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# ON THE HOMOTOPY INVARIANCE OF HIGHER SIGNATURES FOR MANIFOLDS WITH BOUNDARY

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**Abstract**

If $M$ is a compact oriented manifold-with-boundary whose fundamental group is virtually nilpotent or Gromov-hyperbolic, we show that the higher signatures of $M$ are oriented-homotopy invariants. We give applications to the question of when higher signatures of closed manifolds are cut-and-paste invariant.

## 0. Introduction

The Novikov Conjecture hypothesizes that certain numerical invariants of closed oriented manifolds, called higher signatures, are oriented-homotopy invariants. It is natural to ask if there is an extension of the Novikov Conjecture to manifolds with boundary. Such an extension was made in [27], [30]. In this paper we show that if the relevant discrete group is virtually nilpotent or Gromov-hyperbolic then the higher signatures defined in [27], [30] are oriented-homotopy invariants.

Before giving our result, let us recall the statement of Novikov's conjecture. Let $M$ be a closed oriented smooth manifold. Let $L \in \mathrm{H}^*(M;\mathbb{Q})$ be the Hirzebruch $L$-class and let $*L \in \mathrm{H}_*(M;\mathbb{Q})$ be its Poincaré dual. If $\Gamma$ is a finitely-generated discrete group, let $B\Gamma$ denote its classifying space. Recall that $\mathrm{H}^*(B\Gamma;\mathbb{Q}) \cong \mathrm{H}^*(\Gamma;\mathbb{Q})$, the rational group cohomology of $\Gamma$. Let $\nu : M \to B\Gamma$ be a continuous map, defined up to

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homotopy. The Novikov Conjecture hypothesizes that the higher signature $\nu_*(*L) \in \mathrm{H}_*(B\Gamma;\mathbb{Q})$ is an oriented-homotopy invariant of the pair $(M,\nu)$. Equivalently, if $\tau \in \mathrm{H}^*(\Gamma;\mathbb{Q})$ then $\langle \nu_*(*L),\tau\rangle = \langle L \cup \nu^*\tau, [M]\rangle$ should be an oriented-homotopy invariant. If $\Gamma$ is virtually nilpotent or Gromov-hyperbolic then the validity of the Novikov Conjecture was proven by Connes and Moscovici [10, Theorem 6.6].

Now let $M$ be a compact oriented manifold-with-boundary, equipped with a continuous map $\nu : M \to B\Gamma$ which is defined up to homotopy. The formal definition of the higher signature of $M$, from [27, (67)] and [30, Definition 10], is

$$\sigma_M = \left( \int_M L(TM) \wedge \omega \right) - \widetilde{\eta}_{\partial M} \in \overline{\mathrm{H}}_*(\mathcal{B}^\infty). \tag{0.1}$$

The terms of this equation will be defined later in the paper. Briefly,

1. $L(TM) \in \Omega^*(M)$ is the $L$-form of $M$ associated to a Riemannian metric which is a product near the boundary,
2. $\mathcal{B}^\infty$ is a "smooth" subalgebra of the reduced group $C^*$-algebra $C_r^*(\Gamma)$, i.e., $\mathbb{C}\Gamma \subset \mathcal{B}^\infty \subset C_r^*(\Gamma)$ and $\mathcal{B}^\infty$ is closed under the holomorphic functional calculus in $C_r^*(\Gamma)$,
3. $\widetilde{\eta}_{\partial M}$, the higher eta-form [27, Definition 11], is an element of the space $\overline{\Omega}_*(\mathcal{B}^\infty)$ of noncommutative differential forms [18, Sections 1.3 and 4.1] and can be thought of as a boundary correction term,
4. $\omega$ is a certain closed biform in $\Omega^*(M) \widehat{\otimes} \overline{\Omega}_*(\mathbb{C}\Gamma)$ [26, Section V] and
5. $\overline{\mathrm{H}}_*(\mathcal{B}^\infty)$ is the noncommutative de Rham homology of $\mathcal{B}^\infty$ [9, p. 185], [18, Section 4.1].

As in [27, Section 4.7] and [30, Assumption 2], in order to make sense of the higher eta-form $\widetilde{\eta}_{\partial M}$ we must make an assumption about $\partial M$. To be slightly more general, let $F$ be a closed oriented manifold, equipped with a continuous map $\nu_0 : F \to B\Gamma$ which is defined up to homotopy. Associated to $\nu_0$ is a normal $\Gamma$-cover $\pi : F' \to F$ of $F$. There is an associated flat $C_r^*(\Gamma)$-vector bundle $\mathcal{V}_0 = C_r^*(\Gamma) \times_\Gamma F'$ on $F$. Let $\mathrm{H}^*(F;\mathcal{V}_0) = \mathrm{Ker}(d)/\mathrm{Im}(d)$ denote the usual (unreduced) de Rham or simplicial cohomology of $F$, computed using the local system $\mathcal{V}_0$. Let $\overline{\mathrm{H}}^*(F;\mathcal{V}_0) = \mathrm{Ker}(d)/\overline{\mathrm{Im}(d)}$ denote the reduced cohomology. There is an obvious surjection $s : \mathrm{H}^*(F;\mathcal{V}_0) \to \overline{\mathrm{H}}^*(F;\mathcal{V}_0)$.

**Assumption 1.**

a. The map $s : \mathrm{H}^k(F;\mathcal{V}_0) \to \overline{\mathrm{H}}^k(F;\mathcal{V}_0)$ is an isomorphism for $k = \left[\frac{\dim(F)+1}{2}\right]$.

b. If $\dim(F) = 2k$ then $\overline{\mathrm{H}}^k(F; \mathcal{V}_0)$ admits a (stable) Lagrangian subspace.

Assumption 1 is a homotopy-invariant assumption on $F$. If $F$ is endowed with a Riemannian metric then an equivalent formulation of Assumption 1.a. is:

1. If $\dim(F) = 2k$ then the differential form Laplacian on $\Omega^k(F')$ has a strictly positive spectrum on the orthogonal complement of its kernel.

2. If $\dim(F) = 2k - 1$ then the differential form Laplacian on $\Omega^{k-1}(F')/\mathrm{Ker}(d)$ has a strictly positive spectrum.

Given Assumption 1.a, Assumption 1.b. is equivalent to saying that the index of the signature operator of $F$, as an element of $K_0(C_r^*(\Gamma))$, vanishes. As examples,

(a) If $F$ has a cellular decomposition without any cells of dimension $k = \left[\frac{\dim(F)+1}{2}\right]$ then Assumption 1 is satisfied.

(b) If $\Gamma$ is finite and the signature of $F$ vanishes then Assumption 1 is satisfied.

(c) Let $F_1$ and $F_2$ be even-dimensional manifolds, with $F_1$ a connected closed hyperbolic manifold and $F_2$ a closed manifold with vanishing signature. Put $\Gamma = \pi_1(F_1)$. If $F = F_1 \times F_2$ and $\nu_0$ is projection onto the first factor then Assumption 1 is satisfied.

(d) If $\dim(F) = 3$, $F$ is connected, $\Gamma = \pi_1(F)$ and $\nu_0$ is the classifying map for the universal cover of $F$ then, assuming Thurston's geometrization conjecture, Assumption 1 is satisfied if and only if $F$ is a connected sum of spherical space forms, $S^1 \times S^2$'s and twisted circle bundles $S^1 \times_{\mathbb{Z}_2} S^2$ over $\mathbb{R}P^2$.

Suppose that Assumption 1 is satisfied. If $\dim(F) = 2k$, choose a (stable) Lagrangian subspace $L$ of $\overline{\mathrm{H}}^k(F; \mathcal{V}_0)$. Then the higher eta-form $\widetilde{\eta}_F$ is well-defined. In the case of a manifold-with-boundary $M$, let $i : \partial M \to M$ be the boundary inclusion. We take $F = \partial M$ and $\nu_0 = \nu \circ i$. In this case, if Assumption 1.a. holds then Assumption 1.b. holds.

The main result of this paper is the following:

**Theorem 0.1.** *If $\partial M$ satisfies Assumption 1 then $\sigma_M$ is an oriented-homotopy invariant of the pair $(M, \nu)$.*

By oriented-homotopy invariance of $\sigma_M$, we mean the following. Suppose that $h : (M_2, \partial M_2) \to (M_1, \partial M_1)$ is a degree 1 homotopy equivalence of pairs. In particular, $h(\partial M_2) \subset \partial M_1$, but $h\big|_{\partial M_2}$ is not assumed to be a homeomorphism from $\partial M_2$ to $\partial M_1$. Suppose that there are continuous maps $\nu_i : M_i \to B\Gamma$ such that $\nu_2$ is homotopic to $\nu_1 \circ h$. If $\dim(M_i) = 2k+1$, we assume that the (stable) Lagrangian subspaces for the boundaries are related by $(\partial h)^*(L_1) = L_2$. Then $\sigma_{M_1}$, computed using $\nu_1$, equals $\sigma_{M_2}$, computed using $\nu_2$. (If $\dim(M) = 2k+1$ then $\sigma_M$ generally depends on the choice of $L$.)

In order to obtain numerical invariants from $\sigma_M$, we must make an assumption about the smooth subalgebra $\mathcal{B}^\infty$.

**Assumption 2.** Each class $\tau \in \mathrm{H}^*(\Gamma; \mathbb{C})$ has a cocycle representative whose corresponding cyclic cocycle $Z_\tau \in ZC^*(\mathbb{C}\Gamma)$ extends to a continuous cyclic cocycle on $\mathcal{B}^\infty$.

If $\Gamma$ is virtually nilpotent or Gromov-hyperbolic then it is known that smooth subalgebras $\mathcal{B}^\infty$ of $C^*_r(\Gamma)$ satisfying Assumption 2 exist [11, Section 2], [17, Theorem 4.1]. We write $\langle \sigma_M, \tau \rangle$ for the pairing of $\sigma_M$ with $Z_\tau$.

**Corollary 0.2.** *Under Assumptions 1 and 2, the higher signatures $\langle \sigma_M, \tau \rangle$ are oriented-homotopy invariants.*

As special cases of Corollary 0.2, if $\partial M = \emptyset$ then

$$\langle \sigma_M, \tau \rangle = \text{const.} \langle L(TM) \cup \nu^* \tau, [M] \rangle$$

[26, Corollary 2] and so we recover the Connes-Moscovici result [10, Theorem 6.6]. At the other extreme, if $\partial M \neq \emptyset$, $\Gamma = \{e\}$, $\mathcal{B}^\infty = \mathbb{C}$ and $\tau = 1 \in \mathrm{H}^0(\{e\}; \mathbb{C})$ then $\langle \sigma_M, \tau \rangle$ is the Atiyah-Patodi-Singer formula for the signature of $M$ [1, Theorem 4.14], which is clearly an oriented-homotopy invariant of $M$.

Let us emphasize that Theorem 0.1 and Corollary 0.2 are stronger statements than just saying that the index of a signature operator is homotopy-invariant. The latter statement is true (Theorem 6.1 below), but it does not immediately imply Theorem 0.1 or Corollary 0.2. The situation is somewhat analogous to that of closed manifolds, where the homotopy invariance of Novikov's higher signatures, when proven, is a much deeper statement than the homotopy invariance of the symmetric signature.

Let us comment on Assumptions 1 and 2. Assumption 2 is a technical condition on $\Gamma$. Assumption 1 is more germane and is necessary for both analytic and topological reasons. On the analytic side, something like Assumption 1 is necessary in order to make sense of the formal expression for $\widetilde{\eta}_F$. On the topological side, Assumption 1 implies that the higher signature of $F$, with respect to $\nu_0$, vanishes. Of course, if $F = \partial M$ then its higher signature vanishes simply because $\partial M$ is a boundary, but Assumption 1 gives a reason for the vanishing which is intrinsic to $\partial M$.

(For clarity, we note that if we just want to *define* $\langle \sigma_M, \tau \rangle$ then we can get by with something weaker than Assumption 2. Namely, for a connected component $F$ of $\partial M$, put $\Gamma_F = \mathrm{Im}(\pi_1(F) \to \pi_1(M) \to \Gamma)$. Let $\mathcal{B}_F^\infty$ be a smooth subalgebra of $C_r^*(\Gamma_F)$. Then it is enough to assume that for each $F$, $\tau\big|_{\Gamma_F}$ extends to a cyclic cocycle on $\mathcal{B}_F^\infty$. For example, if $\partial M = \emptyset$ then there is no assumption on $\Gamma$ and we recover the Novikov higher signatures $\langle \sigma_M, \tau \rangle$ in full generality. However, in order to *prove* the homotopy-invariance of $\langle \sigma_M, \tau \rangle$, we need Assumption 2.)

From equation (0.1) and the smooth topological invariance of $\sigma_M$, we obtain a "Novikov additivity" for higher signatures.

**Corollary 0.3.** *Let $\Gamma$ satisfy Assumption 2. Let $M$ be a closed oriented manifold and let $F$ be a two-sided hypersurface which separates $M$ into pieces $A$ and $B$. Let $\nu : M \to B\Gamma$ be a continuous map, defined up to homotopy. Let $i : F \to M$ be the inclusion map and put $\nu_0 = \nu \circ i$. Suppose that $F$ satisfies Assumption 1. If $\dim(M) = 2k+1$, choose a (stable) Lagrangian subspace $L$ of $\overline{\mathrm{H}}^k(F; \mathcal{V}_0)$ and use $L$ to define $\sigma_A$, and $-L$ to define $\sigma_B$. Then for any $\tau \in \mathrm{H}^*(\Gamma; \mathbb{C})$, the corresponding higher signature of $M$ satisfies*

$$\text{const.}\, \langle L(TM) \cup \nu^* \tau, [M] \rangle = \langle \sigma_A, \tau \rangle + \langle \sigma_B, \tau \rangle.$$

As a consequence of Corollary 0.3, we obtain a sort of cut-and-paste invariance of the higher signatures of closed manifolds.

**Corollary 0.4.** *Let $\Gamma$ satisfy Assumption 2. Let $M_1$ and $M_2$ be closed oriented manifolds, equipped with continuous maps $\nu_i : M_i \to B\Gamma$ which are defined up to homotopy. Suppose that there are splittings $M_1 = A \cup_F B$ and $M_2 = A \cup_F B$ over separating two-sided hypersurfaces. (That is, both $M_1$ and $M_2$ are constructed by gluing $A$ to $B$, but the gluing diffeomorphisms $\phi_i : \partial A \to \partial B$ can be different.) Suppose that $\nu_1\big|_A$ is homotopic to $\nu_2\big|_A$, $\nu_1\big|_B$ is homotopic to $\nu_2\big|_B$ and $F$ satisfies Assumption 1. If $\dim(M_i) = 2k+1$ then we also assume that the gluing* *diffeomorphisms preserve a (stable) Lagrangian subspace of* $\overline{\mathrm{H}}^k(F;\mathcal{V}_0)$*; see Section 12 for the precise condition. Then the higher signatures of* $M_1$ *and* $M_2$ *coincide.*

Corollary 0.4 is relevant because the higher signatures of closed manifolds are generally not cut-and-paste invariant (over $B\Gamma$). For example, it is not hard to see that this is the case when $\Gamma = \mathbb{Z}$, using [19, Theorem 1.2], and the case $\Gamma = \mathbb{Z}^k$ then follows from [37, Lemma 8]. This shows that some condition like Assumption 1 is necessary if one wants to define higher signatures for manifolds-with-boundary so as to have Novikov additivity. Such a situation does not arise for the usual "lower" signature.

In general, it seems to be an interesting question as to for which groups $\Gamma$ and which cohomology classes $\tau \in \mathrm{H}^*(\Gamma;\mathbb{C})$, the corresponding Novikov higher signature (of closed manifolds) is a cut-and-paste invariant (over $B\Gamma$); see [30, Remark 4.1] and [38, Chapter 30] for further discussion.

We now give a brief description of the proof of Theorem 0.1. In the case of closed manifolds, the analytic proofs of the Novikov Conjecture, as in [10, Theorem 6.6], consist of two steps. First, one shows that the index of the signature operator, as an element of $K_*(C_r^*(\Gamma))$, is an oriented-homotopy invariant. Second, one constructs a pairing of $K_*(C_r^*(\Gamma))$ with $\mathrm{H}^*(\Gamma;\mathbb{C})$ and one verifies that the result is the Novikov higher signature. This last step amounts to proving an index theorem.

In the case of closed manifolds, many of the proofs of the first step implicitly use the cobordism invariance of the index. As even the usual "lower" signatures of manifolds-with-boundary are not cobordism invariant, this method of proof is ruled out for us. Instead, we give a direct proof of the homotopy invariance which, in the closed case, was developed by Hilsum and Skandalis [16]. To use their methods, we need $C_r^*(\Gamma)$-Fredholm signature operators with $C_r^*(\Gamma)$-compact resolvents. For this reason, in our case we would like to cone off the boundary on $M$ to obtain a conical manifold $CM$ (deleting the vertex point) and do analysis on the conical manifold, following Cheeger [8] and Bismut-Cheeger [3]. If $\mathcal{V}$ denotes the canonical flat $C_r^*(\Gamma)$-bundle on $CM$, we would consider the signature operator acting on $\Omega^*(CM;\mathcal{V})$, with its index in $K_*(C_r^*(\Gamma))$. We would then extend homotopy equivalences between manifolds-with-boundary to homotopy equivalences between conical manifolds, in order to compare their indices. However, as the boundary signature operator $\mathcal{D}_{\partial M}$ may well have continuous spectrum which goes down to zero (see [28] for examples), there are serious technical problems in carrying out the conical analysis. (The paper [24] looked at a special case in which $\Gamma$ is of the form $\Gamma' \times G$, with $G$ finite, and $\mathcal{D}_{\partial M}$ can be made invertible by twisting with a nontrivial representation of the finite group $G$. The corresponding index class was proven to be an oriented-homotopy invariant using the results of [20]. The higher APS-index formula of [22] was then applied in order to show that a twisted version of (0.1) was an oriented-homotopy invariant. However, we wish to deal with the general case here.)

In order to get around the problem of low-lying spectrum of $\mathcal{D}_{\partial M}$, we follow the method of proof sketched in [30, Appendix]. We basically add an algebraic complex to cancel out the small spectrum. More precisely, we consider a certain Hermitian complex $\widehat{W}^*$ of finitely-generated projective $C_r^*(\Gamma)$-modules. We form a new complex $C^* = \Omega^*(CM; \mathcal{V}) \oplus \left(\Omega^*(0,2) \widehat{\otimes} \widehat{W}^*\right)$, where the algebraic complex $\Omega^*(0,2) \widehat{\otimes} \widehat{W}^*$ is endowed with a metric which makes it "conical" at 0 and 2. Formally, the complex $\Omega^*(0,2) \widehat{\otimes} \widehat{W}^*$ has vanishing higher signature, and so by adding it we have not changed the putative higher signature of $CM$. Then we perturb the differential of $C^*$ in order to couple $\Omega^*(CM; \mathcal{V})$ and $\Omega^*(0,2) \widehat{\otimes} \widehat{W}^*$ near the endpoint 0. That is, we do a mapping-cone-type construction along the conical end, which is turned on by a function $\phi(x)$ with $\phi(x) = 1$ for $0 < x < 1/4$ and $\phi(x) = 0$ for $1/2 \leq x \leq 2$. This mapping-cone-type construction is done in a way which preserves Poincaré duality, and makes the new boundary operator invertible. The price to be paid is that the new "differential" $D_C$ no longer satisfies $(D_C)^2 = 0$, as $\phi$ is nonconstant. However, by increasing the length of the conical end, we can make $(D_C)^2$ arbitrarily small in norm. Then we can apply the "almost flat" results of [16, Theorem 4.2] to conclude that the signature index class $[\mathcal{D}_C^{\mathrm{conic}}] \in K_*(C_r^*(\Gamma))$ is an oriented-homotopy invariant. The results of [16, Theorem 4.2] were designed to deal with the case of almost-flat vector bundles. We do not have such vector bundles in our case, but we can use the results of [16, Theorem 4.2] nevertheless.

