 4P(X,S)$. Then there is an incompressible surface G homeomorphic to the double of S embedded in Z; also Z is irreducible, thus Z is Haken.

The integer $P(X,S)$ defined in Section 4 is called the *product length* of S in X; it measures how close S is to being a fiber of a fibration of X. It has been conjectured ([19, Problem 3.2]) that every closed irreducible 3-manifold, M, with infinite fundamental group has a finite cover which is Haken. This is known for various special cases; in particular, if M contains an immersed totally geodesic surface, [20], or if M has an orientation reversing involution and also has some finite cover, [22]. See also [2], [3].

Corollary 1.2. *Assume the hypotheses of the theorem. Suppose that $p = \Delta(\alpha,\lambda) \geq 4P(X,S)$. Then $X(\alpha)$ is virtually Haken. Thus $X(\alpha)$ is irreducible and, moreover, the $\mathbf{Z}_p$ abelian cover (dual to S) of $X(\alpha)$ contains an embedded incompressible surface of Euler characteristic $2\chi(S)$.*

It is known that at most 3 Dehn-fillings give a reducible manifold [14], also for a knot in the three sphere, the filling slopes must be integral [15].

Corollary 1.3. *Assume the hypotheses of the theorem. Suppose that α is a slope on T such that the length of α on a maximal cusp of the complete hyperbolic structure on X is greater than 2π. Also suppose that $\Delta(\alpha,\lambda) \geq 4P(X,S)$. Then $X(\alpha)$ is hyperbolic.*

Proof. The 2π theorem ([5, Theorem 4]) implies that $X(\alpha)$ admits a metric of negative sectional curvature. This in turn implies $X(\alpha)$ is atoroidal. By 1.2 $X(\alpha)$ has a cyclic cover, N, which is Haken. Then by Thurston's Uniformization theorem for Haken manifolds, N is hyperbolic. Hence $X(\alpha)$ is homotopy equivalent to a hyperbolic manifold, [10]. By recent work of Gabai [13], and Gabai-Meyerhoff-Thurston, one knows that homotopy equivalence implies homeomorphism in this situation.

Corollary 1.4. *Let K be a knot in the three-sphere with hyperbolic complement and which does not fiber over the circle. Suppose that the genus of K is g. If $p \geq 12g - 4$ then the p-fold cyclic cover, N, of S^3 branched over K contains a closed incompressible surface of genus $2g$. Furthermore, N is hyperbolic.*

Proof. Apply 1.3 to the p-fold cyclic cover, X, of the knot complement. Then N is obtained by Dehn-filling of X along a meridian, $\widetilde{\mu}$, of X. Now $\widetilde{\mu}$ projects to p meridians of $S^3 - K$. Thus the length of $\widetilde{\mu}$ on the boundary of a maximal cusp of X equals p times the length of μ on a maximal cusp of $S^3 - K$ and is thus at least $p > 2\pi$. This completes the proof. In fact, it follows from the Orbifold Theorem of Thurston that if $p \geq 2$ then N has a geometric decomposition. Furthermore, it also follows that if $p \geq 3$ and K is not the figure-eight knot then N is hyperbolic.

The 2π-theorem plus work of Adams [1] implies that there are at most 24 Dehn-fillings of X which give a manifold having no metric of negative sectional curvature ([5, Theorem 6]). With these exceptions, arguing as above we obtain:

Corollary 1.5. *Assume the hypotheses of the theorem and that $\Delta(\alpha, \lambda) \geq 4P(X, S)$. Then except for at most 24 possibilities, $X(\alpha)$ is hyperbolic.*

This paper was inspired by the work of M.H. Freedman and B. Freedman [12]. The first author would like to thank the Université de Rennes 1 for hospitality during completion of this paper. We would like to thank Dave Gabai for suggesting the version of the proof presented here.

2. Proof of Main Theorem

Let X, S be as in Theorem 1.1. Thus X is compact with boundary a torus T. Let $\pi : \widetilde{X} \longrightarrow X$ be the infinite cyclic cover dual to S, and let $\widetilde{S}_0$ be a lift of S to $\widetilde{X}$. Let τ be a generator of the group of covering transformations and set $\widetilde{S}_n = \tau^n \widetilde{S}_0$. Then $\widetilde{X}$ has boundary a union of annuli, and there is at least one component of $\partial \widetilde{S}_i$ on each component of $\widetilde{X}$.

