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OBSTRUCTING FOUR-TORSION IN THE CLASSICAL KNOT CONCORDANCE GROUP

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In 1961 Fox [5] presented the definition of the knot concordance group and summarized the proof that it contains elements of order 2. Details of the argument were presented by Fox and Milnor in [6]. Whether or not there is torsion of other orders was left as an open question. Levine, in his classification of knot concordance [14], showed that 4-torsion exists in higher dimensional concordance groups and offered candidates for 4-torsion in dimension 3. Since then no progress has been made in understanding possible torsion in the classical knot concordance group. Our main result is the following:

0.1 Theorem. *Let K be a knot in S^3 with 2-fold branched cover M_K. If $|H_1(M_K)| = pm$ with p a prime congruent to 3 mod 4 and $gcd(p,m) = 1$, then K is of infinite order in the classical knot concordance group, $\mathcal{C}_1$.*

Our interest in this result is its application to the study of 4-torsion in the concordance group. There are 11 prime knots of 10 or fewer crossings, beginning with the knot 7_7, that represent elements of order 4 in the algebraic concordance group. A simple calculation using this theorem yields:

0.2 Corollary. *No prime knot with fewer than 11 crossings represents an element of order 4 in $\mathcal{C}_1$.*

Of greater interest than obstructing individual knots from being of order 4 is that the obstruction depends only on an abelian invariant of

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the knot. Hence corollaries like the next one concerning the Alexander polynomial of a knot, $\Delta_K(t)$, follow readily.

0.3 Corollary. *If $\Delta_K(t) = 5t^2 - 11t + 5$, then K is of infinite order in $\mathcal{C}_1$.*

By way of contrast, according to Levine [14], **every** knot with $\Delta_K(t) = 5t^2 - 11t + 5$ is of order 4 in $\mathcal{C}_{2k-1}$ for $k > 1$.

1. Introduction

In his classification of the knot concordance groups, Levine [13] defined the algebraic concordance groups, $\mathcal{G}_\pm$, of Witt classes of Seifert matrices and a homomorphism from the odd-dimensional knot concordance groups $\mathcal{C}_{4n\pm1}$ to $\mathcal{G}_\pm$. The homomorphism is induced by the function that assigns to a knot an associated Seifert matrix: it is an isomorphism on $\mathcal{C}_k, k \geq 5$; on $\mathcal{C}_3$ it is injective, onto an index 2 subgroup in the smooth category and surjective in the topological locally flat setting; for $k = 1$ it is surjective. However, Casson and Gordon [1], [2] proved that on $\mathcal{C}_1$ the kernel is nontrivial. (Casson and Gordon's original work applied in the smooth setting, but their results are now known to hold in the topological locally flat setting as well, a fact that follows from the existence of normal bundles in topological 4-manifolds, [7]. Similarly, our work applies in both categories.) Later, Jiang [10] extended Casson and Gordon's work to prove that the kernel of Levine's homomorphism is infinitely generated.

Levine [14] also proved that $\mathcal{G}_\pm$ is isomorphic to an infinite direct sum, $\mathcal{G}_\pm \cong \mathbf{Z}^\infty \oplus \mathbf{Z}_2^\infty \oplus \mathbf{Z}_4^\infty$. There is 2-torsion in $\mathcal{C}_1$ arising from amphicheiral knots, but beyond this little is known concerning torsion in $\mathcal{C}_1$. Fox and Milnor [6], in the paper in which knot concordance is defined, made this observation concerning amphicheiral knots and asked if there is torsion of any order other than 2. This question reappears as problem 1.32 of [11], [12]. In a different direction, in 1977 Gordon [9] (see also [12, Problem 1.94]) asked whether every order 2 class in $\mathcal{C}_1$ is represented by an amphicheiral knot; as of yet the only result bearing on this question is the observation that in higher dimensions the answer is no, and in dimension 3 there are order 2 classes in $\mathcal{G}_-$ that cannot be represented by order 2 knots in $\mathcal{C}_1$ [4].