As $\mathcal{B}^\infty$ is assumed to be a smooth subalgebra of $C_r^*\Gamma$, there is an isomorphism $K_*(C_r^*(\Gamma)) \cong K_*(\mathcal{B}^\infty)$. Hence there is a Chern character $\mathrm{ch}([\mathcal{D}_C^{\mathrm{conic}}]) \in \overline{\mathrm{H}}_*(\mathcal{B}^\infty)$. The second main step in the proof of Theorem 0.1 consists of proving an index theorem, in order to show that $\mathrm{ch}([\mathcal{D}_C^{\mathrm{conic}}])$ is given by the right-hand-side of (0.1). In principle one could do so within the framework of analysis on cone manifolds, but this seems to be very difficult. Instead, we introduce two new $C_r^*(\Gamma)$-Fredholm signature operators, one being an Atiyah-Patodi-Singer (APS)-type operator and the other being a Melrose $b$-type operator. We show that both the conic index and the $b$-index equal the APS-index:

$$[\mathcal{D}_C^{\text{conic}}] = [\mathcal{D}_C^{\text{APS}}] = [\mathcal{D}_C^b] \quad \text{in} \quad K_*(C_r^*(\Gamma)).$$

The advantage of this intermediate step is that we can then compute the Chern character of the $b$-index $[\mathcal{D}_C^b]$ by means of an extension of the higher $b$-pseudodifferential calculus developed in [22] and [24]. Thus we (briefly) develop an enlarged $b$-calculus which takes into account the above mapping-cone construction, and show that the Chern character of the $b$-index class is given by the right-hand-side of (0.1). This completes the proof of Theorem 0.1.

The organization of the paper is as follows. In Section 1 we establish our conventions for signature operators, following [16, Section 3.1]. We also give the product decomposition of the signature operator on a manifold-with-boundary near the boundary. In Section 2 we review the definition of the higher eta-invariant of an odd-dimensional manifold. In Section 3 we review the definition of the higher eta-invariant of an even-dimensional manifold. In Section 4 we discuss the signature operator on a manifold-with-boundary, perturbed by the afore-mentioned algebraic complex $\widehat{W}^*$. In Section 5 we add a conic metric and show that we obtain a well-defined conic index class in $K_0(C_r^*(\Gamma))$. In Section 6 we prove that the conic index class is an oriented-homotopy invariant. In Section 7 we define the APS-index class and prove that it equals the conic index class. In Section 8 we define the (perturbed) $b$-signature operator. In Section 9 we show that the $b$-signature operator has a well-defined index class. In Section 10 we show that the APS-index class and the $b$-index class coincide. In Section 11 we prove an index theorem which computes the index class of the $b$-signature operator. In Section 12 we put the pieces together to prove Theorem 0.1 and Corollaries 0.2–0.4. In the Appendix we sketch an argument which relates the signature class considered in this paper to that defined in [25], using symmetric spectral sections.

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## 1. Signature operators

In this section we establish our conventions for signature operators, following [16, Section 3.1]. The only difference between our conventions and those of [16] is that we deal with left modules, whereas [16] deals with right modules. We also give the product decomposition of the signature operator on a manifold-with-boundary near the boundary.

Let $\Lambda$ be a $C^*$-algebra with unit. Let $\mathcal{B}^\infty$ be a Fréchet locally $m$-convex $*$-subalgebra of $\Lambda$ which is dense in $\Lambda$ and closed under the holomorphic functional calculus in $\Lambda$ [9, Section III.C].

**Definition 1.1.** A graded regular $n$-dimensional Hermitian complex consists of

1. A $\mathbb{Z}$-graded cochain complex $(\mathcal{E}^*, D)$ of finitely-generated projective left $\mathcal{B}^\infty$-modules,
2. A nondegenerate quadratic form $Q : \mathcal{E}^* \times \mathcal{E}^{n-*} \to \mathcal{B}^\infty$ and
3. An operator $\tau \in \mathrm{Hom}_{\mathcal{B}^\infty}(\mathcal{E}^*, \mathcal{E}^{n-*})$

such that

1. $Q(bx, y) = bQ(x, y)$.
2. $Q(x, y)^* = Q(y, x)$.
3. $Q(Dx, y) + Q(x, Dy) = 0$.
4. $\tau^2 = I$.
5. $< x, y > \equiv Q(x, \tau y)$ defines a Hermitian metric on $\mathcal{E}$ ( [29, Definition 7]).

Let $M$ be a closed oriented $n$-dimensional Riemannian manifold. Let $\mathcal{V}^\infty$ be a flat $\mathcal{B}^\infty$-vector bundle on $M$, meaning in particular that its fibers are finitely-generated projective left $\mathcal{B}^\infty$-modules and the transition functions are compatible with the $\mathcal{B}^\infty$-module structures. We assume that the fibers of $\mathcal{V}^\infty$ have $\mathcal{B}^\infty$-valued Hermitian inner products which are compatible with the flat structure. Put $\mathcal{V} = \Lambda \otimes_{\mathcal{B}^\infty} \mathcal{V}^\infty$. It is a flat vector bundle of $\Lambda$-Hilbert modules.

Let $\Omega^*(M; \mathcal{V}^\infty)$ denote the vector space of smooth differential forms with coefficients in $\mathcal{V}^\infty$. If $n = \dim(M) > 0$ then $\Omega^*(M; \mathcal{V}^\infty)$ is not finitely-generated over $\mathcal{B}^\infty$, but we wish to show that it still has all of the formal properties of a graded regular $n$-dimensional Hermitian complex. If $\alpha \in \Omega^*(M; \mathcal{V}^\infty)$ is homogeneous, denote its degree by $|\alpha|$. In what follows, $\alpha$ and $\beta$ will sometimes implicitly denote homogeneous elements of $\Omega^*(M;\mathcal{V}^\infty)$. Given $m \in M$ and $(\lambda_1 \otimes e_1), (\lambda_2 \otimes e_2) \in \Lambda^*(T^*_m M) \otimes \mathcal{V}^\infty_m$, we define $(\lambda_1 \otimes e_1) \wedge (\lambda_2 \otimes e_2)^* \in \Lambda^*(T^*_m M) \otimes \mathcal{B}^\infty$ by

$$(\lambda_1 \otimes e_1) \wedge (\lambda_2 \otimes e_2)^* = (\lambda_1 \wedge \overline{\lambda_2}) \otimes < e_1, e_2 > .$$

Extending by linearity (and antilinearity), given $\omega_1, \omega_2 \in \Lambda^*(T^*_m M) \otimes \mathcal{V}^\infty_m$, we can define $\omega_1 \wedge \omega_2^* \in \Lambda^*(T^*_m M) \otimes \mathcal{B}^\infty$.

Define a $\mathcal{B}^\infty$-valued quadratic form $Q$ on $\Omega^*(M;\mathcal{V}^\infty)$ by

$$Q(\alpha, \beta) = i^{-|\alpha|(n-|\alpha|)} \int_M \alpha(m) \wedge \beta(m)^*.$$

It satisfies $Q(\beta,\alpha) = Q(\alpha,\beta)^*$. Using the Hodge duality operator $*$, define $\tau : \Omega^p(M;\mathcal{V}^\infty) \to \Omega^{n-p}(M;\mathcal{V}^\infty)$ by

$$\tau(\alpha) = i^{-|\alpha|(n-|\alpha|)} * \alpha.$$

Then $\tau^2 = 1$ and the inner product $< \cdot, \cdot >$ on $\Omega^*(M;\mathcal{V}^\infty)$ is given by $< \alpha, \beta > = Q(\alpha, \tau\beta)$. Define $D : \Omega^*(M;\mathcal{V}^\infty) \to \Omega^{*+1}(M;\mathcal{V}^\infty)$ by

$$D\alpha = i^{|\alpha|} d\alpha. \tag{1.1}$$

It satisfies $D^2 = 0$. Its dual $D'$ with respect to $Q$, i.e., the operator $D'$ such that $Q(\alpha, D\beta) = Q(D'\alpha, \beta)$, is given by $D' = -D$. The formal adjoint of $D$ with respect to $< \cdot, \cdot >$ is $D^* = \tau D' \tau = -\tau D \tau$.

**Definition 1.2.** If $n$ is even, the signature operator is

$$\mathcal{D}^{\text{sign}} = D + D^* = D - \tau D \tau. \tag{1.2}$$

It is formally self-adjoint and anticommutes with the $\mathbb{Z}_2$-grading operator $\tau$. If $n$ is odd, the signature operator is

$$\mathcal{D}^{\text{sign}} = -i(D\tau + \tau D). \tag{1.3}$$

It is formally self-adjoint.

Let $\Omega^*_{(2)}(M;\mathcal{V})$ denote the completion of $\Omega^*(M;\mathcal{V})$ in the sense of $\Lambda$-Hilbert modules. If $n$ is even then the triple $(\Omega^*_{(2)}(M;\mathcal{V}), Q, D)$ defines an element of $\mathbf{L}_{nb}(\Lambda)$ in the sense of [16, Définition 1.5].

Now suppose that $M$ is a compact oriented manifold-with-boundary of dimension $n = 2m$. Let $\partial M$ denote the boundary of $M$. We fix a non-negative boundary defining function $x \in C^\infty(M)$ for $\partial M$ and a Riemannian metric on $M$ which is isometrically a product in an (open) collar neighbourhood $\mathcal{U} \equiv (0,2)_x \times \partial M$ of $\partial M$. The signature operator $\mathcal{D}^{\mathrm{sign}}$ still makes sense as a differential operator on $\Omega^*(\mathrm{int}(M); \mathcal{V}^\infty)$. Let $\mathcal{V}_0^\infty$ denote the pullback of $\mathcal{V}^\infty$ from $M$ to $\partial M$; there is a natural isomorphism

$$\mathcal{V}^\infty|_{\mathcal{U}} \cong (0,2) \times \mathcal{V}_0^\infty .$$

Our orientation conventions are such that the volume form on $(0,2) \times \partial M$ is $d\mathrm{vol}_M = dx \wedge d\mathrm{vol}_{\partial M}$. Let $Q_{\partial M}$, $\tau_{\partial M}$, $D_{\partial M}$ and $\mathcal{D}^{\mathrm{sign}}(\partial M)$ denote the expressions defined above on $\Omega^*(\partial M; \mathcal{V}_0^\infty)$. We wish to decompose $Q$, $\tau$, $D$ and $\mathcal{D}^{\mathrm{sign}}$, when restricted to compactly-supported forms on $(0,2) \times \partial M$, in terms of $Q_{\partial M}$, $\tau_{\partial M}$, $D_{\partial M}$ and $\mathcal{D}^{\mathrm{sign}}(\partial M)$.

For notation, we let $\Omega_c^*(0,2)$ denote compactly-supported forms on $(0,2)$. We let $\otimes$ denote a projective tensor product and we let $\widehat{\otimes}$ denote a graded projective tensor product. We write a compactly-supported differential form on $(0,2) \times \partial M$ as $(1 \wedge \alpha(x)) + (dx \wedge \beta(x))$, where for each $x \in (0,2)$, $\alpha(x)$ and $\beta(x)$ are in $\Omega^*(\partial M; \mathcal{V}_0^\infty)$. It is convenient to introduce the notation

$$\widehat{\alpha} = i^{|\alpha|}\alpha$$

for $\alpha \in \Omega^*(\partial M; \mathcal{V}_0^\infty)$. One finds

$$Q(dx \wedge \alpha, 1 \wedge \beta) = \int_0^2 Q_{\partial M}(\alpha(x), \widehat{\beta(x)})dx,$$

$$Q(1 \wedge \alpha, dx \wedge \beta) = \int_0^2 Q_{\partial M}(\widehat{\alpha(x)}, \beta(x))dx,$$

$$\tau(1 \wedge \alpha) = dx \wedge \tau_{\partial M}\widehat{\alpha}, \tag{1.4}$$

$$\tau(dx \wedge \alpha) = 1 \wedge i^{-(2m-1)}\tau_{\partial M}\widehat{\alpha},$$

$$D(1 \wedge \alpha) = (1 \wedge D_{\partial M}\alpha) + (dx \wedge \partial_x \widehat{\alpha}),$$

$$D(dx \wedge \alpha) = dx \wedge -iD_{\partial M}\alpha.$$

Then one can compute that $\mathcal{D}^{\mathrm{sign}}$ takes the form

$$\mathcal{D}^{\mathrm{sign}} = \begin{pmatrix} D_{\partial M} - \tau_{\partial M}D_{\partial M}\tau_{\partial M} & -i^{-|\beta|}\partial_x \\ i^{|\alpha|}\partial_x & -i(D_{\partial M} + \tau_{\partial M}D_{\partial M}\tau_{\partial M}) \end{pmatrix}$$

when acting on $\begin{pmatrix} 1 \wedge \alpha \\ dx \wedge \beta \end{pmatrix}$. That is,

$$\mathcal{D}^{\mathrm{sign}}(1 \wedge \alpha) = (1 \wedge (D_{\partial M} - \tau_{\partial M}D_{\partial M}\tau_{\partial M})\alpha) + (dx \wedge i^{|\alpha|}\partial_x \alpha)$$

and

$$\mathcal{D}^{\mathrm{sign}}(dx \wedge \beta) = (1 \wedge -i^{-|\beta|}\partial_x \beta) + (dx \wedge -i(D_{\partial M} + \tau_{\partial M} D_{\partial M} \tau_{\partial M})\beta).$$

Let us define an operator

$$\Theta : \Omega^*_c((0,2) \times \partial M; \mathcal{V}_0^\infty) \to \Omega^*_c((0,2) \times \partial M; \mathcal{V}_0^\infty)$$

by

$$\Theta((1 \wedge \alpha) + (dx \wedge \beta)) = (1 \wedge -i^{-\beta}\beta) + (dx \wedge i^{|\alpha|}\alpha).$$

Then $\Theta$ anticommutes with $\tau$ and we can write $\mathcal{D}^{\mathrm{sign}} = \Theta(\partial_x + H)$, where $H$ commutes with $\tau$. Acting on the $+1$-eigenvector

$$(dx \wedge \alpha) + \tau(dx \wedge \alpha) = (dx \wedge \alpha) + (1 \wedge i^{-(2m-1-|\alpha|)}\tau_{\partial M}\alpha)$$

of $\tau$, one finds

$$\begin{aligned} H\left((dx \wedge \alpha) + \tau(dx \wedge \alpha)\right) = {} & (dx \wedge -i(D_{\partial M}\tau_{\partial M} + \tau_{\partial M} D_{\partial M})\alpha) \\ & + (1 \wedge -i^{-|\alpha|}(D_{\partial M} - \tau_{\partial M} D_{\partial M} \tau_{\partial M})\alpha). \end{aligned}$$

Let $E^{\pm}$ be the $\pm 1$-eigenspaces of $\tau$ acting on $\Omega^*_c((0,2) \times \partial M; \mathcal{V}_0^\infty)$. We define an isomorphism $\Phi$ from $C^\infty_c(0,2) \otimes \Omega^*(\partial M; \mathcal{V}_0^\infty)$ to $E^+$, by setting

$$\Phi(\alpha) = (dx \wedge \alpha) + \tau(dx \wedge \alpha).$$

We then obtain an isomorphism

$$\Theta \circ \Phi : C^\infty_c(0,2) \otimes \Omega^*(\partial M; \mathcal{V}_0^\infty) \to E^- \, .$$

Denote as usual by $\mathcal{D}^{\mathrm{sign}}_+$ the signature operator on $M$ going from $E^+$ to $E^-$; using the above isomorphisms we easily obtain

$$\Phi^{-1} \circ H|_{E^+} \circ \Phi = \mathcal{D}^{\mathrm{sign}}(\partial M)$$

and

$$\mathcal{D}^{\mathrm{sign}}_+ = \Theta \circ \Phi(\partial_x + \mathcal{D}^{\mathrm{sign}}(\partial M))\Phi^{-1}. \tag{1.5}$$

This shows that $\mathcal{D}^{\mathrm{sign}}(\partial M)$ is the boundary component of $\mathcal{D}^{\mathrm{sign}}$ in the sense of Atiyah-Patodi-Singer [1, (3.1)].

Consider the $\mathbb{Z}_2$-graded vector space

$$(\Omega^*_{(2)}(\partial M; \mathcal{V}_0)) \oplus (dx \wedge \Omega^*_{(2)}(\partial M; \mathcal{V}_0)),$$

where the $\mathbb{Z}_2$-grading comes from the operator $\tau$ of (1.4). The triple $\left((\Omega^*_{(2)}(\partial M; \mathcal{V}_0)) \oplus (dx \wedge \Omega^*_{(2)}(\partial M; \mathcal{V}_0)), Q, \Theta H\right)$ defines an element of $\mathbf{L}_{nb,odd}(\Lambda)$ in the sense of [16, p. 81].

## 2. The higher eta invariant of an odd-dimensional manifold

In this section we review the definition of the higher eta invariant ([27, Definition 11] and [30, Section 3.2]). The material in this section comes from these references, with minor variations. The higher eta invariant is defined for closed oriented Riemannian manifolds of either even or odd dimension. We first treat the case of a closed oriented Riemannian manifold $F$ of dimension $n = 2m-1$.

Let us make a general remark about homotopy equivalences between cochain complexes. Suppose that $(C_1, d_1)$ and $(C_2, d_2)$ are cochain complexes, with homotopy equivalences $f : C_1^* \to C_2^*$ and $g : C_2^* \to C_1^*$. Then one implicitly understands that there are maps $A : C_1^* \to C_1^{*-1}$ and $B : C_2^* \to C_2^{*-1}$ so that $I - gf = d_1 A + A d_1$ and $I - fg = d_2 B + B d_2$. It follows that $gB - Ag : C_2^* \to C_1^{*-1}$ and $fA - Bf : C_1^* \to C_2^{*-1}$ are cochain maps. We will say that such $f$ and $g$ form a *double homotopy equivalence* if, in addition, there are maps $\alpha : C_2^* \to C_1^{*-2}$ and $\beta : C_1^* \to C_2^{*-2}$ such that $gB - Ag = d_1\alpha - \alpha d_2$ and $fA - Bf = d_2\beta - \beta d_1$. One can check that the composition of two double homotopy equivalences is a double homotopy equivalence. The notion of double homotopy equivalence is not strictly needed for this section but will enter in the proof of Theorem 6.1.