Given an integer $m > 0$ define a surface G_m in $\widetilde{X}$ as follows. The surfaces $\widetilde{S}_0, \widetilde{S}_m$ separate $\widetilde{X}$ into three components, one of which has compact closure, P. Define G_m to be the result of pushing the boundary of P into the interior of P. If $\widetilde{X}_p$ is the p-fold cyclic cover of X dual to

S, then G_m projects to an immersed surface in $\widetilde{X}_p$ which is embedded if $p \geq m$. Observe that G_m is homeomorphic to the double of S.

These surfaces were considered in [12], where it was shown that their compression often leads to a non-simply connected incompressible surface. The following proposition, which is proved in Section 4, shows that often no preliminary compression is necessary. The original hyperbolic intuition for this result comes from the fact that two quasi-Fuchsian groups which are "far apart" generate their free-product. The original proof used hyperbolic techniques to construct a certain lamination which turned out to be a finite collection of simple closed curves on S. A conversation with Gabai suggested a purely topological proof based on examining the characteristic submanifold of X cut open along S.

Proposition 2.1. *Suppose X and S are as in theorem 1.1. Then if $m \geq 2P(X,S)$ then the surface G_m in $\widetilde{X}$ constructed above is incompressible.*

Definition 2.2. A *book of I-bundles* is a connected 3-manifold V satisfying the following condition. There are disjoint solid tori $T_1, T_2, \cdots, T_n$ in V such that the closure of $V - \bigcup_i T_i$ is an I-bundle, B, over a compact, not necessarily connected surface, F, with no disc components. Let $\partial_v B$ be the I-bundle restricted to ∂F. We require that each component of $\partial_v B$ is an essential annulus in the boundary of some T_i. Thus, for each i, we have that $B \cap T_i$ is a union of parallel annuli in ∂T_i.

The above definition differs from that in [11] (which permits F to contain disc components) and the following proposition is stated there without proof on p 286.

Proposition 2.3. *Suppose that V is a book of I-bundles as above. For each $1 \leq i \leq n$, let μ_i be a meridian of T_i (the slope on ∂T_i which bounds a disc in T_i). Suppose that for all i that μ_i can not be isotoped to intersect the attaching annuli, $\partial_v B$, in fewer than two arcs. Then V is irreducible and has incompressible boundary.*

Proof. Let $\pi : \widetilde{V} \longrightarrow V$ be the universal cover. Since no component of F is a disc $\pi_1 \partial F \longrightarrow \pi_1 F$ is injective. Since no component of ∂F is glued to a meridian of a solid torus, using Van Kampen's theorem inductively we see that $incl_* : \pi_1(B) \longrightarrow \pi_1 V$ is injective. Thus $\widetilde{V}$ is the union of copies of the universal cover of B glued together along copies of the universal covers of the solid tori T_i. Also V is irreducible thus Haken. Thus $\widetilde{V}$ is homeomorphic to the interior union some of the boundary of a ball. We must show that the components of the boundary of $\widetilde{V}$ are planes.

Consider the dual 1-complex, Γ, to $\widetilde{V}$ coming from the decomposition into components of $\pi^{-1}B$ and $\pi^{-1}(\bigcup T_i)$. Thus Γ has one vertex for each component of $\pi^{-1}B$, and one for each component of $\pi^{-1}(\bigcup_i T_i)$. Two vertices are connected by an edge if the corresponding components intersect.

Then $\widetilde{V}$ retracts to Γ which is therefore a tree. The hypothesis that μ_i must intersect $\partial_v B$ at least twice implies that every vertex of Γ has degree at least 2.