In this paper we will use Casson-Gordon invariants to derive results concerning 4-torsion in the classical knot concordance group. That Casson-Gordon invariants can obstruct an individual knot that is of algebraic order 4 from being of order 4 in concordance may not be surprising, though explicit calculations appeared daunting; the examples presented here are the first. That the method applies to the knot 7_7 is pleasing in that this is the first knot identified by Levine [14] as a candidate to be of order 4. It is far more surprising that the obstructions we find depend only on the abelian invariants of the knot, such as the homology of the cover and the Alexander polynomial. In contrast to this, if a Seifert form represents 0 in $\mathcal{G}_-$, then there is a slice knot having that Seifert form; as a consequence, if a knot is algebraically slice, then there is a slice knot having identical abelian invariants. A related result states that any integral polynomial $\Delta(t)$ with $\Delta(1) = \pm 1$ is an Alexander polynomial of some slice knot if and only if $\Delta(t) = f(t)f(t^{-1})$ for some polynomial f [16].

Outline. In the next section we review the definition of concordance and of Casson-Gordon invariants. Section 3 presents the properties of the 3-dimensional bordism groups $\Omega_3(\mathbf{Z}_p)$ and $\Omega_3(\mathbf{Z}_p \oplus \mathbf{Z})$, and Section 4 describes how bordism invariants can be extracted from Casson-Gordon invariants. An implication of this work is Theorem 4.3, stating that if M^3 is a 3-manifold with $|H_1(M_K)| = pm$ where p is an odd prime and $\gcd(p,m) = 1$, then any Casson-Gordon invariant corresponding to a surjective $\mathbf{Z}_p$ character on M^3 is nontrivial. Section 5 reviews the use of Casson-Gordon invariants to obstruct slicing.

The proof of Theorem 0.1 is fairly technical, so we begin with two special cases. In Section 6 we prove the theorem for the prime 3. This case is especially easy since there is essentially only one $\mathbf{Z}_3$ character on M_K that must be considered. Section 7 presents a restricted form of Theorem 0.1, considering the case $p = 7$ but only proving that K is not of order 4. The point of isolating this case is to indicate how one can deal with the presence of more than one essential character. Finally, in Section 8 the full proof of Theorem 0.1 is presented. Section 9 discusses the corollaries.

2. Concordance and Casson-Gordon invariants

All our work holds in both the smooth and the topological locally flat categories. Throughout this paper p will denote a fixed odd prime. Homology groups are always taken with $\mathbf{Z}$ coefficients unless specifically noted otherwise. Manifolds are all orientable and compact.

2.1. The knot concordance group

A knot K in S^3 is called *slice* if $(S^3, K) = \partial(B^4, D)$ for some embedded 2-disk in B^4. Knots K_1 and K_2 are called *concordant* if $K_1 \# -K_2$ is slice, where $-K$ represents the mirror image of K. The set of concordance classes of knots forms an abelian group under connected sum, denoted $\mathcal{C}_1$.

2.2. Casson-Gordon invariants

Let $(M^3, \bar{\chi})$ be a closed 3-manifold with a homomorphism

$$\bar{\chi}\colon H_1(M^3) \to \mathbf{Z}_p \oplus \mathbf{Z}.$$

The bordism group $\Omega_3(\mathbf{Z}_p \oplus \mathbf{Z}) \cong \mathbf{Z}_p$ (see Section 3 below), so $p(M^3, \bar{\chi}) = \partial(W^4, \bar{\phi})$, where W^4 is a 4-manifold and

$$\bar{\phi}\colon H_1(W^4) \to \mathbf{Z}_p \oplus \mathbf{Z}.$$

Let $t(W^3, \bar{\phi}) \in L_0(\mathbf{C}(t))$ denote the intersection pairing on $H_2(W^4, \mathbf{C}(t))$, where the field coefficients are twisted by the $\mathbf{Z}$ action given by multiplication by t and the $\mathbf{Z}_p$ action given by multiplication by $e^{2\pi i/p}$. This pairing is viewed as an element in the Witt group of nonsingular hermitian forms on finite dimensional $\mathbf{C}(t)$ vector spaces. (In the case that the intersection form is singular, one must first mod out by the radical of the form to achieve a nonsingular pairing.)

The invariant τ is defined by

$$\tau(M^3, \bar{\chi}) = \frac{1}{p}(t(W^4, \bar{\phi}) - t_0(W^4)) \in L_0(\mathbf{C}(t)) \otimes \mathbf{Q},$$

where t_0 is the class represented by the standard intersection form on $H_2(W^4, \mathbf{C})$.