Now let $\Gamma$ be a finitely-generated discrete group. Let $\nu : F \to B\Gamma$ be a continuous map. There is a corresponding normal $\Gamma$-cover $F' \to F$. Let $C_r^*(\Gamma)$ be the reduced group $C^*$-algebra of $\Gamma$. Let $\mathcal{B}^\infty$ be a subalgebra of $C_r^*(\Gamma)$ as in Section 1.

We introduce two flat unitary vector bundles of left modules on $F$ :

$$\mathcal{V} = C_r^*(\Gamma) \times_\Gamma F', \qquad \mathcal{V}^\infty = \mathcal{B}^\infty \times_\Gamma F'.$$

Following [27, Section 4.7], we make an assumption about the de Rham cohomology of $F$, with value in the local system $\mathcal{V}$.

**Assumption 1.** The natural surjection $\mathrm{H}^m(F; \mathcal{V}) \to \overline{\mathrm{H}}^m(F; \mathcal{V})$ is an isomorphism.

**Lemma 2.1.** *If $F$ is equipped with a Riemannian metric then Assumption 1 is equivalent to saying that the differential form Laplacian on $\Omega^{m-1}(F')/\mathrm{Ker}(d)$ has a strictly positive spectrum.*

*Proof.* We give an outline of the proof. Let $\Omega_{(2)}^*(F; \mathcal{V})$ denote the completion of $\Omega^*(F; \mathcal{V})$ as a $C_r^*(\Gamma)$-Hilbert module. Assumption 1 is equivalent to saying that the differential $\widetilde{D}_F : \Omega^{m-1}(F; \mathcal{V}) \to \Omega^m(F; \mathcal{V})$

has a closed image. Using Hodge duality, this is equivalent to saying that $\widetilde{D}_F^* : \Omega^m(F;\mathcal{V}) \to \Omega^{m-1}(F;\mathcal{V})$ has a closed image. Clearly $\widetilde{D}_F^*$ is adjointable, and we obtain an orthogonal decomposition $\Omega^{m-1}(F;\mathcal{V}) = \mathrm{Im}(\widetilde{D}_F^*) \oplus \mathrm{Ker}(\widetilde{D}_F)$. From arguments as in [29, Propositions 10 and 27], this is equivalent to saying that $\widetilde{D}_F^* \widetilde{D}_F$ has a strictly positive spectrum as a densely-defined operator on $\mathrm{Im}\left(\widetilde{D}_F^* : \Omega^m(F;\mathcal{V}) \to \Omega^{m-1}(F;\mathcal{V})\right) \subset \Omega^{m-1}_{(2)}(F;\mathcal{V})/\mathrm{Ker}(\widetilde{D}_F)$; see [40, Theorem 15.3.8] for the analogous result in the case of bounded operators. Put $\mathcal{V}^{(2)} = l^2(\Gamma) \times_\Gamma F'$. Let $\Omega^{m-1}_{(2)}(F;\mathcal{V}^{(2)})$ denote the Hilbert space of square-integrable $\mathcal{V}^{(2)}$-valued $(m-1)$-forms on $F$. We claim that the spectrum of $\widetilde{D}_F^* \widetilde{D}_F$, acting on $\Omega^{m-1}_{(2)}(F;\mathcal{V})/\mathrm{Ker}(\widetilde{D}_F)$, is the same as the spectrum of $d^*d$, acting on $\Omega^{m-1}_{(2)}(F;\mathcal{V}^{(2)})/\mathrm{Ker}(d)$. To see this, by considering $\frac{\widetilde{D}_F^* \widetilde{D}_F}{I+\widetilde{D}_F^* \widetilde{D}_F}$ and $\frac{d^*d}{I+d^*d}$ we can reduce to the case of bounded operators. Using the identification

$$l^2(\Gamma) \otimes_{C_r^*(\Gamma)} \left(\Omega^{m-1}_{(2)}(F;\mathcal{V})/\mathrm{Ker}(\widetilde{D}_F)\right) \cong \Omega^{m-1}_{(2)}(F;\mathcal{V}^{(2)})/\mathrm{Ker}(d),$$

the map $T \to \mathrm{Id} \otimes_{C_r^*(\Gamma)} T$ gives an injective homomorphism from the $C^*$-algebra of bounded adjointable operators on the $C_r^*(\Gamma)$-Hilbert module $\Omega^{m-1}_{(2)}(F;\mathcal{V})/\mathrm{Ker}(\widetilde{D}_F)$ to the $C^*$-algebra of bounded operators on $\Omega^{m-1}_{(2)}(F;\mathcal{V}^{(2)})/\mathrm{Ker}(d)$. The claim follows from the fact that the spectrum of an element does not change under such a homomorphism.

Now $\Omega^{m-1}_{(2)}(F;\mathcal{V}^{(2)})$ is the same as the space $\Omega^{m-1}_{(2)}(F')$ of square-integrable $(m-1)$-forms on $F'$. Since the Laplacian $d^*d + dd^*$ acts on $\Omega^{m-1}_{(2)}(F')/\mathrm{Ker}(d)$ as $d^*d$, the lemma follows. q.e.d.

**Lemma 2.2.**

*(a) If $F$ has a cellular decomposition without any cells of dimension $m$ then Assumption 1 is satisfied.*

*(b) If $\Gamma$ is finite then Assumption 1 is satisfied.*

*(c) If $\dim(F) = 3$, $F$ is connected, $\Gamma = \pi_1(F)$ and $\nu$ is the classifying map for the universal cover of $F$ then, assuming Thurston's geometrization conjecture, Assumption 1 is satisfied if and only if $F$ is a connected sum of spherical space forms, $S^1 \times S^2$'s and twisted circle bundles $S^1 \times_{\mathbb{Z}_2} S^2$ over $\mathbb{R}P^2$.*

*Proof.*

(a) If $F$ has a cellular decomposition without any cells of dimension $m$ then $\mathrm{H}^m(F;\mathcal{V})$ vanishes and Assumption 1 is automatically satisfied.

(b) If $\Gamma$ is finite then $F'$ is compact and from standard elliptic theory, the result of Lemma 2.1 is satisfied.

(c) If $\Gamma = \pi_1(F)$ and $F$ is connected then in the notation of [31], the result of Lemma 2.1 is equivalent to saying that the $m$-th Novikov-Shubin invariant $\alpha_m(F)$ of $F$ is $\infty^+$. In the present case, $m = 2$. Let $F = F_1 \sharp F_2 \sharp \ldots \sharp F_N$ be the connected sum decomposition of $F$ into prime 3-manifolds. From [31, Proposition 3.7.3], $\alpha_2(F) = \min_i \alpha_2(F_i)$. Hence it suffices to characterize the prime closed 3-manifolds $F$ with $\alpha_2(F) = \infty^+$. If $F$ has finite fundamental group then $\alpha_2(F) = \infty^+$ and the geometrization conjecture says that $F$ is a spherical space form. If $F$ has infinite fundamental group and $\alpha_2(F) = \infty^+$ then, assuming the geometrization conjecture, [31, Theorem 0.1.5] implies that $F$ has an $\mathbb{R}^3$, $S^2 \times \mathbb{R}$ or *Sol* structure. From [31, Theorem 0.1.4], if $F$ has an $\mathbb{R}^3$-structure then $\alpha_2(F) = 3$, while if $F$ has an $S^2 \times \mathbb{R}$ structure then $\alpha_2(F) = \infty^+$. Finally, a slight refinement of [28, Corollary 5] shows that if $F$ has a *Sol* structure then $\alpha_2(F) < \infty^+$. The claim follows.

q.e.d.

Hereafter we assume that Assumption 1 is satisfied.

**Lemma 2.3.** *There is a cochain complex* $W^* = \bigoplus_{i=0}^{2m-1} W^i$ *of finitely-generated projective* $\mathcal{B}^\infty$*-modules such that*

1. $W^*$ *is a graded regular* $n$*-dimensional Hermitian complex.*
2. *The differential* $D_W : W^{m-1} \to W^m$ *vanishes.*
3. *There is a double homotopy equivalence*

$$f : \Omega^*(F;\mathcal{V}^\infty) \to W^* \tag{2.1}$$

*which, as an element of* $(\Omega^*(F;\mathcal{V}^\infty))^* \otimes W^*$*, is actually smooth with respect to* $F$*.*

*Proof.* The strategy of the proof is to first establish a double homotopy equivalence between $\Omega^*(F;\mathcal{V}^\infty)$ and a simplicial cochain complex, and then to further homotope the simplicial cochain complex in order to end up with a graded regular Hermitian complex. We will implicitly use results from [29, Proposition 10 and Section 6.1] concerning spectral analysis involving $\mathcal{B}^\infty$. Let $K$ be a triangulation of $F$. Let $(C^*(K;\mathcal{V}^\infty), D_K)$ be the simplicial cochain complex, a complex of finitely-generated free $\mathcal{B}^\infty$-modules. We first construct a cochain embedding from $C^*(K;\mathcal{V}^\infty)$ to $\Omega^*(F;\mathcal{V}^\infty)$, following the work of Whitney [41, Chapter IV.27]. In order to have an embedding into smooth forms, we use the modification of Whitney's formula given in [12, (3.4)]. If $\mathcal{V}^{(2)} = l^2(\Gamma) \times_\Gamma F'$ then the map $W$ of [12, (3.4)], which is defined in the $l^2$-setting, gives a cochain embedding $W : C^*(K;\mathcal{V}^{(2)}) \to \Omega^*(F;\mathcal{V}^{(2)})$ which is a homotopy equivalence. Using the same formula as in [12, (3.4)], but considering cochains and forms with values in the flat bundle $\mathcal{V}^\infty$, we obtain a cochain embedding $w : C^*(K;\mathcal{V}^\infty) \to \Omega^*(F;\mathcal{V}^\infty)$ which is a homotopy equivalence. Give $C^*(K;\mathcal{V}^\infty)$ the induced $\mathcal{B}^\infty$-valued Hermitian inner product.

Using this embedding, let us decompose $\Omega^*(F;\mathcal{V}^\infty)$ as $\Omega^*(F;\mathcal{V}^\infty) = C^*(K;\mathcal{V}^\infty) \oplus C'$ where $C'$ is the orthogonal complement to the finitely-generated free submodule $C^*(K;\mathcal{V}^\infty)$ of $\Omega^*(F;\mathcal{V}^\infty)$. With respect to this decomposition, we can write $w = \begin{pmatrix} I \\ 0 \end{pmatrix}$ and $D_F = \begin{pmatrix} D_K & X \\ 0 & D_{C'} \end{pmatrix}$ for some cochain map $X \in \mathrm{Hom}_{\mathcal{B}^\infty}\left((C')^*, C^{*+1}(K;\mathcal{V}^\infty)\right)$. Then the complex $(C', D_{C'})$ is acyclic. Putting $\widetilde{w} = \mathrm{Id}_{C_r^*(\Gamma)} \otimes_{\mathcal{B}^\infty} w : C^*(K;\mathcal{V}) \to \Omega^*(F;\mathcal{V})$ and doing the analogous constructions, we see that the complex $C_r^*(\Gamma) \otimes_{\mathcal{B}^\infty} C'$ is also acyclic.

As $C'$ is acyclic, there is a operator $\delta_{C'}$ of degree $-1$ such that $(\delta_{C'})^2 = 0$ and $D_{C'}\delta_{C'} + \delta_{C'}D_{C'} = I$. We claim that we can take $\delta_{C'}$ to be continuous. To see this, put $E_F = \begin{pmatrix} 0 & 0 \\ 0 & D_{C'} \end{pmatrix}$. It is an element of the space $\Psi^1_{\mathcal{B}^\infty}(F;\Lambda^*(TF)\otimes\mathcal{V}^\infty, \Lambda^*(TF)\otimes\mathcal{V}^\infty)$ of $\mathcal{B}^\infty$-pseudodifferential operators of order 1, as defined in [29, Section 6.1]. Put $\mathcal{L} = E_F(E_F)^* + (E_F)^*E_F$, an element of $\Psi^2_{\mathcal{B}^\infty}(F;\Lambda^*(TF)\otimes\mathcal{V}^\infty, \Lambda^*(TF)\otimes\mathcal{V}^\infty)$.

As $C_r^*(\Gamma)\otimes_{\mathcal{B}^\infty} C'$ is acyclic, its differentials have closed image. Hence $\mathrm{Id}_{C_r^*(\Gamma)} \otimes_{\mathcal{B}^\infty} E_F : \Omega^*(F;\mathcal{V}) \to \Omega^{*+1}(F;\mathcal{V})$ also has closed image. It follows, as in the proof of [29, Propositions 10 and 27], that 0 is isolated in the spectrum of $\mathcal{L}$, with $\mathrm{Ker}(\mathcal{L}) = C^*(K;\mathcal{V}^\infty) \oplus 0$. Let $G \in \Psi^{-2}_{\mathcal{B}^\infty}(F;\Lambda^*(TF)\otimes\mathcal{V}^\infty, \Lambda^*(TF)\otimes\mathcal{V}^\infty)$ be the Green's operator for $\mathcal{L}$. Then $(E_F)^*G$ is an element of $\Psi^{-1}_{\mathcal{B}^\infty}(F;\Lambda^*(TF)\otimes\mathcal{V}^\infty, \Lambda^*(TF)\otimes\mathcal{V}^\infty)$ and can be written in the form $(E_F)^*G = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{C'} \end{pmatrix}$. As in usual Hodge theory, this operator $\delta_{C'}$ satisfies $(\delta_{C'})^2 = 0$ and $D_{C'}\delta_{C'} + \delta_{C'}D_{C'} = I$.

It is everywhere-defined and continuous, as $(E_F)^*G$ has order $-1$ in the pseudodifferential operator calculus.

Define $q : \Omega^*(F; \mathcal{V}^\infty) \to C^*(K; \mathcal{V}^\infty)$ by $q = \begin{pmatrix} I & -X\delta_{C'} \end{pmatrix}$. Then one can check that $I - qw = 0$ and $I - wq = D_F A + AD_F$, where

$$A = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{C'} \end{pmatrix}.$$

Furthermore, $qA = Aw = 0$. Hence $w$ and $q$ define a double homotopy equivalence between $\Omega^*(F; \mathcal{V}^\infty)$ and $C^*(K; \mathcal{V}^\infty)$.

We now show that $C^*(K; \mathcal{V}^\infty)$ is double homotopy equivalent to an appropriate regular Hermitian complex $W^*$ of finitely-generated projective $\mathcal{B}^\infty$-modules. In the even case, the homotopy equivalence to a regular Hermitian complex was proven in [20, Proposition 2.4]. In order to extend the proof to the odd case, we need Assumption 1.

Let $\widetilde{D}_K$ denote the differential on $C^*(K; \mathcal{V})$, an adjointable operator. Using the homotopy equivalence between $C^*(K; \mathcal{V})$ and $\Omega^*(F; \mathcal{V})$, along with the fact that all of the maps involved in defining the homotopy equivalence are continuous, it follows that Assumption 1 is equivalent to saying that the natural surjection $\mathrm{H}^m(K; \mathcal{V}) \to \overline{\mathrm{H}}^m(K; \mathcal{V})$ is an isomorphism. Equivalently, $\widetilde{D}_K(C^{m-1}(K; \mathcal{V}))$ is closed in $C^m(K; \mathcal{V})$. Let $C^m(K; \mathcal{V}) = \mathrm{Im}(\widetilde{D}_K) \oplus \mathrm{Ker}(\widetilde{D}_K^*)$ be the corresponding orthogonal decomposition [40, Theorem 15.3.8]. Then the operator $\widetilde{D}_K \widetilde{D}_K^*$ is invertible on $\mathrm{Im}(\widetilde{D}_K) \subset C^m(K; \mathcal{V})$ [40, Theorem 15.3.8]. In particular, there is some $\epsilon > 0$ such that the intersection of the spectrum of $\widetilde{D}_K \widetilde{D}_K^*$ (acting on $C^m(K; \mathcal{V})$) with the ball $B_\epsilon(0) \subset \mathbb{C}$ consists at most of the point 0. From [29, Lemma 1], the same is true of the operator $D_K D_K^*$, acting on $C^m(K; \mathcal{V}^\infty)$. Define a continuous operator $G$ on $C^m(K; \mathcal{V}^\infty)$ by

$$G = \frac{1}{2\pi i} \int_\gamma \frac{1}{\lambda} \frac{d\lambda}{D_K D_K^* - \lambda},$$

where $\gamma$ is the circle of radius $\frac{\epsilon}{2}$ around $0 \in \mathbb{C}$, oriented counterclockwise. Then $G$ is the Green's operator for $D_K D_K^*$. Put $\widetilde{G} = \mathrm{Id}_{C_r^*(\Gamma)} \otimes_{\mathcal{B}^\infty} G$, the Green's operator for $\widetilde{D}_K \widetilde{D}_K^*$.

We claim that $D_K(C^{m-1}(K; \mathcal{V}^\infty))$ is closed in $C^m(K; \mathcal{V}^\infty)$. To see this, suppose that $\{z_i\}_{i=1}^\infty$ is a sequence in $C^{m-1}(K; \mathcal{V}^\infty)$ such that $\lim_{i\to\infty} D_K(z_i) = y$ for some $y \in C^m(K; \mathcal{V}^\infty)$. Let $\widetilde{z}_i \in C^{m-1}(K; \mathcal{V})$ and $\widetilde{y} \in C^m(K; \mathcal{V})$ be the corresponding elements. Then

$$\widetilde{D}_K \widetilde{D}_K^* \widetilde{G}(\widetilde{y}) = \lim_{i\to\infty} \widetilde{D}_K \widetilde{D}_K^* \widetilde{G} \widetilde{D}_K(\widetilde{z}_i) = \lim_{i\to\infty} \widetilde{D}_K(\widetilde{z}_i) = \widetilde{y}.$$

It follows that $D_K D_K^* G(y) = y$, showing that $y \in \mathrm{Im}(D_K)$. Equivalently, the surjection $\mathrm{H}^m(K;\mathcal{V}^\infty) \to \overline{\mathrm{H}}^m(K;\mathcal{V}^\infty)$ is an isomorphism. Similarly, using the fact that $D_K^* G D_K$ acts as the identity on $\mathrm{Im}(D_K^*) \subset C^{m-1}(K;\mathcal{V}^\infty)$, one can show that $\mathrm{Im}(D_K^*)$ is closed in $C^{m-1}(K;\mathcal{V}^\infty)$

We recall that $(C^*(K;\mathcal{V}^\infty), D_K)$ is a Hermitian complex. This means that it has a possibly-degenerate quadratic form $Q_K$ which satisfies conditions 1.- 3. of Definition 1.1 and for which the corresponding map $\Phi_K : C^*(K;\mathcal{V}^\infty) \to (C^{2m-1-*}(K;\mathcal{V}^\infty))'$ is a homotopy equivalence. (Here $'$ denotes the antidual space.) *A priori*, $\Phi_K$ may not be an isomorphism. To construct the regular Hermitian complex $W^*$, we need to homotope $C^*(K;\mathcal{V}^\infty)$ so that the map $\Phi_K$ becomes an isomorphism. Using the construction of [32, Proposition 1.3], we can construct a Hermitian complex $Z^*$ which is homotopy equivalent to $C^*(K;\mathcal{V}^\infty)$ and whose map $\Phi_Z : Z^* \to (Z^{2m-1-*})'$ is an isomorphism in degrees other than $m-1$ and $m$. Looking at the diagram in the proof of [32, Proposition 1.3], one sees that $C^*(K;\mathcal{V}^\infty)$ is in fact double homotopy equivalent to $Z^*$. We again have that the surjection $\mathrm{H}^m(Z) \to \overline{\mathrm{H}}^m(Z)$ is an isomorphism, or equivalently, $D_Z(Z^{m-1})$ is closed in $Z^m$. From the diagram in the proof of [32, Proposition 1.3], there is an obvious $\mathcal{B}^\infty$-valued Hermitian inner product on $Z^*$, and $\widetilde{D}_Z = \mathrm{Id}_{C_r^*(\Gamma)} \otimes_{\mathcal{B}^\infty} D_Z$ is adjointable.