Construct a graph Γ_0 in $\partial\widetilde{V}$ by taking one vertex for each component of $\partial\widetilde{V} \cap \pi^{-1}B$ and each component of $\partial\widetilde{V} \cap \pi^{-1}(\bigcup_i T_i)$. Connect two vertices by an edge if the corresponding components intersect. Then Γ_0 is a spine for $\partial\widetilde{V}$. There is simplicial map $\theta : \Gamma_0 \longrightarrow \Gamma$ defined in the obvious way. We claim that θ is locally injective. It thus follows that each component of Γ_0 is simply connected and hence $\partial\widetilde{V}$ is also so.

It suffices to check that the link of a vertex in Γ_0 injects into Γ. For a vertex, v, of Γ_0 corresponding to a component of $\pi^{-1}B$ the link of v is mapped isomorphically by θ to the link of θv. If v is a vertex of Γ_0 corresponding to a component of $\pi^{-1}(\bigcup_i T_i)$, then v has degree 2. The two edges emerging from v go to distinct components of $\pi^{-1}B$ by the hypothesis on μ_i. q.e.d.

Proof of 1.1. Since $p \geq 4P(X,S)$ we may choose $m > 0$ so that $m, p-m \geq 2P(X,S)$. Let $\pi_p : \widetilde{X} \longrightarrow \widetilde{X}_p$ be the covering space projection. By 2.1 it follows that both $G \equiv \pi_p G_m$ and $H \equiv \pi_p \tau^m G_{p-m}$ are disjoint incompressible surfaces in the interior of $\widetilde{X}_p$.

The incompressibility of G in the special case that S, and hence $\widetilde{X}_p$, has one boundary component is an immediate consequence of applying ([9, Theorems 2.4.3, 2.4.5]) to G and H, however we give a direct elementary proof.

Observe that $G \cup H$ separates $\widetilde{X}_p$ into three components.

Let Y be the closure of the component which contains $\partial\widetilde{X}_p$. Let A, B be the closures of the components with $\partial A = G$ and $\partial B = H$. Now Y is a regular neighborhood of $(\partial\widetilde{X}_p) \cup \widetilde{S}_0 \cup \widetilde{S}_m$. Define a manifold

$$V = \overline{Z - (A \cup B)} = Y \cup \bigcup_i T_i,$$

where T_i is a solid torus glued onto the i'th boundary component of $\widetilde{X}_p$. Thus V is a book of I-bundles. The hypothesis in 1.1 that the

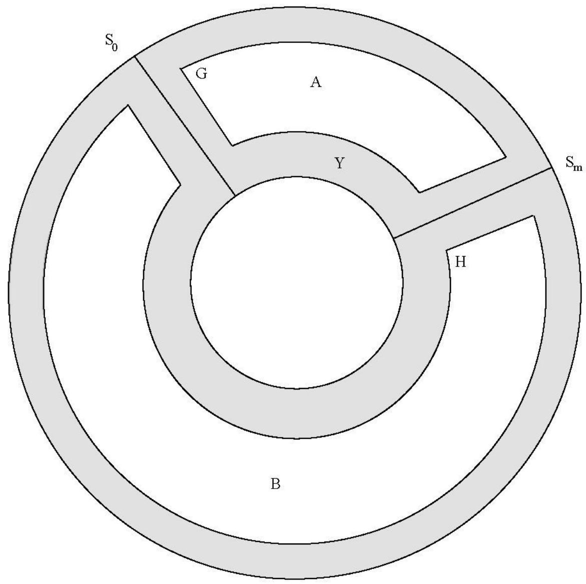

FIGURE 1

Dehn-filling curves are not isotopic to lifts of components of ∂S means that $\widetilde{S}_0$ and $\widetilde{S}_m$ both intersect every meridian curve of each T_i. Thus the hypotheses in 2.3 are satisfied. Hence ∂V is irreducible and has incompressible boundary. Now

$$Z = A \cup V \cup B.$$

Also A and B are both irreducible with incompressible boundary by 2.1. It follows that Z is irreducible and that G is incompressible in Z. q.e.d.