If K is a knot in S^3, M_K its 2-fold branched cover, and χ a character from $H_1(M_K)$ to $\mathbf{Z}_p$, then there is a naturally induced character $\bar{\chi}\colon H_1(M_{K,0}) \to \mathbf{Z}_p \oplus \mathbf{Z}$, where $M_{K,0}$ is the 3-manifold obtained from M_K by performing 0-surgery on the lift of K. This follows from the fact that $H_1(M_{K,0})$ naturally splits as $H_1(M_K) \oplus \mathbf{Z}$ with the generator of the $\mathbf{Z}$ factor given by the meridian of the lift of K. Hence, $\bar{\chi}$ is defined by mapping the meridian to $(0,1) \in \mathbf{Z}_p \oplus \mathbf{Z}$.

2.3 Definition. The *Casson-Gordon* invariant τ is defined by

$$\tau(K, \chi) = \tau(M_{K,0}, \bar{\chi}) \in L_0(\mathbf{C}(t)) \otimes \mathbf{Q}.$$

An associated signature invariant is defined as follows. For a class in $L_0(\mathbf{C}(t))$ the signature is defined by evaluating a representative of the class at a unit complex number and taking the limit of the signature of the resulting complex valued form as the unit complex number approaches 1. This map induces a homomorphism $\sigma\colon L_0(\mathbf{C}(t))\otimes\mathbf{Q}\to\mathbf{Q}$.

2.4 Definition. The Casson-Gordon signature invariant σ is defined by $\sigma(M^3,\bar\chi)=\sigma(\tau(M^3,\bar\chi))$, and $\sigma(K,\chi)=\sigma(\tau(K,\chi))$.

Note, in [1] this is denoted $\sigma_1\tau(K,\chi)$.

2.5. Additivity

Given a knot $K=K_1 \# K_2$, we have $M_K=M_{K_1}\# M_{K_2}$ and any $\mathbf{Z}_p$-valued character χ on $H_1(M_K)$ can be written as $\chi_1\oplus\chi_2$. A key result of Gilmer [8] is that in this situation $\tau(K,\chi)=\tau(K_1,\chi_1)+\tau(K_2,\chi_2)$.

3. Bordism results: the groups $\Omega_3(\mathbf{Z}_p)$ and $\Omega_3(\mathbf{Z}_p\oplus\mathbf{Z})$

The properties of the 3-dimensional bordism groups follow most easily from a consequence of the bordism spectral sequence: $\Omega_3(G)=H_3(G)$. Of course they follow as well from more general bordism theory; [3] is a good reference.

First we have that $\Omega_3(\mathbf{Z}_p)\cong\mathbf{Z}_p$. The map to $\mathbf{Z}_p$ is given as follows. For a pair (M^3,χ) with $\chi\colon H_1(M^3)\to\mathbf{Z}_p$ we view $\chi\in H^1(M^3,\mathbf{Z}_p)$. The quantity $\chi\cdot b(\chi)([M^3]_p)$ is the desired element in $\mathbf{Z}_p$. Here b represents the Bockstein, $b\colon H^1(M^3,\mathbf{Z}_p)\to H^2(M^3,\mathbf{Z}_p)$, the product is the cup product, and $[M]_p$ is the $\mathbf{Z}_p$ reduction of the fundamental class of M.

A useful alternative definition of the isomorphism is given using the linking form, $\beta\colon \mathrm{torsion}(H_1(M^3))\times\mathrm{torsion}(H_1(M^3))\to\mathbf{Q}/\mathbf{Z}$. The restriction of χ to $\mathrm{torsion}(H_1(M^3))$ is given by linking with some element $x\in\mathrm{torsion}(H_1(M^3))$; that is, $\chi(y)=\beta(x,y)$ for all $y\in\mathrm{torsion}(H_1(M^3))$. The self-linking of $x,\beta(x,x)$, is in $\mathbf{Q}/\mathbf{Z}$, but since it is p-torsion, it can be viewed as an element in $\mathbf{Z}_p$.