Put

$$W^i = \begin{cases} Z^i & \text{if } i < m-1, \\ \mathrm{Ker}\left(D_Z : Z^{m-1} \longrightarrow Z^m\right) & \text{if } i = m-1, \\ Z^m / \mathrm{Im}\left(D_Z : Z^{m-1} \longrightarrow Z^m\right) & \text{if } i = m, \\ Z^i & \text{if } i > m. \end{cases} \tag{2.2}$$

We give $W^*$ the differential induced from $Z^*$ in degrees other than $m-1$, and the zero differential in degree $m-1$.

Using the fact that $D_Z(Z^{m-1})$ is closed in $Z^m$, it follows as before that there is some $\epsilon > 0$ such that the intersection of the spectrum of $D_Z D_Z^*$ (acting on $Z^m$) with the ball $B_\epsilon(0) \subset \mathbb{C}$ consists at most of the point 0. Then with $\gamma$ as before, the projection operator $\frac{1}{2\pi i}\int_\gamma \frac{d\lambda}{\lambda - D_K D_K^*}$ gives a direct sum decomposition into closed $\mathcal{B}^\infty$-submodules :

$$Z^m = \mathrm{Im}\left(D_Z : Z^{m-1} \longrightarrow Z^m\right) \oplus \mathrm{Ker}\left(D_Z^* : Z^m \longrightarrow Z^{m-1}\right).$$

Using this decomposition, we can identify $W^m$ with $\mathrm{Ker}\Big(D_Z^* : Z^m \longrightarrow Z^{m-1}\Big)$. It also follows as before that $D_Z^*(Z^m)$ is closed in $Z^{m-1}$, and there is a direct sum decomposition into closed $\mathcal{B}^\infty$-submodules :

$$Z^{m-1} = \operatorname{Ker}\left(D_Z : Z^{m-1} \longrightarrow Z^m\right) \oplus \operatorname{Im}\left(D_Z^* : Z^m \longrightarrow Z^{m-1}\right).$$

Let $p : Z^* \longrightarrow W^*$ be the corresponding projection operator and let $i : W^* \longrightarrow Z^*$ be the inclusion operator. Let $L : Z^* \to Z^{*-1}$ be the map which is an inverse to

$$D_Z : \operatorname{Im}\left(D_Z^* : Z^m \longrightarrow Z^{m-1}\right) \to \operatorname{Im}\left(D_Z : Z^{m-1} \longrightarrow Z^m\right)$$

on $\operatorname{Im}\left(D_Z : Z^{m-1} \longrightarrow Z^m\right) \subset Z^m$, i.e., $L = D_Z^* \left(D_Z D_Z^*\big|_{\operatorname{Im}(D_Z)}\right)^{-1}$, and which vanishes on
$\operatorname{Ker}\left(D_Z^* : Z^m \longrightarrow Z^{m-1}\right) \subset Z^m$ and on the rest of $Z^*$. Then one can check that $p$ and $i$ are cochain maps, that $p \circ i = I$ and that $i \circ p = I - D_Z L - L D_Z$. Also, $Li = pL = 0$. Thus $Z^*$ and $W^*$ are doubly homotopy equivalent. We give $W^*$ the structure of a Hermitian complex by saying that $\Phi_W = i' \circ \Phi_Z \circ i$. Equivalently, $Q_W$ is the quadratic form induced from $Q_Z$ under $i : W^* \to Z^*$. Then $W^*$ and $Z^*$ are homotopy equivalent as Hermitian complexes.

We claim that $\Phi_W$ is an isomorphism. This is clear when $\Phi_W$ acts on $W^*$, $* \notin \{m-1, m\}$, as $\Phi_Z$ is an isomorphism in those degrees. Hence we must prove the following result:

**Sublemma 2.4.** *Suppose that we have a homotopy equivalence $\Phi^* : W^* \to (W^{2m-1-*})'$*

$$\begin{array}{ccccccccccc}
\dots & \to & W^{m-2} & \to & W^{m-1} & \to & W^m & \to & W^{m+1} & \to & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \to & (W^{m+1})' & \to & (W^m)' & \to & (W^{m-1})' & \to & (W^{m-2})' & \to & \dots
\end{array}$$

*such that $\Phi^*$ is an isomorphism for $* \notin \{m-1, m\}$ and $D_W^{m-1} : W^{m-1} \to W^m$ vanishes. Then $\Phi^{m-1}$ and $\Phi^m$ are isomorphisms.*

*Proof.* We first show that $\Phi^{m-1}$ is injective. Suppose that $x \in W^{m-1}$ and $\Phi^{m-1}(x) = 0$. As $\Phi^{m-1}$ is an isomorphism on cohomology, and $[\Phi^{m-1}(x)]$ vanishes in cohomology, there is a $y \in W^{m-2}$ such that $x = D_W^{m-2} y$. Then

$$(D_W^m)'(\Phi^{m-2}(y)) = \Phi^{m-1}(D_W^{m-2} y) = 0.$$

Hence $[\Phi^{m-2}(y)]$ represents a cohomology class and, as $\Phi^{m-2}$ is an isomorphism on cohomology, there are some $z \in W^{m-2}$ and $u \in (W^{m+2})'$ such that $D_W^{m-2} z = 0$ and

$$\Phi^{m-2}(y) - \Phi^{m-2}(z) = (D_W^{m+1})'(u).$$

Put $v = (\Phi^{m-3})^{-1}(u)$. Then

$$\begin{aligned}\Phi^{m-2}(y - z - D_W^{m-3} v) &= (D_W^{m+1})'(u) - \Phi^{m-2}(D_W^{m-3} v) \\ &= (D_W^{m+1})'(u) - (D_W^{m+1})'(\Phi^{m-3}(v)) \\ &= 0.\end{aligned}$$

Thus $y - z - D_W^{m-3} v = 0$ and

$$x = D_W^{m-2} y = D_W^{m-2}(z + D_W^{m-3} v) = 0,$$

which shows that $\Phi^{m-1}$ is injective.

We now show that $\Phi^m$ is injective. Suppose that $x \in W^m$ and $\Phi^m(x) = 0$. Then

$$\Phi^{m+1}(D_W^m x) = (D_W^{m-2})'(\Phi^m(x)) = 0,$$

so $D_W^m x = 0$. Thus $x$ represents a cohomology class. As $\Phi^m$ is an isomorphism on cohomology, and $[\Phi^m(x)]$ vanishes in cohomology, it follows that $x \in \mathrm{Im}(D_W^{m-1}) = 0$. This shows that $\Phi^m$ is injective.

We now show that $\Phi^{m-1}$ is surjective. Suppose that $x \in (W^m)'$. As $(D_W^{m-1})'(x) = 0$, there is a cohomology class represented by $[x]$. As $\Phi^{m-1}$ is an isomorphism on cohomology, there are some $y \in W^{m-1}$ and $z \in (W^{m+1})'$ such that $x = \Phi^{m-1}(y) + (D_W^m)'(z)$. Put $w = (\Phi^{m-2})^{-1}(z)$. Then

$$\begin{aligned}x = \Phi^{m-1}(y) + (D_W^m)'(\Phi^{m-2}(w)) &= \Phi^{m-1}(y) + \Phi^{m-1}(D_W^{m-2} w) \\ &= \Phi^{m-1}(y + D_W^{m-2} w),\end{aligned}$$

which shows that $\Phi^{m-1}$ is surjective.

We finally show that $\Phi^m$ is surjective. Suppose that $x \in (W^{m-1})'$. Put $y = (\Phi^{m+1})^{-1}((D_W^{m-2})'(x))$. As $[\Phi^{m+1}(y)]$ vanishes in cohomology, and $\Phi^{m+1}$ is an isomorphism on cohomology, there is some $z \in W^m$ such that $y = D_W^m z$. Then

$$(D_W^{m-2})'(x - \Phi^m(z)) = \Phi^{m+1}(D_W^m z) - (D_W^{m-2})'(\Phi^m(z)) = 0.$$

Thus $x - \Phi^m(z)$ represents a cohomology class. As $\Phi^m$ is an isomorphism on cohomology, there is some $w \in W^m$ such that $D_W^m w = 0$ and $x - \Phi^m(z) = \Phi^m(w)$. Hence $x = \Phi^m(z + w)$, which proves the sublemma.

q.e.d.

To finish the proof of Lemma 2.3, as in [20, Proposition 2.6], one can introduce a grading $\tau_W$ so that $(W^*, Q_W, \tau_W)$ satisfies Definition

1.1. Hence we have constructed the desired complex $W^*$, along with a double homotopy equivalence $f : \Omega^*(F; \mathcal{V}^\infty) \to W^*$. From this, $f$ is an element of $(\Omega^*(F; \mathcal{V}^\infty))^* \otimes_{\mathcal{B}^\infty} W^*$. *A priori*, it could be distributional with respect to $F$. However, in the proof we constructed $f$ to actually be smooth on $F$, i.e., $f \in C^\infty(F; \mathrm{Hom}_{\mathcal{B}^\infty}(\Lambda^* TF \otimes \mathcal{V}^\infty, W^*))$. The lemma follows. q.e.d.

Following [30, (3.23)], we define a new $n$-dimensional complex $\widehat{W}^*$ by

$$\widehat{W}^i = \begin{cases} W^{i+1} & \text{if } -1 \leq i \leq m-2, \\ 0 & \text{if } i = m-1 \text{ or } m, \\ W^{i-1} & \text{if } m+1 \leq i \leq 2m. \end{cases} \tag{2.3}$$

The differential $D_W$ induces a differential $D_{\widehat{W}}$ in an obvious way. We also obtain a Hermitian form $Q_{\widehat{W}}(\cdot, \cdot)$ on $\widehat{W}$ by putting

$$Q_{\widehat{W}}(v^j, z^{(2m-1)-j}) = Q_W(v^j, z^{(2m-1)-j})$$

for $v^j \in \widehat{W}^j$, $z^{(2m-1)-j} \in \widehat{W}^{(2m-1)-j}$, and a duality operator $\tau_{\widehat{W}} : \widehat{W}^j \to \widehat{W}^{(2m-1)-j}$ by putting

$$\tau_{\widehat{W}}(v^j) = \tau_W(v^j).$$

The signature operator of $\widehat{W}^*$ is defined to be

$$\mathcal{D}^{\mathrm{sign}}_{\widehat{W}} = i(D_{\widehat{W}} \tau_{\widehat{W}} + \tau_{\widehat{W}} D_{\widehat{W}}). \tag{2.4}$$

(The right-hand-side of (2.4) differs from the right-hand-side of (1.3) by a sign; the reason for this will become apparent in the formula for $\mathcal{D}^{\mathrm{sign}}_C(\epsilon)$ given below.)

Let $g : W^* \to \Omega^*(F; \mathcal{V}^\infty)$ be the dual to $f$ with respect to the Hermitian forms, i.e.,

$$Q_W(f(\alpha), z) = Q_F(\alpha, g(z)).$$

Then we leave to the reader the proof of the following lemma.

**Lemma 2.5.** *$g$ commutes with the differentials. If $f^*$ denotes the adjoint of $f$ with respect to the inner products $< \cdot, \cdot >_F$ and $< \cdot, \cdot >_W$ then $f^* = \tau_F g \tau_W$.*

Using the isomorphism between $W^*$ and $\widehat{W}^*$ in (2.3), let $\widehat{f} : \Omega^*(F;\mathcal{V}^\infty) \to \widehat{W}^*$ and $\widehat{g} : \widehat{W}^* \to \Omega^*(F;\mathcal{V}^\infty)$ be the obvious extensions of $f$ and $g$. Define a cochain complex $C^* = \bigoplus_{k=-1}^{2m} C^k$ by $C^k = \Omega^k(F;\mathcal{V}^\infty) \oplus \widehat{W}^k$. Given $\epsilon \in \mathbb{R}$, define a differential $D_C$ on $C^*$ by

$$\begin{aligned} D_C &= \begin{pmatrix} D_F & \epsilon\widehat{g} \\ 0 & -D_{\widehat{W}} \end{pmatrix} \text{ if } * < m - \frac{1}{2}, \\ D_C &= \begin{pmatrix} D_F & 0 \\ -\epsilon\widehat{f} & -D_{\widehat{W}} \end{pmatrix} \text{ if } * > m - \frac{1}{2}, \end{aligned} \tag{2.5}$$

where $D_F$ has been defined in (1.1). Since $D_{\widehat{W}}\widehat{f} = \widehat{f} D_F$ and $D_F \widehat{g} = \widehat{g} D_{\widehat{W}}$, we have $(D_C)^2 = 0$.

If $\epsilon > 0$ then the complex $(C^*, D_C)$ has vanishing cohomology, as can be seen by Lemma 2.3 and the mapping-cone nature of the construction of $(C^*, D_C)$. Define a duality operator $\tau_C$ on $C^*$ by

$$\tau_C = \begin{pmatrix} \tau_F & 0 \\ 0 & \tau_{\widehat{W}} \end{pmatrix}. \tag{2.6}$$

There is also a Hermitian form $Q_C : C^k \times C^{(2m-1)-k} \to \mathcal{B}^\infty$ given by

$$Q_C((\alpha, v), (\beta, z)) = Q_F(\alpha, \beta) + Q_{\widehat{W}}(v, z).$$

Note that $C$ has formal dimension $2m-1$. We obtain a Hermitian inner product on $C^*$ by

$$< \cdot, \cdot >_C = Q_C(\cdot, \tau_C \cdot).$$

The signature operator of $(C^*, D_C)$ is defined to be

$$\mathcal{D}_C^{\mathrm{sign}}(\epsilon) = -i(\tau_C D_C + D_C \tau_C)$$

and is given on the degree-$j$ subspace by

$$\begin{aligned} \mathcal{D}_C^{\mathrm{sign}}(\epsilon) = (-i) &\begin{pmatrix} D_F\tau_F + \tau_F D_F & 0 \\ 0 & -(D_{\widehat{W}}\tau_{\widehat{W}} + \tau_{\widehat{W}} D_{\widehat{W}}) \end{pmatrix} \\ + (-i) &\begin{cases} \begin{pmatrix} 0 & \epsilon\tau_F\widehat{g} \\ -\epsilon\widehat{f}\tau_F & 0 \end{pmatrix} & \text{if } j < m - \frac{1}{2} \\ \begin{pmatrix} 0 & \epsilon\widehat{g}\tau_{\widehat{W}} \\ -\epsilon\tau_{\widehat{W}}\widehat{f} & 0 \end{pmatrix} & \text{if } j > m - \frac{1}{2}. \end{cases} \end{aligned} \tag{2.7}$$

If $\epsilon > 0$, it follows from the vanishing of the cohomology of $C^*$ that $\mathcal{D}_C^{\mathrm{sign}}(\epsilon)$ is an invertible self-adjoint $\mathcal{B}^\infty$-operator. Namely, $\left(\mathcal{D}_C^{\mathrm{sign}}(\epsilon)\right)^2$

is the Laplace operator $D_C D_C^* + D_C^* D_C$ on $C^*$. From the method of proof of [29, Propositions 10 and 27], the vanishing of the cohomology of $C^*$ implies the invertibility of $D_C D_C^* + D_C^* D_C$.

We are now in a position to recall the definition of the higher eta invariant. Suppose that $F$ satisfies Assumption 1. Define the space $\overline{\Omega}_*(\mathcal{B}^\infty)$ of noncommutative differential forms as in [26, Section II]. Define a rescaling operator $\mathcal{R}$ on $\overline{\Omega}_*(\mathcal{B}^\infty)$ which acts on $\overline{\Omega}_{2j}(\mathcal{B}^\infty)$ as multiplication by $(2\pi i)^{-j}$ and acts on $\overline{\Omega}_{2j-1}(\mathcal{B}^\infty)$ as multiplication by $(2\pi i)^{-j}$.

Let

$$\nabla^\Omega : \Omega^*(F; \mathcal{V}^\infty) \to \Omega_1(\mathcal{B}^\infty) \otimes_{\mathcal{B}^\infty} \Omega^*(F; \mathcal{V}^\infty)$$

be the connection constructed in [26, Proposition 9], in terms of a function $h \in C_0^\infty(F')$ such that $\sum_{\gamma \in \Gamma} \gamma \cdot h = 1$. (Recall that $F'$ is a normal $\Gamma$-cover of $F$.) As in [30, (3.28)], let

$$\nabla^W : W^* \to \Omega_1(\mathcal{B}^\infty) \otimes_{\mathcal{B}^\infty} W^* \tag{2.8}$$

be a connection on $W^*$ which is invariant under $\tau_W$ and preserves $Q_W$. Let $\nabla^{\widehat{W}^*}$ be the obvious extension of $\nabla^W$ to $\widehat{W}^*$ and put $\nabla^C = \nabla^\Omega \oplus \nabla^{\widehat{W}^*}$.

Let $\mathrm{Cl}(1)$ be the complex Clifford algebra of $\mathbb{C}$ generated by 1 and $\sigma$, with $\sigma^2 = 1$, and let $\mathrm{STR}_{\mathrm{Cl}(1)}$ be the supertrace as in [22]. Let $\epsilon \in C^\infty(0, \infty)$ now be a nondecreasing function such that $\epsilon(s) = 0$ for $s \in (0, 1]$ and $\epsilon(s) = 1$ for $s \in [2, +\infty)$. Consider

$$\begin{aligned}\widetilde{\eta}_F(s) = \frac{1}{\sqrt{\pi}} \mathcal{R}\, \mathrm{STR}_{\mathrm{Cl}(1)} \left( \frac{d}{ds}[\sigma s \mathcal{D}_C^{\mathrm{sign}}(\epsilon(s)) + \nabla^C] \right) \\ \cdot \exp[-(\sigma s \mathcal{D}_C^{\mathrm{sign}}(\epsilon(s)) + \nabla^C)^2] \in \overline{\Omega}_{\mathrm{even}}(\mathcal{B}^\infty).\end{aligned} \tag{2.9}$$

The higher eta invariant of $F$ is, by definition,

$$\widetilde{\eta}_F = \int_0^\infty \widetilde{\eta}_F(s) ds \in \overline{\Omega}_{\mathrm{even}}(\mathcal{B}^\infty)/d\overline{\Omega}_{\mathrm{odd}}(\mathcal{B}^\infty). \tag{2.10}$$

It is shown in [30, Proposition 14] that $\widetilde{\eta}_F$ is independent of the particular choices of the function $\epsilon$, the perturbing complex $W^*$ and the self-dual connection $\nabla^W$. Definition (2.10) can be seen as a way of regularizing the *a priori* divergent integral

$$\frac{1}{\sqrt{\pi}} \mathcal{R} \int_0^\infty \mathrm{STr}_{\mathrm{Cl}(1)} \left( \frac{d}{ds}[\sigma s \mathcal{D}^{\mathrm{sign}} + \nabla] \right) \exp[-(\sigma s \mathcal{D}^{\mathrm{sign}} + \nabla)^2]\, ds \tag{2.11}$$

coming from the signature operator $\mathcal{D}^{\mathrm{sign}}$ of $F$. It is not clear that the integrand in (2.11) is integrable for large $s$, as the spectrum of $\mathcal{D}^{\mathrm{sign}}$ may include zero. To get around this problem, we have first added the complex $\widehat{W}^*$, whose higher eta-invariant formally vanishes by a duality argument. Then we have perturbed the direct sum differential so that for large $s$, we are dealing with the invertible signature operator $\mathcal{D}_C^{\mathrm{sign}}(1)$.