3. Characteristic product regions

Definition 3.1. Let N be a connected 3-manifold with connected boundary. Suppose that $\partial_v N$ is a subsurface of ∂N such that each component is a compact annulus. Suppose that $\partial_v N$ separates ∂N into two incompressible components with closure S_0, S_1 and which are diffeomorphic. Also suppose that each annulus has one boundary component in each of S_0 and S_1. We call $\partial_v N$ the **vertical** boundary of N, and

$\partial_h N = S_0 \cup S_1$ the **horizontal boundary.** We call N a **relative cobordism** between S_0 and S_1. A **vertical arc** is an arc in N with one endpoint in each of S_0 and S_1. A **vertical square** is a disc D properly embedded in N which intersects $\partial_v N$ in two vertical arcs which are called the **vertical boundary** of D and written $\partial_v D$. The **horizontal boundary** of D is $\partial_h D = D \cap \partial_h N$. A vertical square is called **essential** if it is not isotopic rel boundary into the boundary of N. An annulus embedded in N is called **vertical** if it has one boundary component in each of S_0 and S_1. We denote the unit interval as $I = [0, 1]$. A **product region** for N is a submanifold $\Phi \times I$ of N such that Φ is a compact, possibly disconnected, surface and for $i = 0, 1$ we have $\Phi \times i$ is an incompressible subsurface of S_i. We also require that $\Phi \times I$ contains $\partial_v N$, and that each component of $\Phi \times I$ contains at least one component of $\partial_v N$.

Our first goal is now to show that there is a characteristic product region which is unique up to isotopy. This assertion follows from the existence and properties of the characteristic submanifold of N. We will give a direct proof instead of deducing it from standard statements: [18], [17].

Lemma 3.2. *Suppose that N is an orientable, irreducible relative cobordism and D is a vertical square in N. If A is an incompressible vertical annulus properly embedded in N, then we may isotop D rel $\partial_v D$ so that $D \cap A$ consists of vertical arcs. If D' is another vertical square in N and if $\partial_v D$ and $\partial_v D'$ are disjoint, then we may isotop D rel $\partial_v D$ so that $D \cap D'$ consists of vertical arcs.*

Proof. Since $\partial A = \partial_h A$ we may isotop D rel $\partial_v D$ so that A and D are transverse and their intersection has the smallest number of components. Suppose C is a component of $A \cap D$. If C is a circle then C bounds a disc, Δ, in D. Since A is incompressible, C also bounds a disc in A. If C is innermost on A then the union of these discs is a sphere, which bounds a ball, B, because N is irreducible. The interior of B is disjoint from D. Thus the disc D can be isotoped across B to remove C. This isotopy is fixed on the boundary. Since the intersection was minimal, there are no circles in $A \cap D$.

If C is an arc with both endpoints on S_0, then there is a disc Δ in A with boundary C plus an arc, α, in ∂A. We may choose an innermost such disc so that the interior of Δ is disjoint from D. There is also a disc Δ' in D with boundary C plus an arc, α', in ∂D. Then $\alpha \cup \alpha'$ is a simple loop in S_0 which bounds the disc $\Delta \cup \Delta'$, and this loop bounds a disc in S_0, since S_0 is incompressible. We may choose $\alpha \cup \alpha'$ to be an innermost loop in S_0. But then C can be removed by an isotopy of D. Similarly, arcs with both endpoints in S_1 can be removed. Hence $A \cap D$ consists of vertical arcs.

A similar argument works for a disc D' in place of the annulus A. q.e.d.

Theorem 3.3. *Suppose that N is a relative cobordism between S_0 and S_1, and assume that N is orientable and irreducible. Suppose that $\Phi \times I$ is a product region in N with $|\chi(\Phi)|$ maximal. Then every vertical square in N can be properly isotoped into $\Phi \times I$.*