We have the following result:

3.1 Theorem. *If $|H_1(M^3)|=pm$ with p and m relatively prime, then for any nontrivial $\mathbf{Z}_p$ character, $[M^3,\chi]$ is nonzero in $\Omega_3(\mathbf{Z}_p)$.*

Proof. For such a manifold the Bockstein is an isomorphism. By Poincaré duality the cup product is nontrivial in $H^3(M^3,\mathbf{Z}_p)$. Hence, $\chi\cdot b(\chi)$ is nontrivial in $H^3(M^3,\mathbf{Z}_p)$ and the result follows.

We also will be using the fact that map $\Omega_3(\mathbf{Z}_p) \to \Omega_3(\mathbf{Z}_p \oplus \mathbf{Z})$ induced by inclusion is an isomorphism, with inverse induced by projection. This follows from either the Kunneth formula on homology or Kunneth results on bordism. Again, see [3].

4. Casson-Gordon invariants as bordism invariants

If $(M_1^3, \bar{\chi}_1)$ and $(M_2^3, \bar{\chi}_2)$ represent the same element in the bordism group $\Omega_3(\mathbf{Z}_p \oplus \mathbf{Z})$, then $\tau(M_1^3, \bar{\chi}_1)$ and $\tau(M_2^3, \bar{\chi}_2)$ differ by an element in $L_0(\mathbf{C}(t))$. Hence the difference $\sigma(M_1^3, \bar{\chi}_1) - \sigma(M_2^3, \bar{\chi}_2)$ is an integer. It follows that σ defines a homomorphism $\sigma' \colon \Omega_3(\mathbf{Z}_p \oplus \mathbf{Z}) \to \mathbf{Q}/\mathbf{Z}$. This homomorphism takes values in $((\frac{1}{p})\mathbf{Z})/\mathbf{Z}$, and so can be viewed as a homomorphism $\sigma_p \colon \Omega_3(\mathbf{Z}_p \oplus \mathbf{Z}) \to \mathbf{Z}_p$. The induced homomorphism on $\Omega_3(\mathbf{Z}_p)$ will also be denoted σ_p.

4.1 Theorem. *For p odd, σ_p is an isomorphism.*

Proof. A calculation of [1] shows that for the lens space $L(p,1)$ given by p-surgery on the unknot in S^3 with $\mathbf{Z}_p$ character taking value 1 on the meridian, $\sigma_p = 2$. Since p is odd, the result follows.

4.2 Theorem. *For a knot K and $\mathbf{Z}_p$ character χ on $H_1(M_K)$, $\sigma_p(M_{K,0}, \bar{\chi}) = \sigma_p(M_K, \chi)$. Equivalently,*

$$p\sigma(K, \chi) = \sigma_p(M_K, \chi) \mod p.$$

Proof. Since $\Omega_3(\mathbf{Z}_p \oplus \mathbf{Z}) \to \Omega_3(\mathbf{Z}_p)$ induces an isomorphism, it follows that $\sigma_p(M_{K,0}, \bar{\chi}) = \sigma_p(M_{K,0}, \chi \oplus (0))$, where (0) represents the trivial character to $\mathbf{Z}$. A $\mathbf{Z}_p$-bordism from $(M_{K,0}, \chi \oplus (0))$ to (M_K, χ) is constructed from $M_K \times [0,1]$ by adding a 2-handle with 0 framing to $M_K \times \{1\}$. Note that χ extends over this bordism since the 2-handle is added along a null homologous curve (the lift of K), and provides the desired $\mathbf{Z}_p$-bordism.

We can now make one of our key observations.

4.3 Theorem. *If $H_1(M_K) = pm$ with $gcd(p,m) = 1$ and p odd, then for any nontrivial $\chi \colon H_1(M_K) \to \mathbf{Z}_p$, $\sigma(K,\chi) \neq 0$.*

Proof. By Theorem 4.2, the nontriviality of $\sigma(K,\chi)$ will follow from that of $\sigma_p(M_K, \chi)$. Theorem 3.1 implies that $[M^3, \chi]$ is nontrivial in $\Omega_3(\mathbf{Z}_p)$. So Theorem 4.1 gives the desired result.

5. Casson-Gordon knot slicing obstructions

There are a number of formulations of how Casson-Gordon invariants provide obstructions to slicing. For our purposes the following fairly simple statement will suffice. Let p be an odd prime and let H_p denote the p-primary summand of $H_1(M_K)$. Again, we let β denote the torsion linking form.