The invertibility of $\mathcal{D}_C^{\mathrm{sign}}(1)$ ensures that the integrand in (2.10) is integrable for large $s$; see the proof of [29, Proposition 28] in the analogous but more difficult case of the analytic torsion form. From [27, Proposition 26], the integrand in (2.10) is integrable for small $s$.

**Remark.** A different regularization of (2.11) has been proposed in [25] using the notion of *symmetric spectral section*. See the Appendix for an informal argument showing the equality of the two regularizations.

The higher eta-invariant satisfies

$$d\widetilde{\eta}_F = \int_F L(R^F/2\pi) \wedge \omega, \tag{2.12}$$

where the closed biform $\omega \in \Omega^*(F) \widehat{\otimes} \overline{\Omega}_*(\mathcal{B}^\infty)$ is given in [26, Section V]. In fact, $\omega$ is the image of an element of $\Omega^*(F) \widehat{\otimes} \overline{\Omega}_*(\mathbb{C}\Gamma)$ under the map $\overline{\Omega}_*(\mathbb{C}\Gamma) \to \overline{\Omega}_*(\mathcal{B}^\infty)$. (This follows from the fact that the function $h$ used to define $\nabla^\Omega$ and $\omega$ from [26, (40)] has compact support on $F'$.) By abuse of notation, we will also denote this element of $\Omega^*(F) \widehat{\otimes} \overline{\Omega}_*(\mathbb{C}\Gamma)$ by $\omega$. It satisfies the property that if $Z_\tau$ is a cyclic cocycle which represents a cohomology class $\tau \in \mathrm{H}^*(\Gamma; \mathbb{C})$ then $\langle \omega, Z_\tau \rangle \in \Omega^*(F)$ is an explicit closed form on $F$ whose de Rham cohomology class is a nonzero constant (which only depends on the degree of $\tau$) times $\nu^*\tau$.

**Conventions.** Let us take this occasion to establish our conventions for Chern characters. If $\nabla$ is a connection on a vector bundle then its Chern character is

$$\mathrm{ch}(\nabla) = \mathrm{TR}\left(e^{-\frac{\nabla^2}{2\pi i}}\right) = \mathcal{R}\ \mathrm{TR}\left(e^{-\nabla^2}\right).$$

The de Rham cohomology class of $\mathrm{ch}(\nabla)$ is the representative of a *rational* cohomology class. Similarly, the de Rham cohomology class of the $L$-form $L(R^F/2\pi)$ lies in the image of the map $\mathrm{H}^*(F;\mathbb{Q}) \to \mathrm{H}^*(F;\mathbb{R})$. If $\mathbb{A}$ is a superconnection then its Chern character is $\mathrm{ch}(\mathbb{A}) = \mathcal{R}\,\mathrm{STR}\left(e^{-\mathbb{A}^2}\right)$.

## 3. The higher eta invariant of an even-dimensional manifold

Before dealing with the case of even-dimensional $F$, we introduce the notion of a Lagrangian subspace of a $\mathcal{B}^\infty$-module. Let H be a finitely-generated projective $\mathcal{B}^\infty$-module with a nondegenerate quadratic form $Q_{\mathrm{H}} : \mathrm{H} \times \mathrm{H} \to \mathcal{B}^\infty$ such that $Q_{\mathrm{H}}(bx, y) = bQ_{\mathrm{H}}(x, y)$ and $Q_{\mathrm{H}}(x, y)^* = Q_{\mathrm{H}}(y, x)$. A *Lagrangian subspace* of H is a finitely-generated projective $\mathcal{B}^\infty$-submodule $L$ on which $Q_{\mathrm{H}}$ vanishes, such that $L$ equals $L^\perp$, its orthogonal space with respect to $Q_{\mathrm{H}}$. Equivalently, let $L$ be a finitely-generated projective $\mathcal{B}^\infty$-submodule of H. Let $L'$ be the antidual to $L$, i.e., the set of $\mathbb{R}$-linear maps $l' : L \to \mathcal{B}^\infty$ such that $l'(bl) = l'(l)b^*$ for all $b \in \mathcal{B}^\infty$ and $l \in L$. Here $L'$ is also a left $\mathcal{B}^\infty$-module, with the multiplication given by $(al')(l) = al'(l)$. Then for $L$ to be a Lagrangian subspace of H amounts to the existence of a short exact sequence

$$0 \longrightarrow L \longrightarrow \mathrm{H} \longrightarrow L' \longrightarrow 0 \tag{3.1}$$

whose maps are an injection $i : L \to \mathrm{H}$ and its antidual (with respect to $Q_{\mathrm{H}}$) $i' : \mathrm{H} \to L'$.

If $\mathcal{L}$ is a finitely-generated projective $\mathcal{B}^\infty$-module then there is a canonical quadratic form on $\mathcal{L} \oplus \mathcal{L}'$ given by

$$Q(l_1 + l_1', l_2 + l_2') = l_1'(l_2) + (l_2'(l_1))^*.$$

It has a canonical Lagrangian subspace given by $\mathcal{L}$. (In what follows, it will in fact suffice to take $\mathcal{L}$ of the form $(\mathcal{B}^\infty)^N$.) A *stable Lagrangian subspace* of H is a Lagrangian subspace $L$ of $\mathcal{H} = \mathrm{H} \oplus (\mathcal{L} \oplus \mathcal{L}')$ for some $\mathcal{L}$ as above. We say that two stable Lagrangian subspaces of H, $L_1 \subset \mathcal{H}_1$ and $L_2 \subset \mathcal{H}_2$, are equivalent if there are $\mathcal{L}_3$ and $\mathcal{L}_4$, and an isomorphism $j : \mathcal{H}_1 \oplus \mathcal{L}_3 \oplus \mathcal{L}_3' \to \mathcal{H}_2 \oplus \mathcal{L}_4 \oplus \mathcal{L}_4'$ of quadratic form spaces, such that $j(L_1 \oplus \mathcal{L}_3) = L_2 \oplus \mathcal{L}_4$.

Now suppose that $F$ is a closed oriented manifold of dimension $n = 2m$. Let $\nu : F \to B\Gamma$ be a continuous map as before. We make the following assumption:

**Assumption 1.a.** The natural surjection $\mathrm{H}^m(F; \mathcal{V}) \to \overline{\mathrm{H}}^m(F; \mathcal{V})$ is an isomorphism.

**Lemma 3.1.** *If $F$ is equipped with a Riemannian metric then Assumption 1.a is equivalent to saying that the differential form Laplacian on $\Omega^m(F')$ has a strictly positive spectrum on the orthogonal complement of its kernel.*

*Proof.* We give an outline of the proof. Let $\Omega^*_{(2)}(F;\mathcal{V})$ denote the completion of $\Omega^*(F;\mathcal{V})$ as a $C^*_r(\Gamma)$-Hilbert module. Assumption 1 is equivalent to saying that the differential $D_F : \Omega^{m-1}(F;\mathcal{V}) \to \Omega^m(F;\mathcal{V})$ has a closed image. From arguments as in [29, Propositions 10 and 27], this is equivalent to saying that $D_F D_F^*$ has a strictly positive spectrum on $\mathrm{Im}(D_F : \Omega^{m-1}_{(2)}(F;\mathcal{V}) \to \Omega^m_{(2)}(F;\mathcal{V}))$. Then by Hodge duality, $D_F^* D_F$ has a strictly positive spectrum on $\Omega^m_{(2)}(F;\mathcal{V})/\mathrm{Ker}(D_F)$. Again as in the proof of [29, Propositions 10 and 27], under Assumption 1.a there is an orthogonal direct sum decomposition of closed $C^*_r(\Gamma)$-Hilbert modules

$$\Omega^m_{(2)}(F;\mathcal{V}) = \mathrm{Ker}(D_F D_F^* + D_F^* D_F) \oplus \mathrm{Im}(D_F) \oplus \Omega^m_{(2)}(F;\mathcal{V})/\mathrm{Ker}(D_F).$$

Thus $D_F D_F^* + D_F^* D_F$ has a strictly positive spectrum on the orthogonal complement of its kernel. The lemma now follows as in the rest of the proof of Lemma 2.1. q.e.d.

Hereafter we assume that Assumption 1.a is satisfied. The proof of the next lemma is similar to that of Lemma 2.3, but easier, and will be omitted.

**Lemma 3.2.** *There is a cochain complex $W^* = \bigoplus_{i=0}^{2m} W^i$ of finitely-generated projective $\mathcal{B}^\infty$-modules such that*

*1. $W^*$ is a graded regular $n$-dimensional Hermitian complex.*

*2. The differentials $D_W : W^{m-1} \to W^m$ and $D_W : W^m \to W^{m+1}$ vanish.*

*3. There is a double homotopy equivalence*

$$f : \Omega^*(F;\mathcal{V}^\infty) \to W^* \tag{3.2}$$

*which, as an element of $(\Omega^*(F;\mathcal{V}^\infty))^* \otimes W^*$, is actually smooth with respect to $F$.*

For brevity, let us denote $\mathrm{H}^m(F;\mathcal{V}^\infty)$ by $\mathrm{H}$. Then $\mathrm{H}$ is a finitely-generated projective $\mathcal{B}^\infty$-module which is isomorphic to the module $W^m$ of Lemma 3.2. The quadratic form $Q_F$ restricts to a nondegenerate quadratic form $Q_{\mathrm{H}}$. The grading operator $\tau_F$ induces a grading operator $\tau_{\mathrm{H}}$ on $\mathrm{H}$. Let $\mathrm{H}^\pm$ be the $\pm 1$-eigenspace of $\tau_{\mathrm{H}}$. Put $\overline{\mathrm{H}} = C^*_r(\Gamma) \otimes_{\mathcal{B}^\infty} \mathrm{H}$, and similarly for $\overline{\mathrm{H}}^\pm$.

**Lemma 3.3.** *The index of the signature operator on $F$ equals*

$$\left[\overline{\mathrm{H}}^+\right] - \left[\overline{\mathrm{H}}^-\right] \in K_0(C^*_r(\Gamma)).$$

*Proof.* Put $\overline{W}^* = C_r^*(\Gamma) \otimes_{\mathcal{B}^\infty} W^*$. The index of the signature operator of $F$ equals the index of the signature operator of the complex $\overline{W}^*$ [20, Theorem 4.1] and is independent of the choice of grading operator $\tau$ [20, Proposition 3.6]. Hence we may work with the complex $\overline{W}^*$. Consider the regular Hermitian complex $\widetilde{W}^* = \oplus_{i \neq m} \overline{W}^i$. It is enough to show that the index of the signature operator $\mathcal{D}^{\mathrm{sign},+}_{\widetilde{W}}$ of $\widetilde{W}^*$ vanishes. To show this, define an operator $\mu$ on $\widetilde{W}^*$ by

$$\mu(w) = \begin{cases} w & \text{if } w \in \widetilde{W}^i, i < m, \\ -w & \text{if } w \in \widetilde{W}^i, i > m. \end{cases}$$

Then $\mu^2 = 1$, $\mu \mathcal{D}^{\mathrm{sign}}_{\widetilde{W}} = \mathcal{D}^{\mathrm{sign}}_{\widetilde{W}} \mu$ and $\mu \tau_{\widetilde{W}} + \tau_{\widetilde{W}} \mu = 0$. Let $\widetilde{W}^*_\pm$ be the $\pm 1$ eigenspaces of $\tau_{\widetilde{W}}$. Then $\mu$ induces an isomorphism from $\widetilde{W}^*_+$ to $\widetilde{W}^*_-$ and so $\mathrm{Ind}(\mathcal{D}^{\mathrm{sign},+}_{\widetilde{W}}) = [\widetilde{W}^*_+] - [\widetilde{W}^*_-] = 0$. q.e.d.

We make the following further assumption.

**Assumption 1.b.** $\mathrm{H}^m(F; \mathcal{V}^\infty)$ admits a (stable) Lagrangian subspace.

**Lemma 3.4.** *Given Assumption 1.a, Assumption 1.b is equivalent to saying that the index of the signature operator on $F$ vanishes in $K_0(C_r^*(\Gamma))$.*

*Proof.* In general, if $\mathcal{L}$ is a finitely-generated projective $\mathcal{B}^\infty$-module, put $\overline{\mathcal{L}} = C_r^*(\Gamma) \otimes_{\mathcal{B}^\infty} \mathcal{L}$. If $h$ is a $\mathcal{B}^\infty$-valued Hermitian metric on $\mathcal{L}$, let $\overline{h}$ be its extension to a $C_r^*(\Gamma)$-valued Hermitian metric on $\overline{\mathcal{L}}$. Define $\mathcal{I} : \overline{\mathcal{L}} \to \overline{\mathcal{L}}'$ by

$$(\mathcal{I}(l_1))\,(l_2) = \overline{h}(l_1, l_2). \tag{3.3}$$

Put

$$\tau_{\overline{\mathcal{L}} \oplus \overline{\mathcal{L}}'} = \begin{pmatrix} 0 & \mathcal{I}^{-1} \\ \mathcal{I} & 0 \end{pmatrix}.$$

If $\left(\overline{\mathcal{L}} \oplus \overline{\mathcal{L}}'\right)^\pm$ denotes the $\pm 1$ eigenspaces of $\tau_{\overline{\mathcal{L}} \oplus \overline{\mathcal{L}}'}$ then $(\overline{\mathcal{L}} \oplus \overline{\mathcal{L}}')^+$ is isomorphic to $(\overline{\mathcal{L}} \oplus \overline{\mathcal{L}}')^-$, under $x + \mathcal{I}(x) \to x - \mathcal{I}(x)$.

Suppose that Assumption 1.b is satisfied. Then there is some finitely-generated projective $\mathcal{B}^\infty$-module $\mathcal{L}$ such that $\mathrm{H} \oplus \mathcal{L} \oplus \mathcal{L}'$ has a Lagrangian subspace $L$. Give $\overline{\mathrm{H}} \oplus \overline{\mathcal{L}} \oplus \overline{\mathcal{L}}'$ the grading operator $\tau_{\overline{\mathrm{H}}} \oplus \tau_{\overline{\mathcal{L}} \oplus \overline{\mathcal{L}}'}$. Then from what has been said,

$$[\overline{\mathrm{H}}^+] - [\overline{\mathrm{H}}^-] = [(\overline{\mathrm{H}} \oplus \overline{\mathcal{L}} \oplus \overline{\mathcal{L}}')^+] - [(\overline{\mathrm{H}} \oplus \overline{\mathcal{L}} \oplus \overline{\mathcal{L}}')^-] \tag{3.4}$$

in $K_0(C^*_r(\Gamma))$. However, as the element of $K_0(C^*_r(\Gamma))$ coming from $\overline{\mathrm{H}} \oplus \overline{\mathcal{L}} \oplus \overline{\mathcal{L}}'$ is independent of the choice of the grading operator $\tau$, we can use $\overline{L}$ to define a grading operator on $\overline{\mathrm{H}} \oplus \overline{\mathcal{L}} \oplus \overline{\mathcal{L}}'$, as in (3.3), to see that $[(\overline{\mathrm{H}} \oplus \overline{\mathcal{L}} \oplus \overline{\mathcal{L}}')^+] - [(\overline{\mathrm{H}} \oplus \overline{\mathcal{L}} \oplus \overline{\mathcal{L}}')^-] = 0$ in $K_0(C^*_r(\Gamma))$. From (3.4) and Lemma 3.3, this implies that the index of the signature operator on $F$ vanishes in $K_0(C^*_r(\Gamma))$.

Now suppose that the index of the signature operator on $F$ vanishes in $K_0(C^*_r(\Gamma))$. Then there is some $N \geq 0$ such that $\overline{\mathrm{H}}^+ \oplus C^*_r(\Gamma)^N$ is isomorphic to $\overline{\mathrm{H}}^- \oplus C^*_r(\Gamma)^N$. We can take the isomorphism $\overline{j} : \overline{\mathrm{H}}^+ \oplus C^*_r(\Gamma)^N \to \overline{\mathrm{H}}^- \oplus C^*_r(\Gamma)^N$ to be an isometry. Using arguments as in [22, Appendix A], we can assume that $\overline{j} = j \otimes_{\mathcal{B}^\infty} I$ for some isometric isomorphism $j : \mathrm{H}^+ \oplus (\mathcal{B}^\infty)^N \to \mathrm{H}^- \oplus (\mathcal{B}^\infty)^N$. Then graph$(j)$ is a Lagrangian subspace of $\mathrm{H} \oplus (\mathcal{B}^\infty)^N \oplus (\mathcal{B}^\infty)^N$. Thus Assumption 1.b is satisfied. q.e.d.

**Corollary 3.5.** *Given Assumption 1.a, if $F$ is the boundary of a compact oriented manifold $M$ and $\nu$ extends over $M$ then Assumption 1.b is satisfied.*

*Proof.* This follows from the cobordism invariance of the index, along with Lemma 3.4. q.e.d.

**Lemma 3.6.**

*(a) If $F$ has a cellular decomposition without any cells of dimension $m$ then Assumptions 1.a and 1.b are satisfied.*

*(b) If $\Gamma$ is finite and the signature of $F$ vanishes then Assumptions 1.a and 1.b are satisfied.*

*(c) Let $F_1$ and $F_2$ be even-dimensional manifolds, with $F_1$ a connected closed hyperbolic manifold and $F_2$ a closed manifold with vanishing signature. Put $\Gamma = \pi_1(F_1)$. If $F = F_1 \times F_2$ and $\nu$ is projection onto the first factor then Assumptions 1.a and 1.b are satisfied.*

*Proof.*

(a) If $F$ has a cellular decomposition without any cells of dimension $m$ then $\mathrm{H}^m(F; \mathcal{V})$ vanishes and Assumptions 1.a and 1.b are automatically satisfied

(b) If $\Gamma$ is finite then $F'$ is compact and from standard Hodge theory, the result of Lemma 3.1 is satisfied. From Lemma 3.4, it remains to show that $F$ has vanishing index in $K_0(C_r^*(\Gamma))$. Now $K_0(C_r^*(\Gamma))$ is isomorphic to the ring of complex virtual representations of $\Gamma$. Given a representation $\rho : \Gamma \to U(N)$, the corresponding component of the $K_0(C_r^*(\Gamma))$-index is the usual index of the signature operator acting on $\Omega^*(F') \otimes_\rho \mathbb{C}^N$. By the Atiyah-Singer index theorem this equals $N$ times the signature of $F$, and hence vanishes.