Proof. Observe that a regular neighborhood of $\partial_v N$ is a product region for N. Consider a product region $\Phi \times I$ of N with minimal Euler characteristic. We will regard Φ as a subsurface of S_0. Since every component of Φ contains a component of ∂S_0, the number of components of Φ is bounded by $|\partial S_0|$. Since Φ is incompressible it follows that $\chi(\Phi) \geq \chi(S_0)$ and so there is such a Φ with minimal Euler characteristic. Observe that $(\partial\Phi) \times I$ is a collection of vertical annulli. Now given a vertical square D, we can isotop it so that the intersection of D with $(\partial\Phi) \times I$ consists of vertical arcs, and has the minimum number of such arcs. Since a product region must contain $\partial_v N$ and since D contains two vertical arcs in its boundary, it follows that either D is contained in $\Phi \times I$ or else $\partial_h D$ intersects $\partial\Phi$ in the interior of $\partial_n N$. In the latter case, there is an arc, γ, in S_0 with endpoints in $\partial\Phi$ and interior disjoint from Φ. Then γ is not isotopic rel endpoints into Φ for otherwise we could isotop D to reduce the number of vertical arcs of intersection of D with A. Let δ_1, δ_2 be the two vertical arcs in $D \cap A$ which have endpoints in common with γ. Then there is a sub-disc D_- of D with boundary $\delta_1 \cup \gamma \cup \delta_2 \cup \gamma'$ where γ' is contained in $D \cap S_1$.

Define R to be a regular neighborhood of $\Phi \times I \cup D_-$. Then $R = \Psi \times I$ where Ψ is a regular neighborhood of $\Phi \cup \gamma$ in S_0. Since γ is not isotopic rel endpoints into Φ, it follows that Ψ is incompressible thus R is a product region with smaller Euler characteristic, a contradiction.

q.e.d.

We shall call such a product region a **characteristic product region.** This terminology is justified by the following two results.

Theorem 3.4. *Suppose that N is a relative cobordism between S_0 and S_1 and assume that N is orientable and irreducible. Suppose that*

$\Phi \times I$ is a characteristic product region for N and that $\Psi \times I$ is any product region in N. Then there is an ambient isotopy of N which takes $\Psi \times I$ into $\Phi \times I$.

Proof. Suppose that there is a component, A, of $\partial_v \Psi \times I$ which is not contained in $\Phi \times I$. Then A is a vertical annulus. Because each component of Ψ contains a component of ∂S_0, we may choose an arc, γ, in Ψ from ∂S_0 to a point, x, on $A \cap S_0$. There is an path, δ, which runs along γ then runs round $A \cap S_0$ and then back along γ, and by moving this path slightly we may suppose δ is an arc embedded in Ψ. The vertical square $\delta \times I$ contained in $\Psi \times I$ may be isotoped rel vertical boundary into $\Phi \times I$. It is now easy to isotop the rest of A into $\Phi \times I$. We do this for each component of $\partial_v \Psi \times I$. After these isotopies $\partial_v \Psi \times I$ is contained in $\Phi \times I$ and hence $\Psi \times I$ is contained in $\Phi \times I$. q.e.d.

Theorem 3.5. *Suppose that N is a relative cobordism and that N is orientable and irreducible. If $\Phi \times I$ and $\Psi \times I$ are characteristic product regions of N, then there is an ambient isotopy of N which takes $\Psi \times I$ onto $\Phi \times I$.*

Proof. By 3.4 we can isotop $\Psi \times I$ into $\Phi \times I$ and we may also isotop $\Phi \times I$ into $\Psi \times I$. Combining these, we may isotop $\Psi \times I$ so that it is contained in $\Phi \times I$ and contains this manifold minus a collar. Uniqueness of collars now shows that $\Psi \times I$ may be isotoped to equal $\Phi \times I$. q.e.d.

4. The incompressibility of G_m

Suppose that X and S are as in Theorem 1.1. Define N to be the 3-manifold obtained by removing from X a regular neighborhood of S and taking the closure. There are two copies, S_0 and S_1, of S in the boundary of N. We will regard $N - (S_0 \cup S_1)$ as equal to $X - S$. Thus X is obtained from N by identifying S_0 with S_1 via a homeomorphism $\phi : S_1 \longrightarrow S_0$. Since S is incompressible in X, it follows that N is a relative cobordism between S_0 and S_1. Let $\pi : \widetilde{X} \longrightarrow X$ be the infinite cyclic cover dual to S, and let $\widetilde{S}_0$ be a lift of S to $\widetilde{X}$. Let τ be a generator of the group of covering transformations and set $\widetilde{S}_n = \tau^n \widetilde{S}_0$. We can regard $\widetilde{X}$ as $\bigcup_i N_i$ where N_i is a copy of N with $N_i \cap N_{i+1} = \widetilde{S}_i$ and $\tau N_i = N_{i+1}$. Define Y_n to be the submanifold $Y_n = \bigcup_{i=1}^n N_i$ of $\widetilde{X}$. Then Y_n is a relative cobordism between $\widetilde{S}_0$ and $\widetilde{S}_n$. Since X is irreducible and has incompressible boundary, it follows that $\widetilde{X}$ and Y_n are both irreducible. Define P_n to be a characteristic product region in Y_n.