5.1 Theorem. *If K is slice, there is a subgroup (or* metabolizer*) $L_p \subset H_p$ with $|L_p|^2 = |H_p|$, $\beta(L_p, L_p) = 0$, and $\tau(K, \chi) = 0$ for all χ vanishing on L_p.*

6. The Main Theorem: p = 3

We now have the required material to prove Theorem 0.1. To simplify notation, for any abelian group A, let A_p denote the p-primary summand of A.

6.1 Theorem. *If $|H_1(M_K)| = 3m$, where $gcd(3, m) = 1$, then K is of infinite order in $\mathcal{C}_1$.*

Proof. We have that $H_1(M_K)_3$ is isomorphic to $\mathbf{Z}_3$, generated by an element x with $\beta(x, x) = \pm(\frac{1}{3}) \in \mathbf{Q}/\mathbf{Z}$. Suppose now that K is of order d in $\mathcal{C}_1$. Any nontrivial metabolizing element for $H_1(M_{dK})_3 \cong (\mathbf{Z}_3)^d$ is of the form $(x_i)_{i=1,\ldots,d}$, where $x_i = \pm x$ for r values of i and is 0 otherwise. Hence, Theorem 5.1 yields that $\tau(\#_d M_K, (\chi_{x_i})_{i=1,\ldots,d}) = 0$ with exactly r of the $x_i = \pm 1$ and the rest 0. Here χ_y denotes the character given by linking with y.

Now, applying Gilmer's additivity theorem and taking signatures we have that $r\sigma(K, \chi_1) = 0$, where we have used that $\sigma(M_K, 0) = 0$ and $\sigma(M_K, \chi_{-1}) = \sigma(M_K, \chi_1)$. It follows of course that $\sigma(K, \chi_1) = 0$, contradicting Theorem 4.3.

7. The Case p = 7

The case of $p = 7$ introduces an issue that we want to consider before dealing with the general case. One interesting observation about the following argument is that the result does not follow from knowing the vanishing of the Casson-Gordon invariants for a spanning set of metabolizing elements, or even their multiples. This demonstrates the very nonlinear property of these invariants, and it is the first application we know of in which that nonlinearity plays such an essential role.

7.1 Theorem. *If $|H_1(M_K)| = 7m$, where $gcd(7, m) = 1$, then K is not of order 4 in $\mathcal{C}_1$.*

Proof. For such a knot K, a metabolizer for $H_1(M_{4K})_7 \cong (\mathbf{Z}_7)^4$ can be seen to be generated by a pair of elements $< (1, 0, 2, 3), (0, 1, -3, 2) >$. (There are other possibilities differing only in order and sign from this one.) Denoting by χ_a the $\mathbf{Z}_7$-character that takes value a on fixed generator of $H_1(M_K)_7$, we find from either of these metabolizing vectors that $\tau(M_K, \chi_1) + \tau(M_K, \chi_2) + \tau(M_K, \chi_3) = 0$. However, adding the generators we see that the metabolizer must also contain the vector $(1, 1, 6, 5)$ and its multiples, $(2, 2, 5, 3)$ and $(3, 3, 4, 1)$. From this we get the relations $3\tau(M_K, \chi_1) + \tau(M_K, \chi_2) = 0$, $3\tau(M_K, \chi_2) + \tau(M_K, \chi_3) = 0$, and $3\tau(M_K, \chi_3) + \tau(M_K, \chi_1) = 0$. Combining these one finds that $28\tau(M_K, \chi_1) = 0$ again contradicting Theorem 4.3.

8. The general case

We now prove Theorem 0.1.

8.1 Theorem. *If $|H_1(M_K)| = pm$ with p a prime congruent to 3 mod 4 and $gcd(p, m) = 1$, then K is of infinite order in $\mathcal{C}_1$.*

Proof. Suppose that dK is slice. The existence of a Z_p-metabolizer implies that d is a multiple of 4. (The linking form of $H_1(M_K)$ represents an element of order 4 in the Witt group of Z_p linking forms.) We begin by setting up some formalism to simplify the sort of linear algebra that appeared in the previous section. The example below illustrates the notation we develop next.