(c) From [13], the spectrum of the differential form Laplacian on the hyperbolic space $\widetilde{F_1}$ is strictly positive on the orthogonal complement of its kernel (which is concentrated in the middle degree). Then by separation of variables and using the fact that the universal cover $\widetilde{F_2}$ is compact, it follows that the result of Lemma 3.1 is satisfied. From Lemma 3.4 and the multiplicativity of the index, along with the vanishing of the signature of $F_2$, we obtain that Assumption 1.b is satisfied.

q.e.d.

Hereafter we assume that $\mathrm{H}$ admits a stable Lagrangian subspace. Let $L \subset \mathrm{H} \oplus \mathcal{L} \oplus \mathcal{L}'$ be one such. We define a new complex $\widehat{W}^*$ by

$$\widehat{W}^i = \begin{cases} W^{i+1} & \text{if } -1 \le i \le m-2, \\ L & \text{if } i = m-1 \\ 0 & \text{if } i = m \\ L' & \text{if } i = m+1 \\ W^{i-1} & \text{if } m+2 \le i \le 2m+1. \end{cases} \tag{3.5}$$

There is an obvious extension of $D_W$ to a differential $D_{\widehat{W}}$ and obvious extensions of $Q_W$ and $\tau_W$ to $\widehat{W}$, at least on the part of $\widehat{W}$ that does not involve $L$ or $L'$. Define $Q_{\widehat{W}} : L \times L' \to \mathcal{B}^\infty$ by $Q_{\widehat{W}}(l, l') = (l'(l))^*$. Let $h$ be the $\mathcal{B}^\infty$-valued Hermitian metric on $L$ induced from $\mathrm{H} \oplus \mathcal{L} \oplus \mathcal{L}'$. Define $\tau_{\widehat{W}} : L \to L'$ by $\left(\tau_{\widehat{W}}(l_1)\right)(l_2) = h(l_1, l_2)$. Let $\tau_{\widehat{W}} : L' \to L$ be the inverse. Then we obtain a well-defined triple $(D_{\widehat{W}}, Q_{\widehat{W}}, \tau_{\widehat{W}})$ on $\widehat{W}$.

Let $\Omega^*(F; \mathcal{V}^\infty) \oplus \mathcal{L} \oplus \mathcal{L}'$ be the direct sum cochain complex, with $\mathcal{L} \oplus \mathcal{L}'$ concentrated in degree $m$. Recall the notation $i$ and $i'$ for the maps in (3.1), where $\mathrm{H}$ is again $\mathrm{H}^m(F; \mathcal{V}^\infty) = \overline{\mathrm{H}}^m(F; \mathcal{V}^\infty)$. Let $\mathcal{I} : \mathrm{H}^m(F; \mathcal{V}^\infty) \to \Omega^m(F; \mathcal{V}^\infty)$ be the inclusion coming from Hodge theory and let $\mathcal{I}^* : \Omega^m(F; \mathcal{V}^\infty) \to \mathrm{H}^m(F; \mathcal{V}^\infty)$ be orthogonal projection. Define

$\widehat{f} : \Omega^*(F;\mathcal{V}^\infty) \oplus \mathcal{L} \oplus \mathcal{L}' \to \widehat{W}^*$ to be the obvious extension of $f$ outside of degree $m$, and to be given in degree $m$ by $\widehat{f}(\omega + l + l') = (i' \circ \mathcal{I}^*)(\omega) \in \widehat{W}^{m+1}$. Define $\widehat{g} : \widehat{W}^* \to \Omega^*(F;\mathcal{V}^\infty) \oplus \mathcal{L} \oplus \mathcal{L}'$ to be the obvious extension of $g$ outside of degrees $m-1$, $m$ and $m+1$, to be given in degree $m-1$ by $\widehat{g}(l) = (\mathcal{I} \circ i)(l) \in \Omega^m(F;\mathcal{V}^\infty)$ and to vanish in degrees $m$ and $m+1$. Define a cochain complex $C = \bigoplus_{k=-1}^{2m+1} C^k$ by $C^* = \Omega^*(F;\mathcal{V}^\infty) \oplus \mathcal{L} \oplus \mathcal{L}' \oplus \widehat{W}^*$. Given $\epsilon \in \mathbb{R}$, define a differential $D_C$ on $C$ by

$$(3.6) \qquad \begin{aligned} D_C &= \begin{pmatrix} D_F & \epsilon\widehat{g} \\ 0 & -D_{\widehat{W}} \end{pmatrix} \text{ if } * \leq m-1, \\ D_C &= \begin{pmatrix} D_F & 0 \\ -\epsilon\widehat{f} & -D_{\widehat{W}} \end{pmatrix} \text{ if } * \geq m. \end{aligned}$$

We can then define $\tau_C$, $Q_C$ and $\mathcal{D}_C^{\text{sign}}(\epsilon)$ in analogy to what we did in the odd-dimensional case.

We put

$$(3.7) \qquad \begin{aligned} \widetilde{\eta}_F(s) &= \mathcal{R}\,\mathrm{STR}\left(\frac{d}{ds}[s\mathcal{D}_C^{\text{sign}}(\epsilon(s)) + \nabla^C]\right) \\ &\qquad\qquad \cdot \exp[-(s\mathcal{D}_C^{\text{sign}}(\epsilon(s)) + \nabla^C)^2] \\ &\in \overline{\Omega}_{\text{odd}}(\mathcal{B}^\infty) \end{aligned}$$

where STR is the supertrace and $\nabla^C$ is a self-dual connection as before. The function $\epsilon(s)$ is the same as in the odd-dimensional case.

The higher eta invariant of $F$ is, by definition,

$$(3.8) \qquad \widetilde{\eta}_F = \int_0^\infty \widetilde{\eta}_F(s)ds \in \overline{\Omega}_{\text{odd}}(\mathcal{B}^\infty)/d\overline{\Omega}_{\text{even}}(\mathcal{B}^\infty).$$

As in [30, Proposition 14], $\widetilde{\eta}_F$ is independent of the particular choices of $\epsilon$, the perturbing complex $W^*$ and the self-dual connection $\nabla^W$. It satisfies (2.12).

Let us consider how $\widetilde{\eta}_F$ depends on the choice of (stable) Lagrangian subspace $L$. For the moment, let us denote the dependence by $\widetilde{\eta}_F(L)$. From equation (2.120, if $L_1$ and $L_2$ are two (stable) Lagrangian subspaces then $d(\widetilde{\eta}_F(L_1) - \widetilde{\eta}_F(L_2)) = 0$. Thus $\widetilde{\eta}_F(L_1) - \widetilde{\eta}_F(L_2)$ represents an element of $\overline{\mathrm{H}}_{\text{odd}}(\mathcal{B}^\infty)$. To describe it, we construct a characteristic class coming from two (stable) Lagrangian subspaces.

Let H be a finitely-generated projective $\mathcal{B}^\infty$-module as above, equipped with a quadratic form $Q_{\mathrm{H}}$. For simplicity, we will only deal with honest

Lagrangian subspaces of H; the case of stable Lagrangian subspaces can be dealt with by replacing H by $\mathrm{H} \oplus (\mathcal{B}^\infty)^N \oplus (\mathcal{B}^\infty)^N$.

As in the proof of Lemma 3.4, after choosing a grading $\tau_H$, the set of Lagrangian subspaces of H can be identified with $\mathrm{Isom}_{\mathcal{B}^\infty}(\mathrm{H}^+, \mathrm{H}^-)$, the set of isometric isomorphisms from $\mathrm{H}^+$ to $\mathrm{H}^-$. If $j_1, j_2 \in \mathrm{Isom}_{\mathcal{B}^\infty}(\mathrm{H}^+, \mathrm{H}^-)$ then $j_1 \circ j_2^{-1} \in \mathrm{Isom}_{\mathcal{B}^\infty}(\mathrm{H}^-, \mathrm{H}^-)$. Now $\mathrm{Isom}_{\mathcal{B}^\infty}(\mathrm{H}^-, \mathrm{H}^-)$ is homotopy-equivalent to $\mathrm{GL}_{\mathcal{B}^\infty}(\mathrm{H}^-)$. Hence given two Lagrangian subspaces $L_1$ and $L_2$ of H, we obtain an element of $\pi_0(\mathrm{GL}_{\mathcal{B}^\infty}(\mathrm{H}^-))$ represented by $j_1 \circ j_2^{-1}$. Let $[L_1 - L_2]$ denote its image in $K_1(\mathcal{B}^\infty) \cong \pi_0(\mathrm{GL}_{\mathcal{B}^\infty}(\infty))$.

**Proposition 3.7.**

$$\widetilde{\eta}_F(L_1) - \widetilde{\eta}_F(L_2) = \mathrm{ch}([L_1 - L_2]) \qquad \text{in } \overline{\mathrm{H}}_{\mathrm{odd}}(\mathcal{B}^\infty).$$

*Proof.* Fix, for the moment, a Lagrangian subspace $L$ of H. Writing $L = \mathrm{graph}(j)$ with $j \in \mathrm{Isom}_{\mathcal{B}^\infty}(\mathrm{H}^+, \mathrm{H}^-)$, we can identify $L$, and hence $L'$, with $\mathrm{H}^+$. Under these identifications, the short exact sequence (3.1) becomes

$$0 \longrightarrow \mathrm{H}^+ \longrightarrow \mathrm{H}^+ \oplus \mathrm{H}^- \longrightarrow \mathrm{H}^+ \longrightarrow 0.$$

To describe the maps involved explicitly, let us consider this to be a graded regular $2m$-dimensional Hermitian complex $\mathcal{E}^*$ concentrated in degrees $m-1$, $m$ and $m+1$. Then the maps are given by saying that if $h_+ \in \mathcal{E}^{m-1}$ then $D_\mathcal{E}(h_+) = \frac{1}{\sqrt{2}}(h_+, j(h_+))$, while if $(h_+, h_-) \in \mathcal{E}^m$ then $D_\mathcal{E}(h_+, h_-) = \frac{1}{\sqrt{2}}(-h_+ + j^{-1}(h_-))$. If $(h_+, h_-), (k_+, k_-) \in \mathcal{E}^m$ then $Q_\mathcal{E}((h_+, h_-), (k_+, k_-)) = \langle h_+, k_+ \rangle - \langle h_-, k_- \rangle$, while if $h_+ \in \mathcal{E}^{m-1}$ and $k_+ \in \mathcal{E}^{m+1}$ then $Q_\mathcal{E}(h_+, k_+) = \langle h_+, k_+ \rangle$. If $(h_+, h_-) \in \mathcal{E}^m$ then $\tau_\mathcal{E}(h_+, h_-) = (h_+, -h_-)$, while $\tau_\mathcal{E} : \mathcal{E}^{m\pm1} \to \mathcal{E}^{m\mp1}$ is the identity map on $\mathrm{H}^+$.

The connection $\nabla^{\mathrm{H}}$, induced from $\nabla^\Omega$, breaks up as a direct sum $\nabla^{\mathrm{H}^+} \oplus \nabla^{\mathrm{H}^-}$. We choose to put the connection $\nabla^{\mathrm{H}^+}$ on both $\mathcal{E}^{m-1}$ and $\mathcal{E}^{m+1}$. We obtain a self-dual connection $\nabla^\mathcal{E}$ on $\mathcal{E}$.

It is convenient to perform a change of basis by means of the isomorphism

$$\mathcal{K} : \mathcal{E}^{m-1} \oplus \mathcal{E}^{m+1} \to \mathrm{H}^+ \oplus \mathrm{H}^-$$

given by

$$\mathcal{K}(o_1, o_2) = \frac{1}{\sqrt{2}}(o_1 - o_2, j(o_1 + o_2)).$$

One can compute that the signature operator $\mathcal{D}_\mathcal{E} = D_\mathcal{E} - \tau_\mathcal{E} D_\mathcal{E} \tau_\mathcal{E}$ acts as $\mathcal{K}$ on $\mathcal{E}^{m-1} \oplus \mathcal{E}^{m+1}$, and as $\mathcal{K}^{-1}$ on $\mathcal{E}^m = \mathrm{H}^+ \oplus \mathrm{H}^-$. Thus using the isomorphism $\mathcal{K}$ to identify $\mathcal{E}^{m-1} \oplus \mathcal{E}^{m+1}$ with $\mathrm{H}^+ \oplus \mathrm{H}^-$, the signature operator $\mathcal{D}_{\mathcal{E}}$ acts on the total space

$$\mathcal{E}^m \oplus (\mathcal{E}^{m-1} \oplus \mathcal{E}^{m+1}) \cong (\mathrm{H}^+ \oplus \mathrm{H}^-) \oplus (\mathrm{H}^+ \oplus \mathrm{H}^-)$$

as $\begin{pmatrix} 0 & I \oplus I \\ I \oplus I & 0 \end{pmatrix}$. We note that this is indeed an odd operator with respect to the $\mathbb{Z}_2$-grading, as the induced duality operator on the second $\mathrm{H}^+ \oplus \mathrm{H}^-$ factor is $\mathcal{K}\tau_{\mathcal{E}}\mathcal{K}^{-1} = (-I, I)$. One can compute that in the new basis, the connection $\nabla^{\mathcal{E}}$ becomes

$$\mathcal{K}\nabla^{\mathcal{E}}\mathcal{K}^{-1} = (\nabla^{\mathrm{H}^+} \oplus \nabla^{\mathrm{H}^-}) \oplus (\nabla^{\mathrm{H}^+} \oplus j\nabla^{\mathrm{H}^+} j^{-1}).$$

We now consider the complex $C^* = \Omega^*(F; \mathcal{V}^\infty) \oplus \mathcal{L} \oplus \mathcal{L}' \oplus \widehat{W}^*$ used to define $\widetilde{\eta}_F$. Then $\mathcal{E}^*$ is a subcomplex of $C^*$ and there is a direct sum decomposition $C^* = \mathcal{E}^* \oplus (\mathcal{E}^*)^\perp$. As a $\mathbb{Z}_2$-graded vector space, we have shown that $C^*$ is isomorphic to $(\mathrm{H}^+ \oplus \mathrm{H}^-) \oplus (\mathrm{H}^+ \oplus \mathrm{H}^-) \oplus (\mathcal{E}^*)^\perp$, regardless of $L$. For $s > 0$, put $\mathbb{A}_s = s\mathcal{D}_C^{\mathrm{sign}}(\epsilon(s)) + \nabla^C$, thought of as a superconnection on this $\mathbb{Z}_2$-graded vector space. With our identifications, the 0-th order part of $\mathbb{A}_s$, namely $s\mathcal{D}_C^{\mathrm{sign}}(\epsilon(s))$ is independent of $L$. Furthermore, for large $s$, the operator $\mathcal{D}_C^{\mathrm{sign}}(\epsilon(s))$ is invertible. However, the connection part of $\mathbb{A}_s$,

$$\nabla^C = (\nabla^{\mathrm{H}^+} \oplus \nabla^{\mathrm{H}^-}) \oplus (\nabla^{\mathrm{H}^+} \oplus j\nabla^{\mathrm{H}^+} j^{-1}) \oplus \nabla^{\mathcal{E}^\perp}, \tag{3.9}$$

does depend on $L$ through the map $j : \mathrm{H}^+ \to \mathrm{H}^-$.

We are reduced to studying how $\widetilde{\eta}_F$ depends on the connection part of the superconnection. In general, if $\{\mathbb{A}(u)\}_{u \in [0,1]}$ is a smooth 1-parameter family of superconnections of the form

$$\mathbb{A}(u) = \sum_{j=0}^{\infty} \mathbb{A}_{[j]}(u)$$

and we put

$$\mathbb{A}_s(u) = \sum_{j=0}^{\infty} s^{1-j} \mathbb{A}_{[j]}(u)$$

then from [27, (49)], modulo exact forms,

$$\frac{d}{du}\widetilde{\eta}(s) = \frac{d}{ds} \mathcal{R}\,\mathrm{STR}\, \frac{dA_s}{du}\, e^{-A_s^2}.$$

Hence, when it can be justified,

$$\frac{d}{du}\widetilde{\eta} = \mathcal{R}\,\mathrm{STR}\,\frac{dA_s}{du}\,e^{-A_s^2}\Big|_{s=\infty} - \mathcal{R}\,\mathrm{STR}\,\frac{dA_s}{du}\,e^{-A_s^2}\Big|_{s=0}.$$

Now consider two Lagrangian subspaces $L_1$ and $L_2$ of H. Choose a 1-parameter family $\{\nabla^{\widehat{W}^*(u)}\}_{u\in[0,1]}$ of self-dual connections on $\widehat{W}^*$ such that $\nabla^{\widehat{W}^*}(1)$ is the connection coming from $L_1$ and $\nabla^{\widehat{W}^*}(0)$ is the connection coming from $L_2$. In our case, the invertibility of $\mathcal{D}_C^{\mathrm{sign}}(\epsilon(\infty))$ implies that $\mathcal{R}\,\mathrm{STR}\,\frac{dA_s}{du}\,e^{-A_s^2}\Big|_{s=\infty} = 0$. For small $s$, the complexes $\Omega^*(F;\mathcal{V}^\infty)$ and $\widehat{W}^*$ decouple. After making a change of basis as above, the only $u$-dependence of $\mathbb{A}_s(u)$ arises from the $u$-dependence of $\nabla^{\widehat{W}^*}$. Hence

$$\mathcal{R}\,\mathrm{STR}\,\frac{dA_s}{du}\,e^{-A_s^2}\Big|_{s=0} = \mathcal{R}\,\mathrm{STR}\,\frac{d\nabla^{\widehat{W}^*}}{du}\,e^{-(\nabla^{\widehat{W}^*})^2}$$

and so

$$\widetilde{\eta}_F(L_1) - \widetilde{\eta}_F(L_2) = -\int_0^1 \mathcal{R}\,\mathrm{STR}\,\frac{d\nabla^{\widehat{W}^*}}{du}\,e^{-(\nabla^{\widehat{W}^*})^2}du.$$

Let $j_1, j_2 \in \mathrm{Isom}_{\mathcal{B}^\infty}(\mathrm{H}^+, \mathrm{H}^-)$ be the maps corresponding to $L_1$ and $L_2$. Recall that $[L_1 - L_2]$ denote the element of $K_1(\mathcal{B}^\infty) \cong \pi_0\left(\mathrm{GL}_{\mathcal{B}^\infty}(\infty)\right)$ corresponding to $j_1 \circ j_2^{-1}$. As $-\int_0^1 \mathcal{R}\,\mathrm{STR}\,\frac{d\nabla^{\widehat{W}^*}}{du}\,e^{-(\nabla^{\widehat{W}^*})^2}du$ is the Chern character of $j_1 \circ j_2^{-1}$ [14, Definition 1.1], the proposition follows. q.e.d.