Lemma 4.1. *With the hypotheses of 1.1, after an isotopy of P_k in Y_k we may arrange that for all $i \leq k$ that $P_k \cap Y_i$ is contained in the characteristic product region P_i of Y_i.*

Proof. Suppose that P is a component of P_k. The first step is to arrange that for all $0 \leq i \leq k$, that $R = \widetilde{S}_i \cap P$ is a connected incompressible surface in P. This is already true for $i = 0, k$. Now for $0 < i < k$ we have that ∂R is contained in $\partial_v P$ which is a union of annuli. Suppose there is a component, C, of ∂R which bounds a disc, D, in $\partial_v P$. Since $\widetilde{S}_i$ is incompressible, C bounds a disc, Δ, in $\widetilde{S}_i$. If we choose C innermost on $\widetilde{S}_i$, then $D \cup \Delta$ is a sphere which bounds a ball, B, in Y_k since Y_k is irreducible. Thus the interior of B is disjoint from $\widetilde{S}_i$. Then we can isotop D across the ball to remove C. We may thus assume that every component of ∂R is essential in $\partial_v P$. Suppose that D is a compressing disc for R in P. Since $\widetilde{S}_i$ is incompressible in Y_k, there is a disc Δ in $\widetilde{S}_i$ with the same boundary as D. Then Δ intersects $\partial_v P$ in circles, and since $\partial_v P$ is incompressible in Y_k, these circles bound discs in $\partial_v P$. But these circles are in ∂R and so there are no such circles. Hence Δ is contained in P, and thus also in R, so R is incompressible in P.

Now $P \cong I \times \Phi$ for some compact connected surface Φ, and R is an incompressible surface, possibly disconnected, in $I \times \Phi$ such that ∂R is contained in $I \times \partial\Phi$. Suppose that R_0 is a component of R. The projection of $I \times \Phi$ onto $0 \times \Phi$ maps R_0 π_1-injectively and sends boundary to boundary. Applying Theorem 13.1 of [16] to R_0 either this map is homotopic to a covering, or R_0 is an annulus and the map is homotopic rel boundary into $\partial\Phi$. In this case, both boundary components of R_0 are on the same annulus component of $I \times \partial\Phi$. Thus there is an annulus, B, in $\partial_v P$ with the same boundary as R_0, and $B \cup R_0$ is the boundary of a solid torus in P. Then, after choosing an innermost such solid torus, we can isotop B across this solid torus to remove R_0 from R. Hence we may assume the map of $R_0 \longrightarrow \Phi$ is homotopic to a covering. Define C to be a component of $\partial\widetilde{S}_0 \cap \Phi$. Recall that every component of a product region contains a component of $\partial\widetilde{S}_0$. In particular there is a component, C', of ∂R_0 which maps onto C. But C' is the only component of $\partial\widetilde{S}_i$ which maps to C and so $R_0 = R$ is connected.

Now P is a product $I \times \Phi$. Also R is an incompressible connected surface in P which separates P. In addition, R is disjoint from $\partial I \times \Phi$. It follows from the homotopy cobordism theorem for Haken manifolds that the submanifold of P between $0 \times \Phi$ and R is a product region for Y_i. Since Y_i is irreducible, 3.4 applied to Y_i implies that R may be isotoped into the charcteristic product region, P_i, of Y_i. q.e.d.

It now follows from 4.1 that there is an ambient isotopy of Y_k taking $P_{k+1} \cap Y_k$ into P_k, so we may assume that

$$Y_k \cap P_{k+p} \subset P_k.$$

Thus the compact connected subsurfaces $A_k = \widetilde{S}_0 \cap P_k$ are decreasing, in other words $A_{k+1} \subset A_k$, and each is π_1-injective. We will define $A_0 = \widetilde{S}_0$.