Any metabolizing vector for the linking form on $H_1(M_K, \mathbf{Z}_p)$ (in the subgroup L_p given by Theorem 5.1) can be written as $x = (x_i)_{i=1,\dots,d} \in (\mathbf{Z}_p)^d$. The condition that a corresponding Casson-Gordon invariant vanishes yields $\sum_{x_i \neq 0} \tau(M_K, \chi_{x_i}) = 0$. Now the x_i are in the cyclic group of nonzero elements in $\mathbf{Z}_p$. Denoting a generator for this group by g, each nonzero x_i corresponds to g^{α_i} for some α_i. If we introduce further shorthand, setting $t^{\alpha_i} = \tau(M_K, \chi_{x_i})$, we find that each metabolizing vector leads to a relation $\sum_{x_i \neq 0} t^{\alpha_i} = 0$. Note that at this point the symbol t^α does not represent a power of any element "t", it is purely symbolic. However it does permit us to view the relations as being elements in the ring $\mathbf{Z}[\mathbf{Z}_{p-1}]$. Furthermore, since $\tau_{x_i} = \tau_{p-x_i}$, we have that $t^j = t^{j+(p-1)/2}$. (Recall that $g^{(p-1)/2} = -1$.) Hence, we can view the relations as sitting in $\mathbf{Z}[\mathbf{Z}_q]$, where $q = (p-1)/2$.

Suppose that the metabolizing vector x corresponds to the relation $f = 0$, where f is represented by an element in $\mathbf{Z}[\mathbf{Z}_q]$. Then a calculation shows that ax corresponds to the relation $t^\alpha f$ where $g^\alpha = a$. Hence it follows that the relations between Casson-Gordon invariants generated by a given element $x \in L_p$ and its multiples forms an ideal in $\mathbf{Z}[\mathbf{Z}_q]$ generated by the polynomial f. Before applying this to complete the proof, we should pause for an example.

Example. Consider the metabolizing vector $x = (2, 3, 15, 16)$ in $(\mathbf{Z}_{19})^4$. The nonzero elements of $\mathbf{Z}_{19}$ are generated by 2, and we have $2 = 2^1$, $3 = 2^{13}$, $15 = 2^{11}$, and $16 = 2^4$. Hence in the notation just given, the vanishing of the corresponding Casson-Gordon invariant can be written as $t^1 + t^{13} + t^{11} + t^4 = 0$. Here we are in $\mathbf{Z}[\mathbf{Z}_{18}]$. Switching to $\mathbf{Z}[\mathbf{Z}_9]$ we have that $t^1 + t^4 + t^2 + t^4 = 0$. Notice that χ_{15} and χ_4 yield the same Casson-Gordon invariant, and that χ_{15} corresponds to t^{11}, while χ_4 to t^2 (since $2^2 = 4$) and $t^{11} = t^2 \in \mathbf{Z}[\mathbf{Z}_9]$.

Now consider the metabolizing vector $5x = (10, 15, 18, 4)$. Since $10 = 2^{17}, 15 = 2^{11}, 18 = 2^9$, and $4 = 2^2$, all mod 19, the corresponding relation in $\mathbf{Z}[\mathbf{Z}_{18}]$ is $t^{17} + t^{11} + t^9 + t^2 = 0$. Reducing to $\mathbf{Z}[\mathbf{Z}_9]$ gives $t^8 + t^2 + 1 + t^2$.

Hence by multiplying x by 5 we have gone from the equation $t^1 + t^4 + t^2 + t^4 = 0$ to $t^8 + t^2 + 1 + t^2 = 0$. Notice that the second polynomial is obtained from the first by multiplication by t^7. Finally $5 = 2^{16}$ mod 19, and $t^{16} = t^7 \in \mathbf{Z}[\mathbf{Z}_9]$.

To return to the proof, we must analyze the possible metabolizers L_p for $(\mathbf{Z}_p)^{4k}$, where $d = 4k$. Such a metabolizer must be generated by $2k$ elements. Applying the Gauss-Jordan algorithm to a basis for L_p, and perhaps reordering, we find a generating set $\{v_i\}_{i=1\ldots 2k}$ where the first $2k$ components of v_i are 0, except the i-component which is 1. Summing this basis produces the element $(1, 1, \ldots, 1, a_1, \ldots, a_{2k}) \in L_p$ where the first $2k$ entries are 1 and the a_i are unknown.