**Remark.** In Assumption 1.b of the introduction, instead of assuming that the index $\mathrm{Ind}_F$ of the signature operator vanishes in $K_0(C_r^*(\Gamma))$, for our purposes it suffices to assume that it vanishes in $K_0(C_r^*(\Gamma))\otimes_{\mathbb{Z}}\mathbb{Q}$. If this is the case then there is an integer $N > 0$ such that $N\,\mathrm{Ind}_F$ vanishes in $K_0(C_r^*(\Gamma))$. We can then take $N$ disjoint copies of $F$, choose a (stable) Lagrangian subspace of $\mathbb{C}^N \otimes \mathrm{H}$, go through the previous construction of $\widetilde{\eta}_F$ and divide by $N$.

## 4. Manifolds with boundary: the perturbed signature operator

Let $M$ be a compact oriented manifold-with-boundary of dimension $2m$. We fix a non-negative boundary defining function $x \in C^\infty(M)$ for $\partial M$ and a Riemannian metric on $M$ which is isometrically a product in an (open) collar neighbourhood $\mathcal{U} \equiv (0,2)_x \times \partial M$ of the boundary.

We let $\Omega_c^*(M)$ denote the compactly-supported differential forms on the interior of $M$.

Let $\Gamma$ be a finitely-generated discrete group. Consider a continuous map $M \to B\Gamma$, with corresponding normal $\Gamma$-cover $M' \to M$. Let $\mathcal{B}^\infty$ be a subalgebra of $C_r^*\Gamma$ as in Section 1.

We consider the following bundles of left modules over $M$:

$$\mathcal{V} = C_r^*(\Gamma) \times_\Gamma M' \qquad \mathcal{V}^\infty = \mathcal{B}^\infty \times_\Gamma M',$$

and denote their restrictions to the boundary $\partial M$ by $\mathcal{V}_0$ and $\mathcal{V}_0^\infty$. We suppose that Assumption 1 of the introduction is satisfied, with $F = \partial M$. Under this assumption, we shall define, following [30, Appendix A], a perturbation of the differential complex on $M$. We shall also give the product structure, near the boundary, of the associated signature operator.

Using Assumption 1 and following the construction of the previous section with $F = \partial M$, we obtain a perturbed differential complex on the boundary $\partial M$; this complex is constructed in terms of a graded regular Hermitian complex $W^*$ which is homotopy equivalent to $\Omega^*(\partial M; \mathcal{V}_0^\infty)$.

**Notation.** We shall denote the complex on the boundary by $C_0^*$; thus $C_0^* = \Omega^*(\partial M; \mathcal{V}_0^\infty) \oplus \widehat{W}^*$. In general, we let the subscript 0 denote something living on $\partial M$.

Equation (2.7), with $F = \partial M$, defines an *invertible* boundary-signature operator

$$\mathcal{D}_C^{\text{sign}}(1)_0 : \Omega^*(\partial M; \mathcal{V}_0^\infty) \oplus \widehat{W}^* \to \Omega^*(\partial M; \mathcal{V}_0^\infty) \oplus \widehat{W}^*.$$

We wish to realize $\mathcal{D}_C^{\text{sign}}(1)_0$ as the boundary component of the signature operator $\mathcal{D}_C^{\text{sign}}(1)$ associated to a *perturbed complex* $(C^*, D_C)$ on $M$.

To this end, consider the Hermitian $\mathcal{B}^\infty$-cochain complex $\Omega_c^*(0,2) \widehat{\otimes} \widehat{W}$. We imitate the results of (1.4), thinking of $\widehat{W}$ as algebraically similar to $\Omega^*(\partial M; \mathcal{V}_0^\infty)$. Thus, we have objects $Q_{\text{alg}}$, $\tau_{\text{alg}}$ and $D_{\text{alg}}$ defined on $\Omega_c^*(0,2) \widehat{\otimes} \widehat{W}$ by the formulas in (1.4), replacing the "$\partial M$" on the right-hand-side of (1.4) by "$\widehat{W}$" and changing the sign of $D_{\widehat{W}}$. Recalling that $\widehat{\alpha} = i^{|\alpha|}\alpha$, we thus have

$$Q_{\text{alg}}(dx \wedge \alpha, 1 \wedge \beta) = \int_0^2 Q_{\widehat{W}}(\alpha(x), \widehat{\beta(x)})dx,$$

$$Q_{\text{alg}}(1 \wedge \alpha, dx \wedge \beta) = \int_0^2 Q_{\widehat{W}}(\widehat{\alpha(x)}, \beta(x))dx,$$

$$\begin{aligned}\tau_{\mathrm{alg}}(1 \wedge \alpha) &= dx \wedge \tau_{\widehat{W}}\widehat{\alpha},\\ \tau_{\mathrm{alg}}(dx \wedge \alpha) &= 1 \wedge i^{-(2m-1)}\tau_{\widehat{W}}\widehat{\alpha},\\ D_{\mathrm{alg}}(1 \wedge \alpha) &= (1 \wedge -D_{\widehat{W}}\alpha) + (dx \wedge \partial_x\widehat{\alpha}),\\ D_{\mathrm{alg}}(dx \wedge \alpha) &= dx \wedge iD_{\widehat{W}}\alpha.\end{aligned} \tag{4.1}$$

One easily checks that the dual to $D_{\mathrm{alg}}$, with respect to $Q_{\mathrm{alg}}$, is $D'_{\mathrm{alg}} = -D_{\mathrm{alg}}$.

Define a new $\mathcal{B}^\infty$-cochain complex $C^*$ by

$$C^* = \Omega^*_c(M;\mathcal{V}^\infty) \oplus (\Omega^*_c(0,2) \,\widehat{\otimes}\, \widehat{W}^*).$$

It inherits objects $Q_C$, $\tau_C$ and $D_C$ from the direct sum decomposition. Consider the open collar $\mathcal{U}$ of $\partial M$, with $\mathcal{U} \cong (0,2) \times \partial M$. The bundle $\mathcal{V}^\infty|_{\mathcal{U}}$ is isomorphic to $(0,2) \times \mathcal{V}^\infty_0$. Using this isomorphism, we can identify the elements of $\Omega^*_c(M;\mathcal{V}^\infty)$ with support in $\mathcal{U}$, with $\Omega^*_c(0,2) \,\widehat{\otimes}\, \Omega^*(\partial M;\mathcal{V}^\infty_0)$.

Now we construct a perturbation of the differential $D_C$ to an "almost" differential on the complex $C^*$. Let $\phi \in C^\infty(0,2)$ be a nonincreasing function satisfying $\phi(x) = 1$ for $0 < x \leq \frac{1}{4}$ and $\phi(x) = 0$ for $\frac{1}{2} \leq x < 2$. Using Assumption 1, we construct a homotopy equivalence $f : \Omega^*(\partial M;\mathcal{V}^\infty_0) \to W^*$ with adjoint $g : W^* \to \Omega^*(\partial M;\mathcal{V}^\infty_0)$, exactly as in (2.1) of Section 2 (but with $F = \partial M$). We define $\widehat{f} : \Omega^*(\partial M;\mathcal{V}^\infty_0) \to \widehat{W}^*$ and $\widehat{g} : \widehat{W}^* \to \Omega^*(\partial M;\mathcal{V}^\infty_0)$ as in Section 2. We extend $\widehat{f}$ and $\widehat{g}$ to act on $\Omega^*_c(0,2) \,\widehat{\otimes}\, \Omega^*(\partial M;\mathcal{V}^\infty_0)$ and $\Omega^*_c(0,2) \,\widehat{\otimes}\, \widehat{W}^*$, respectively, by

$$\widehat{f}(\omega_0 + dx \wedge \omega_1) \;=\; \widehat{f}(\omega_0) - \,i\,dx \wedge \widehat{f}(\omega_1)$$

and

$$\widehat{g}(w_0 + dx \wedge w_1) \;=\; \widehat{g}(w_0) - \,i\,dx \wedge \widehat{g}(w_1).$$

Using the cutoff function $\phi$ and the product structure on $\mathcal{U}$, it makes sense to define an operator on $C^*$ by

$$D_C = \begin{cases} \begin{pmatrix} D_M & \phi\widehat{g} \\ 0 & D_{\mathrm{alg}} \end{pmatrix} & \text{if } * \leq m-1,\\[2ex] \begin{pmatrix} D_M & 0 \\ 0 & D_{\mathrm{alg}} \end{pmatrix} & \text{if } * = m,\\[2ex] \begin{pmatrix} D_M & 0 \\ -\phi\widehat{f} & D_{\mathrm{alg}} \end{pmatrix} & \text{if } * \geq m+1. \end{cases} \tag{4.2}$$

Note that $(D_C)^2 \neq 0$, as $\phi$ is nonconstant. With our conventions, we have

$$D_C + (D_C)^* = D_C - \tau_C D_C \tau_C .$$

The next lemma follows from the same calculations as at the end of Section 1.

**Lemma 4.1.** *Define an operator* $\Theta$ *on* $\Omega_c^*(0, 1/4) \widehat{\otimes} C_0^*$ *by*

$$\Theta((1 \wedge \alpha) + (dx \wedge \beta)) = (1 \wedge -i^{-|\beta|}\beta) + (dx \wedge i^{|\alpha|}\alpha).$$

*Then when restricted to* $\mathcal{U}$, *we can write* $D_C + (D_C)^*$ *in the form* $D_C + (D_C)^* = \Theta(\partial_x + H)$. *Define an isomorphism* $\Phi$ *from* $C_c^\infty(0, 1/4) \otimes C_0^*$ *to the +1-eigenspace* $E^+ \subset \Omega_c^*(0, 1/4) \otimes C_0^*$ *of* $\tau_C$ *by*

$$\Phi(\alpha) = (dx \wedge \alpha) + \tau_C(dx \wedge \alpha).$$

*Then*

$$\Phi^{-1} H\big|_{E^+} \Phi = \mathcal{D}_C^{\mathrm{sign}}(1)_0$$

*and*

$$(D_C + (D_C)^*)^+ = \Theta \cdot \Phi(\partial_x + \mathcal{D}_C^{\mathrm{sign}}(1)_0)\Phi^{-1} .$$

**Notation.** We shall denote the signature operator $D_C + (D_C)^*$ by $\mathcal{D}_C^{\mathrm{sign}}$. The content of the above lemma is that the boundary operator corresponding to $\mathcal{D}_C^{\mathrm{sign}}$ is precisely the odd perturbed signature operator (2.7) introduced in Section 2 for closed manifolds, with $\epsilon = 1$.

## 5. The conic index class

We wish to apply the formalism of Hilsum-Skandalis [16] to show that the higher signatures of manifolds-with-boundary are homotopy-invariant. The approach of [16] is to show that the index of an appropriate Fredholm operator is homotopy-invariant. In particular, to apply the results of [16, Sections 1 and 2], we need to have an operator with a $C_r^*(\Gamma)$-compact resolvent. For this reason, we will replace the product metric on $(0, 2) \times \partial M$ with a conic metric. Recall that $M$ has dimension $2m$.

We keep the notation of Section 4 and assume that $\partial M$ satisfies Assumption 1. We take an (open) collar neighborhood of $\partial M$ which is diffeomorphic to $(0, 2) \times \partial M$. Let $\varphi \in C^\infty(0, 2)$ be a nondecreasing function such that $\varphi(x) = x$ if $x \leq 1/2$ and $\varphi(x) = 1$ if $x \geq 3/2$. Given

$t > 0$, consider a Riemannian metric on $\mathrm{int}(M)$ whose restriction to $(0,2) \times \partial M$ is

$$g_M = t^{-2} dx^2 + \varphi^2(x) g_{\partial M}. \tag{5.1}$$

We have a triple $(Q, D, \tau)$ for $\Omega_c^*(M; \mathcal{V}^\infty)$ as in (1.4), with the difference that $\tau$ is now given on $\Omega_c^*(0,2) \,\widehat{\otimes}\, \Omega^*(\partial M)$ by

$$\tau(1 \wedge \alpha) = dx \wedge t^{-1} \varphi(x)^{2m-1-2|\alpha|} \tau_{\partial M} \widehat{\alpha},$$

$$\tau(dx \wedge \alpha) = 1 \wedge i^{-(2m-1)} t \varphi(x)^{2m-1-2|\alpha|} \tau_{\partial M} \widehat{\alpha}.$$

We define $Q_{\mathrm{alg}}$ and $D_{\mathrm{alg}}$ as in (4.1). We modify $\tau_{\mathrm{alg}}$ of (4.1) to act on $\Omega_c^*(0,2) \,\widehat{\otimes}\, \widehat{W}^*$ by

$$\tau_{\mathrm{alg}}(1 \wedge \alpha) = dx \wedge t^{-1} (\varphi(x)\varphi(2-x))^{2m-1-2|\alpha|} \tau_{\widehat{W}} \widehat{\alpha},$$

$$\tau_{\mathrm{alg}}(dx \wedge \alpha) = 1 \wedge i^{-(2m-1)} t (\varphi(x)\varphi(2-x))^{2m-1-2|\alpha|} \tau_{\widehat{W}} \widehat{\alpha}.$$

That is, we metrically cone off the algebraic complex at both 0 and 2. Then we obtain a direct sum duality operator $\tau_C$ on

$$C^* = \Omega_c^*(M; \mathcal{V}^\infty) \oplus \left( \Omega_c^*(0,2) \,\widehat{\otimes}\, \widehat{W}^* \right),$$

and a corresponding "conic" inner product on $C^*$.

By the definition of the $\mathcal{B}^\infty$-module associated to $\Omega^*(M; \mathcal{V}^\infty)$ (endowed with the conic metric above), the following maps are isometries:

$$\begin{aligned} J'_p : C_c^\infty(0,1/2) \otimes \left( \Omega^{p-1}(\partial M; \mathcal{V}_0^\infty) \oplus \Omega^p(\partial M; \mathcal{V}_0^\infty) \right) \\ \to \Omega_c^p((0,1/2) \times \partial M; \mathcal{V}^\infty) \end{aligned}$$

$$\begin{aligned} J'_p(\phi_{p-1}, \phi_p) = & \left( dx \wedge t^{-1/2} x^{p-1-(2m-1)/2} \phi_{p-1} \right) \\ & + \left( 1 \wedge t^{1/2} x^{p-(2m-1)/2} \phi_p \right). \end{aligned}$$

Similarly, by definition, the following maps are isometries:

$$\widehat{J}_p : C_c^\infty(0,1/2) \otimes \left( \widehat{W}^{p-1} \oplus \widehat{W}^p \right) \to \bigoplus_{q+r=p} \left( \Omega_c^q(0,1/2) \,\widehat{\otimes}\, \widehat{W}^r \right)$$

$$\begin{aligned} \widehat{J}_p(w^{p-1}, w^p) = & \left( dx \wedge t^{-1/2} x^{p-1-(2m-1)/2} w^{p-1} \right) \\ & + \left( 1 \wedge t^{1/2} x^{p-(2m-1)/2} w^p \right). \end{aligned}$$

Put

$$J = \begin{pmatrix} J' & 0 \\ 0 & \widehat{J} \end{pmatrix} .$$

We define an "almost" differential $D_C^{\mathrm{cone}}$ on the conic complex

$$C^* = \Omega_c^*(M; \mathcal{V}^\infty) \oplus \left( \Omega_c^*(0,2) \,\widehat{\otimes}\, \widehat{W}^* \right)$$

by the same formula as in (4.2). Let $C^*_{(2)}$ denote the completion of $C^*$ in the sense of $C_r^*(\Gamma)$-Hilbert modules.

**Lemma 5.1.** *$D_C^{\mathrm{cone}}$ is a regular operator in the sense of [2, Definition 1.1] when acting on $C^*_{(2)}$.*

*Proof.* We give a sketch of the proof and omit some computational details. We will use throughout the general fact that if $T$ is a regular operator and $a$ is an adjointable bounded operator then $T + a$ is a regular operator. This is proven in [39, Lemma 1.9] when $T$ and $a$ are self-adjoint, but one can check that the proof goes through without this additional assumption. In addition, it follows from [15, Lemme 2.1] that a compactly-supported change in the Riemannian metric does not affect the regularity question. Hence, for simplicity, we will only specify our Riemannian metrics up to a compactly-supported perturbation.

We define three new complexes. Put

$$\begin{aligned} C_1^* &= \left(\Omega_c^*(-\infty,\infty) \,\widehat{\otimes}\, \Omega^*(\partial M; \mathcal{V}_0)\right) \oplus \left(\Omega_c^*(-\infty,\infty) \,\widehat{\otimes}\, \widehat{W}^*\right), \\ C_2^* &= \left(\Omega_c^*(0,\infty) \,\widehat{\otimes}\, \Omega^*(\partial M; \mathcal{V}_0)\right) \oplus \left(\Omega_c^*(0,\infty) \,\widehat{\otimes}\, \widehat{W}^*\right), \\ C_3^* &= \Omega_c^*(((-\infty,0] \times \partial M) \cup_{\partial M} M; \mathcal{V}) \oplus \left(\Omega_c^*(-\infty,2) \,\widehat{\otimes}\, \widehat{W}^*\right). \end{aligned}$$

For $1 \le i \le 3$, the differentials $D_{C_i}$ will roughly be of the form (4.2), but the "coupling" between the geometric and algebraic subcomplexes will depend on $i$. Namely, $D_{C_1}$ will be uncoupled on $(-\infty, 0)$ and fully coupled on $(1/4, \infty)$. The differential $D_{C_2}$ will always be fully coupled. The differential $D_{C_3}$ will always be completely uncoupled. The metric on $C_1^*$ will be product-like on $(-\infty, 0)$ and conic on $(1/2, \infty)$. The metric on $C_2^*$ will be fully conic on $(0, \infty)$. The metric on $C_3^*$ will be product-like on $(-\infty, 0)$, and conic on its algebraic part for $(3/2, 2)$. Let $C^*_{i,(2)}$ be the completion of $C_i^*$ in the sense of $C_r^*(\Gamma)$-Hilbert modules. By abuse of notation, we will also let $D_{C_i}$ denote the densely-defined differential on $C^*_{i,(2)}$.

We claim that $D_{C_2}$ is regular, when acting on $C^*_{2,(2)}$. It is not hard to show that $D_{C_2}$ is closable and that $D^*_{C_2}$ is densely defined. It remains to show that $I + D^*_{C_2} D_{C_2}$ is surjective. To see this, we can use separation of variables and adapt the functional calculus of [8, Section 3] to our setting. That is, it is possible to write down an explicit inverse to $I + D^*_{C_2} D_{C_2}$. Namely, as in [8, p. 586], we can span $\Omega^*(0,\infty) \widehat{\otimes} \left(\Omega^*(\partial M; \mathcal{V}_0) \oplus \widehat{W}^*\right)$ by forms of type 1-4, E and O. As in [8, p. 587], the operator $I + D^*_{C_2} D_{C_2}$ acts as the identity on forms of type 2, 4 and O and as $I + D^*_{C_2} D_{C_2} + D_{C_2} D^*_{C_2}$ on forms of type 1, 3 and E. Then using the equivalent of [8, (3.37) and (3.40)], one can write down an explicit inverse to $I + D^*_{C_2} D_{C_2} + D_{C_2} D^*_{C_2}$ when acting on forms of type 1, 3 and E. (The spectrum of the transverse Laplacian is discrete in [8], but in our case the spectrum of the Laplacian on $\Omega^*(\partial M; \mathcal{V}_0) \oplus \widehat{W}^*$ is generally not discrete. Thus one must make the notational change of replacing the eigenvalue sums in [8] by a functional calculus.) This proves the claim.