Lemma 4.2. *With the hypotheses of 1.1, then for each* $k \geq 0$, *one of the following occurs:*

- A_k *is a regular neighborhood of* $\partial\widetilde{S}_0$ *in* $\widetilde{S}_0$.
- A_{k+1} *is not isotopic to* A_k.

We say that A_k *is a* **strictly decreasing** *sequence of surfaces.*

Proof. Suppose that A_{k+1} is isotopic to A_k. Let

$$\tau : \widetilde{X} \longrightarrow \widetilde{X}$$

be the generator of the group of covering transformations such that $\tau\widetilde{S}_0 = \widetilde{S}_1$. Then $P_{k+1} \cap \tau Y_k$ is a product region in τY_k and by 4.1 can therefore be isotoped in τY_k into the characteristic product region which is τP_k. Therefore the surface $R = P_{k+1} \cap \widetilde{S}_1$ can be isotoped in $\widetilde{S}_1$ into τA_k. Since P_{k+1} is a product, R is diffeomorphic to A_{k+1}, hence to A_k. Now R is a π_1-injective subsurface of τA_k, and is diffeomorphic to τA_k. Therefore R can be isotoped in $\widetilde{S}_1$ so that $R = \tau A_k$. The submanifold $Q = P_{k+1} \cap Y_1$ now has boundary

$$\partial Q \cong A_k \cup (I \times \partial A_k) \cup \tau A_k.$$

The image, πQ, of Q in X is obtained from Q by identifying A_k and τA_k via τ. Thus πQ has boundary consisting of tori

$$\partial(\pi Q) \cong S^1 \times \partial A_k.$$

Since every essential torus in X is boundary parallel, $\partial(\pi Q)$ is contained in a regular neighborhood of ∂X, hence ∂A_k is contained in a regular neighborhood of $\partial\widetilde{S}_0$. Since X does not fiber over the circle with fiber S, the characteristic product region of N is not all of N, so A_k is a proper subsurface of $\widetilde{S}_0$. Thus A_k is contained in a regular neighborhood of $\partial\widetilde{S}_0$. q.e.d.

Lemma 4.3. *Suppose that S is a compact hyperbolic surface of genus g with b boundary component. A set Λ, of disjoint, pairwise non-parallel, non-boundary-parallel simple closed curves on S has at most $3g-3+b$ components.*

Proof. Enlarge the set Λ so that it is maximal subject to the above conditions. Then cut open S along Λ to obtain a surface S^-. The components of S^- are pairs of pants, otherwise some component has a simple closed curve which is not boundary parallel. Since the Euler characteristic of a pair of pants is -1, the number of pairs of pants is $|\chi(F)| = 2g-2+b$. These pants have $3(2g-2+b)$ boundary curves. Now b of these curves are in the boundary of S. Each of the components of Λ appears twice among the $3(2g-2+b)-b$ remaining boundary curves. Hence the number of closed curves is at most $3g-3+b$. q.e.d.

We say that the charcteristic product region of N is *trivial* if it is a regular neighborhood of the vertical boundary $\partial_v N$.

Corollary 4.4. *With the hypotheses of 1.1, then for each $k \geq 3g-2+b$, the charcteristic product region of Y_k is trivial.*

Proof. Suppose that $A_0 \supset A_1 \supset A_2 \supset \cdots A_k$ is a strictly decreasing sequence of surfaces with A_k a regular neighborhood of $\partial\widetilde{S}_0$. A sequence of maximal length is obtained if A_{i+1} is obtained from A_i be deleting a neighborhood of an essential simple closed curve which is not parallel to ∂A_i. The number of such curves by 4.3 is at most $3g-3+b$. Thus if $k > 3g-3+b$, then P_k is a product. q.e.d.

Definition 4.5. With X and S as in Theorem 1.1 the product length $P(X,S)$ is the smallest k for which the characteristic product region of Y_k is trivial. If S is a fiber, then $P(X,S)$ is ∞.