The corresponding relation is of the form $f = 2k + \sum_{i=1}^{k'} t^{\alpha_i} = 0$. (The sum may not contain $2k$ terms if any of the $a_i = 0$; hence k' is less than or equal to $2k$.) We next show that the ideal generated by f in $\mathbf{Z}[\mathbf{Z}_q]$ contains a nonzero integer. This will follow from the fact that f and $t^q - 1$ are relatively prime, which will be the case unless f vanishes at some q-root of unity, ω; however, by considering norms and the triangle inequality we see that this will be the case only if $k' = 2k$

and $\omega^{a_i} = -1$ for all i. But since q is odd, no power of ω can equal -1.

Since we now have that f and $t^q - 1$ are relatively prime, it follows that with $\mathbf{Q}$ coefficients (so that we are working over a PID) there is a polynomial g satisfying $gf = 1 \bmod (t^q - 1)$. Clearing denominators we find that for some integral polynomial h, $hf = n \bmod (t^q - 1)$ for some positive integer n.

The proof of the theorem is concluded by observing that we now have the relation corresponding to $n \in \mathbf{Z}[\mathbf{Z}_q]$. That is, $n\tau(M_K, \chi_1) = 0$. As before, this would imply that $\sigma(M_K, \chi_1) = 0$, contradicting Theorem 4.3

9. Corollaries

9.1. Low crossing number knots

Based on the work of Levine, Morita [15] developed an algorithm to determine the order of a knot in the algebraic concordance group using only its Alexander polynomial. Based on this, he enumerated all prime knots of 10 or fewer crossings that are of algebraic order 4. There are eleven such knots, including 7_7, 9_{34}, and nine 10 crossing knots. Of these, seven have $H_1(M^3)$ satisfying our criteria for $p = 3$. Three more satisfy the condition for $p = 7$, and the last, 10_{86}, has $H_1(M^3) = \mathbf{Z}_{83}$. Hence, Corollary 0.2 follows.

9.2. Polynomial conditions

A special case of Levine's results states that a knot K with $\Delta_K(t)$ quadratic is of finite order if $\Delta_K(t) = at^2 - (1+2a)t + a$ for some $a > 0$, and in that case it is of order 4 if for some prime $p = 3 \bmod 4$, $\Delta_K(-1) = p^\alpha m$ with α odd and $\gcd(p, m) = 1$. Our theorem applies only in the case that α can be assumed to be 1. However that is sufficient to give an infinite family of examples, beginning with $\Delta_K(t) = 5t^2 - 11t + 5$, where $\Delta_K(-1) = (3)(7)$.

9.3. Infinitely many linearly independent examples

Knots formed as twisted doubles of the unknot we among the first knots used to construct algebraically slice knots that are not slice [1], [2]. Jiang [10] used these knots to demonstrate that the set of algebraically slice knots contains a infinite set of knots that is linearly independent in concordance. Here we demonstrate that twisted doubles also provide such independent families of knots that are of algebraic order 4.

To achieve independence within a family of examples, our theorem must be extended somewhat. Here is the statement we need.

9.4 Theorem. *If $|H_1(M_K)| = pm$ with p a prime congruent to 3 mod 4 and $gcd(p, m) = 1$, and if J is any knot with $|H_1(M_J)| = q$, where $gcd(q, p) = 1$, then $dK\#J$ is not slice for all nonzero integers d.*

Proof. If we consider $\mathbf{Z}_p$ characters on the 2-fold branched cover, the characters all vanish on $H_1(M_J)$ so by the additivity of Casson-Gordon invariants we are reduced to considering the character restricted to $H_1(M_{dK})$, which places us in the setting of the proof of Theorem 0.1 in Section 8.

To apply this, let K_n denote the $(-n)$-twisted double of the unknot, with $n > 0$. Then $\Delta_{K_n}(t) = nt^2 - (1+2n)t + n$, and $H_1(M_{K_n}) = \mathbf{Z}_{4n+1}$. To pick an appropriate set of these knots, let $\{p_i\}$ be an enumeration of the primes that are congruent to 3 mod 4. Let $n_i = (p_{2i-1}p_{2i} - 1)/4$. Then the previous theorem quickly yields the following.

9.5 Corollary. *The subset of the set of twisted doubles of the unknot given by $\{K_{n_i}\}$, is a linearly independent set in the concordance group and consists only of knots of algebraic order 4.*

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