The following argument was told to us by Gabai.

Lemma 4.6. *Suppose that M and N are irreducible relative cobordisms which do not contain essential vertical squares. Suppose that*

$$\partial_h M = S_0 \cup S_1, \qquad \partial_h N = T_0 \cup T_1,$$

and that $\phi : S_1 \longrightarrow T_0$ is a homeomorphism. Let P be the 3-manifold formed from M and N by identifying S_1 with T_0 via ϕ. Then P has incompressible boundary.

Proof. Suppose that D is a compressing disc for P. Note that P is a relative cobordism between S_0 and T_1. We may isotop D so that:

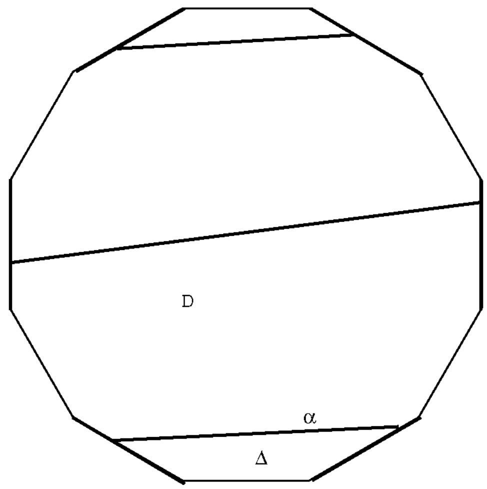

FIGURE 2

- D is transverse to $S_1 \equiv T_0$.
- The intersection of ∂D with $\partial_v P$ consists of vertical arcs.

Choose D, subject to the above, so that it has the minimum number of vertical arcs in its boundary. If there are no vertical arcs, then D is either contained in M or in N. Without loss of generality, we suppose that D is contained in M. Then D is a compressing disc for S_0 in M, but the hypothesis that M is a relative cobordism includes that S_0 is incompressible. Hence we have a contradiction. The boundary of D is made up of horizontal and vertical arcs which alternate. This is shown in Figure 2, where the vertical arcs are shown thicker.

Observe that since M and N are irreducible and they are glued together along incompressible surfaces, then P is also irreducible and $S_1 \equiv T_0$ is incompressible in P. The intersection of D with S_1 consists of arcs and circles. Since S_1 is incompressible in P, each such circle bounds a disc in S_1 and so we may isotop D to remove the circles. Thus each vertical side of D has an endpoint of exactly one arc of intersection in $D \cap S_1$. Now choose an outermost (on D) arc, α, of the intersection of D with S_1. We claim that the situation is as shown in Figure 2.

Thus there is a disc, Δ, in D with boundary the union of α and an arc β in ∂D, and β intersects exactly two vertical arcs, γ, δ in ∂D. Now Δ is contained in either M or in N, without loss of generality we will assume that Δ is contained in M. Thus Δ is a vertical disc in M and so is boundary parallel. Hence D can be isotoped to reduce the number of vertical arcs in its boundary, a contradiction. q.e.d.

Proof of 2.1. Since $m \geq 2P(X,S)$ it follows that $m = k + k'$ with $k, k' \geq P(X,S)$. Then by 4.4 the characteristic product region in Y_k and $Y_{k'}$ are both trivial. Thus by 3.3 there are no essential vertical squares in Y_k or $Y_{k'}$. So $M = Y_k$ and $N = \tau^k Y_{k'}$ satisfy the hypotheses of 4.6. Hence $Y_m = (M \cup N)$ has incompressible boundary. Now

$$\widetilde{X} = \left(\bigcup_{i\leq 0} N_i\right) \cup_{\widetilde{S}_0} Y_m \cup_{\widetilde{S}_m} \left(\bigcup_{i>m} N_i\right),$$

and $\widetilde{S}_0$ and $\widetilde{S}_m$ are both incompressible in $\widetilde{X}$. It follows that ∂Y_m is incompressible in $\widetilde{X}$. Now ∂Y_m is isotopic to G_m. q.e.d.

Remark. The main result of this paper has been used in [4] and [21] to prove the existence of immersed boundary slopes. Related ideas are used in [24].

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