We claim that $D_{C_3}$ is also regular, when acting on $C^*_{3,(2)}$. To see this, let $d_{C_3}$ denote the differential on $\Omega^*_{(2)}(((-\infty,0] \times \partial M) \cup_{\partial M} M)$. It follows from the analysis in [1] that $d_{C_3}$ is regular. Hence, for any $N \in \mathbb{N}$, $d_{C_3} \otimes \mathrm{Id}_N$ is regular when acting on $\Omega^*_{(2)}(((-\infty,0] \times \partial M) \cup_{\partial M} M) \otimes C^*_r(\Gamma)^N$. Now we can find some $N > 0$ and a projection $p \in C^\infty(M; M_N(C^*_r(\Gamma)))$ so that $\mathcal{V} = \mathrm{Im}(p)$. Taking $p$ to be a product near $\partial M$, we can extend it to a projection $p \in C^\infty(((-\infty,0] \times \partial M) \cup_{\partial M} M; M_N(C^*_r(\Gamma)))$. As $p(d_{C_3} \otimes \mathrm{Id}_N)p + (1-p)(d_{C_3} \otimes \mathrm{Id}_N)(1-p)$ differs from $d_{C_3} \otimes \mathrm{Id}_N$ by an adjointable bounded operator, it follows that $p(d_{C_3} \otimes \mathrm{Id}_N)p + (1 - p)(d_{C_3} \otimes \mathrm{Id}_N)(1-p)$ is regular. Hence, $p(d_{C_3} \otimes \mathrm{Id}_N)p$ is regular. Now $p(d_{C_3} \otimes \mathrm{Id}_N)p$ differs from the differential on $\Omega^*_{(2)}(((-\infty,0] \times \partial M) \cup_{\partial M} M; \mathcal{V})$ by an adjointable bounded operator. Finally, one can show by hand that the differential on $\left(\Omega^*_c(-\infty,2) \widehat{\otimes} \widehat{W}^*\right)_{(2)}$ is regular. Thus $D_{C_3}$ is regular.

We now define a certain unitary operator $U$ from $C^*_{2,(2)} \oplus C^*_{3,(2)}$ to $C^*_{(2)} \oplus C^*_{1,(2)}$. The construction of $U$ is as in [6, Section 3.2], with some obvious changes in notation. We refer to [6, Section 3.2] for the details. Clearly $U(D_{C_2} \oplus D_{C_3})U^{-1}$ is regular. From the method of construction of [6, Section 3.2], one sees that $U(D_{C_2} \oplus D_{C_3})U^{-1}$ differs from $D^{\mathrm{cone}}_C \oplus D_{C_1}$ by an adjointable bounded operator. Hence, $D^{\mathrm{cone}}_C \oplus D_{C_1}$ is regular when acting on $C^*_{(2)} \oplus C^*_{1,(2)}$. In particular, $D^{\mathrm{cone}}_C$ is regular when acting on $C^*_{(2)}$. q.e.d.

The perturbed conic signature operator $\mathcal{D}_C^{\mathrm{sign,cone}} = D_C^{\mathrm{cone}} + (D_C^{\mathrm{cone}})^*$ satisfies

$$\mathcal{D}_C^{\mathrm{sign,cone}} = D_C^{\mathrm{cone}} - \tau D_C^{\mathrm{cone}} \tau.$$

A straightforward calculation shows that on the part of $C^*$ corresponding to $x \in (0, 1/4)$, we have

$$\begin{aligned}(5.2)\qquad & \left(J^{-1}\, \mathcal{D}_C^{\mathrm{sign,cone}}\, J\right)^+ \\ &\quad = \Theta \cdot \Phi \left( t \left( \partial_x + \frac{m - \frac{1}{2} - \mathrm{degree}}{x} \right) + \frac{\mathcal{D}_C^{\mathrm{sign}}(1)_0}{x} \right) \Phi^{-1},\end{aligned}$$

where $\Phi$, $\Theta$ and $\mathcal{D}_C^{\mathrm{sign}}(1)_0$ are as in Lemma 4.1, and "degree" is the $\mathbb{Z}$-grading operator. As $\mathcal{D}_C^{\mathrm{sign}}(1)_0$ is invertible, we can evidently choose a $t > 0$ small enough so that for any $s \in [0, 1]$,

$$(5.3)\qquad \mathrm{Spec}\left( s(m - \frac{1}{2} - \mathrm{degree}) + t^{-1} \mathcal{D}_C^{\mathrm{sign}}(1)_0 \right) \cap (-1, 1) = \emptyset\,.$$

In the rest of this section, we will fix such a number $t$.

**Proposition 5.2.** *For $t > 0$ small enough the triple $\left(C^*_{(2)}, Q_C, D_C^{\mathrm{cone}}\right)$ defines an element of $\mathbf{L}_{nb}(C_r^*(\Gamma))$ in the sense of [16, Définition 1.5].*

*Proof.* We have shown in Lemma 5.1 that $D_C^{\mathrm{cone}}$ is regular. We must show in addition that

1. $(D_C^{\mathrm{cone}})' + D_C^{\mathrm{cone}}$ is $C_r^*(\Gamma)$-bounded.
2. $(D_C^{\mathrm{cone}})^2$ is $C_r^*(\Gamma)$-bounded
3. There are $C_r^*(\Gamma)$-compact operators $S$ and $T$ such that $SD_C^{\mathrm{cone}}$ is $C_r^*(\Gamma)$-bounded, $\mathrm{Im}(T) \subset \mathrm{Dom}(D_C^{\mathrm{cone}})$, $D_C^{\mathrm{cone}} T$ is $C_r^*(\Gamma)$-bounded and $SD_C^{\mathrm{cone}} + D_C^{\mathrm{cone}} T - I$ is $C_r^*(\Gamma)$-compact.

For 1., we have $(D_C^{\mathrm{cone}})' + D_C^{\mathrm{cone}} = 0$. For 2., we have

$$(5.4)\qquad (D_C^{\mathrm{cone}})^2 = \begin{cases} \begin{pmatrix} 0 & \#\, dx\, (\partial_x \phi)\, \widehat{g} \\ 0 & 0 \end{pmatrix} & \text{if } * < m - \frac{1}{2}, \\ \\ \begin{pmatrix} 0 & 0 \\ \#\, dx\, (\partial_x \phi)\, \widehat{f} & 0 \end{pmatrix} & \text{if } * > m - \frac{1}{2}, \end{cases}$$

where # denotes a power of $i$. This is clearly a bounded operator.

Following [5, Section 2], for $f \in L^2(0,\infty)$, put

$$P_1(s)[f](x) = \int_0^x (y/x)^s f(y)dy, \quad s > -1/2$$

$$P_2(s)[f](x) = \int_1^x (y/x)^s f(y)dy, \quad s < 1/2$$

Let $\phi_1(s)$, $\phi_2(s) \in C^\infty(\mathbb{R})$ be such that $\phi_1(s) = 1$ for $s \geq 1$, $\phi_1(s) = 0$ for $s \leq 1/2$, $\phi_2(s) = 1$ for $s \leq -1$ and $\phi_1(s) = 0$ for $s \geq -1/2$. Moreover, put

$$X = \frac{1}{t}\mathcal{D}_C^{\mathrm{sign}}(1)_0 + (m - \frac{1}{2} - \mathrm{degree}).$$

There is a standard interior parametrix for $\mathcal{D}_C^{\mathrm{sign,cone}}$. Furthermore, as in [5, Theorem 2.1],

$$t^{-1}\left(P_1(\phi_1(X)) + P_2(\phi_2(X))\right)$$

is a parametrix for $t(\partial_x + X)$ on $(0, 1/4) \times \partial M$. Finally, if $z = 2 - x$ then with an evident notation, when acting on $\Omega_c^*(7/4, 2) \mathbin{\widehat{\otimes}} \widehat{W}^*$, we can write

$$\begin{aligned} &\left(J^{-1}\,\mathcal{D}_C^{\mathrm{sign,cone}}\,J\right)^+ \\ (5.5)\qquad &= \Theta \cdot \Phi\left(t\left(\partial_z + \frac{m - \frac{1}{2} - \mathrm{degree}}{z}\right) + \frac{\mathcal{D}_{\widehat{W}}^{\mathrm{sign}}}{z}\right)\Phi^{-1}. \end{aligned}$$

We remark that, as in [5, Lemma 5.4], the middle-dimensional vanishing in (2.3) ensures that the conic operator (5.5) exists without a further choice of boundary condition. Put

$$X' = \frac{1}{t}\mathcal{D}_{\widehat{W}}^{\mathrm{sign}} + (m - \frac{1}{2} - \mathrm{degree}).$$

Then a parametrix for $t(\partial_x + X')$ on $\Omega_c^*(7/4, 2) \mathbin{\widehat{\otimes}} \widehat{W}^*$ is given by

$$t^{-1}\left(P_1(\phi_1(X')) + P_2(\phi_2(X'))\right).$$

One constructs an (adjointable) parametrix $G$ for $\mathcal{D}_C^{\mathrm{sign,cone}}$ by patching these three parametrices together, using (5.2) and (5.5). Put $S = T = G(D_C^{\mathrm{cone}})^* G$. The proposition follows. q.e.d.

From [16, Proposition 1.6], the conic signature operator defines a higher index class $\mathrm{Ind}(\mathcal{D}_C^{\mathrm{sign,cone},+}) \in K_0(C_r^*(\Gamma))$ which depends neither on the choice of Riemannian metric on $M$ nor on $t$ (provided that $t$ is sufficiently small for the constructions to make sense). As in [30, Proposition 14], it is also independent of the choices of $\phi$, $W$ and the homotopy equivalence $f$.

## 6. Homotopy invariance of the conic index class

We keep the notation of Section 4. In this section alone, we put $\mathcal{B}^\infty = C_r^*(\Gamma)$. Let $M_1$ and $M_2$ be compact oriented manifolds-with-boundary. Suppose that we have oriented-homotopy equivalences $h_1 : (M_1, \partial M_1) \to (M_2, \partial M_2)$ and $h_2 : (M_2, \partial M_2) \to (M_1, \partial M_1)$ which are homotopy inverses to each other. We can homotope $h_1$ and $h_2$ to assume that they are product-like near the boundaries. That is, for $i \in \{1, 2\}$, put $\partial h_i = h_i\big|_{\partial M_i}$. Then when restricted to the collar neighborhood $\mathcal{U}_i = (0, 2) \times \partial M_i$, we assume that $h_i(x, b_i) = (x, \partial h_i(b_i))$ for $x \in (0, 2)$ and $b_i \in \partial M_i$.

We assume that $\partial M_1$ and $\partial M_2$ satisfy Assumption 1. Let $\widehat{W}_i^*$ be cochain complexes as in (2.3), with corresponding maps

$$\widehat{f}_i : \Omega^*(\partial M_i; (\mathcal{V}_i)_0) \to \widehat{W}_i^*$$

and

$$\widehat{g}_i : \widehat{W}_i^* \to \Omega^*(\partial M_i; (\mathcal{V}_i)_0).$$

We would like to compare $\Omega^*_{(2)}(M_2; \mathcal{V}_2)$ with $\Omega^*_{(2)}(M_1; \mathcal{V}_1)$ using the maps $h_i^*$, but there is the technical problem that $h_i^*$, as originally defined on smooth forms, need not be $L^2$-bounded if $h_i$ is not a submersion. As in [16, p. 90], we modify $\{h_i^*\}_{i=1}^2$ to obtain $L^2$-bounded cochain homotopy equivalences between $\Omega^*_{(2)}(M_2; \mathcal{V}_2)$ and $\Omega^*_{(2)}(M_1; \mathcal{V}_1)$ as follows. From [16, p. 90], for suitably large $N$, there is a submersion $H_i : B^N \times M_i \to M_{3-i}$ such that $H_i(0, m_i) = h_i(m_i)$. Here $B^N$ is an open ball in an euclidean space of dimension $N$. Furthermore, from the construction in [16, p. 90], we may assume that $H_i$ is product-like near $\partial M_i$. Fix $v \in \Omega_c^N(B^N)$ with $\int_{B^N} v = 1$. Define a bounded cochain homotopy equivalence

$$t_i : \Omega^*_{(2)}(M_i; \mathcal{V}_i) \to \Omega^*_{(2)}(M_{3-i}; \mathcal{V}_{3-i})$$

by $t_i(\omega) = \int_{B^N} v \wedge H_{3-i}^*(\omega)$. Let $\partial t_i$ be the analogous map from $\Omega^*_{(2)}(\partial M_i; (\mathcal{V}_i)_0)$ to $\Omega^*_{(2)}(\partial M_{3-i}; (\mathcal{V}_{3-i})_0)$.

As $f_i$ and $g_i$ are homotopy inverses, there are bounded operators

$$A_i : \Omega^*(\partial M_i; (\mathcal{V}_i)_0) \to \Omega^{*-1}(\partial M_i; (\mathcal{V}_i)_0)$$

and

$$B_i : W_i^* \to W_i^{*-1}$$

such that

$$I - g_i \circ f_i = D_{\partial M_i} A_i + A_i D_{\partial M_i}$$

and

$$I - f_i \circ g_i = D_{W_i} B_i + B_i D_{W_i}.$$

As $(f_i)' = g_i$, $(D_{\partial M_i})' = -D_{\partial M_i}$ and $(D_{W_i})' = -D_{W_i}$, we can assume that $(A_i)' = -A_i$ and $(B_i)' = -B_i$. Let $\widehat{B}_i$ denote the obvious extension of $B_i$ to a map from $\widehat{W}_i^*$ to $\widehat{W}_i^{*-1}$. Put

$$C_i^* = \Omega^*_{(2)}(M_i; \mathcal{V}_i) \,\oplus\, \left(\Omega^*_{(2)}(0,2) \,\widehat{\otimes}\, \widehat{W}_i^*\right).$$

**Theorem 6.1.** *The index class* $\operatorname{Ind}(\mathcal{D}^{C,+}_{\text{sign,cone}}) \in K_0(C_r^*(\Gamma))$ *is the same for* $M_1$ *and* $M_2$.

*Proof.* We will show that [16, Lemme 2.4] applies. From [16, pp.90-91], there are bounded operators $y_i : \Omega^*(M_i; \mathcal{V}_i) \to \Omega^{*-1}(M_i; \mathcal{V}_i)$ such that

$$1 - t_i' t_i = D_{M_i} y_i + y_i D_{M_i}.$$

Similarly, there are bounded operators $z_i : \Omega^*(M_i; \mathcal{V}_i) \to \Omega^{*-1}(M_i; \mathcal{V}_i)$ such that

$$1 - t_i t_i' = D_{M_i} z_{3-i} + z_{3-i} D_{M_i}.$$

(This follows from the proof of [16, Proposition 2.5] in the case $\alpha = 0$, with the operator $S$ being the analogue of [16, p. 94 b -7].) We can assume that $y_i$ and $z_i$ are product-like near $\partial M_i$. Let $\partial y_i$ and $\partial z_i$ denote their boundary restrictions.

Define $T_i : C_i^* \to C_{3-i}^*$ by

(6.1)

$$T_i = \begin{cases} \begin{pmatrix} t_i & \phi(x) \,\widehat{\otimes}\, (A_{3-i} \circ \partial t_i \circ \widehat{g}_i) \\ 0 & \operatorname{Id} \widehat{\otimes}\, (\widehat{f}_{3-i} \circ \partial t_i \circ \widehat{g}_i) \end{pmatrix} & \text{if } * < m - \frac{1}{2}, \\[2em] \begin{pmatrix} t_i & 0 \\ -\phi(x) \,\widehat{\otimes}\, (\widehat{f}_{3-i} \circ \partial t_i \circ A_i) & \operatorname{Id} \widehat{\otimes}\, (\widehat{f}_{3-i} \circ \partial t_i \circ \widehat{g}_i) \end{pmatrix} & \text{if } * > m - \frac{1}{2}. \end{cases}$$

Define $Y_i : C_i^* \to C_i^{*-1}$ by

(6.2)

$$Y_i = \begin{cases} \begin{pmatrix} y_i & \phi(x)\,\widehat{\otimes}\,\widehat{\alpha}_i \\ 0 & -\mathrm{Id}\,\widehat{\otimes}\,(\widehat{B}_i + \widehat{f}_i \circ \partial y_i \circ \widehat{g}_i + \widehat{f}_i \circ (\partial t_i)' \circ A_{3-i} \circ \partial t_i \circ \widehat{g}_i) \end{pmatrix} & \text{if } * < m - \frac{1}{2}, \\ \begin{pmatrix} y_i & 0 \\ -\phi(x)\,\widehat{\otimes}\,\widehat{\beta}_i & -\mathrm{Id}\,\widehat{\otimes}\,(\widehat{B}_i + \widehat{f}_i \circ \partial y_i \circ \widehat{g}_i + \widehat{f}_i \circ (\partial t_i)' \circ A_{3-i} \circ \partial t_i \circ \widehat{g}_i) \end{pmatrix} & \text{if } * > m - \frac{1}{2}, \end{cases}$$

where

$$\alpha_i : W_i^* \to \Omega^{*-2}(\partial M_i; (\mathcal{V}_i)_0)$$

is a map which satisfies $D_{\partial M_i}\alpha_i - \alpha_i D_{W_i} = g_i B_i - A_i g_i$, $\widehat{\alpha}_i$ is its extension to $\widehat{W}_i^*$,

$$\beta_i : \Omega^*(\partial M_i; (\mathcal{V}_i)_0) \to W_i^{*-2}$$

is a map which satisfies $D_{W_i}\beta_i - \beta_i D_{\partial M_i} = -B_i f_i + f_i A_i$ and $\widehat{\beta}_i$ is the extension of $\beta_i$ to $\widehat{W}_i^*$. Here $\alpha_i$ and $\beta_i$ exist because of Lemma 2.3, and from the proof of Lemma 2.3, we can take them to be continuous. Define $Z_i : C_i^* \to C_i^{*-1}$ by

(6.3)

$$Z_i = \begin{cases} \begin{pmatrix} z_i & \phi(x)\,\widehat{\otimes}\,\widehat{\alpha}_i \\ 0 & -\mathrm{Id}\,\widehat{\otimes}\,(\widehat{B}_i + \widehat{f}_i \circ \partial z_i \circ \widehat{g}_i + \widehat{f}_i \circ \partial t_{3-i} \circ A_{3-i} \circ (\partial t_{3-i})' \circ \widehat{g}_i) \end{pmatrix} & \text{if } * < m - \frac{1}{2}, \\ \begin{pmatrix} z_i & 0 \\ -\phi(x)\,\widehat{\otimes}\,\widehat{\beta}_i & -\mathrm{Id}\,\widehat{\otimes}\,(\widehat{B}_i + \widehat{f}_i \circ \partial z_i \circ \widehat{g}_i + \widehat{f}_i \circ \partial t_{3-i} \circ A_{3-i} \circ (\partial t_{3-i})' \circ \widehat{g}_i) \end{pmatrix} & \text{if } * > m - \frac{1}{2}. \end{cases}$$

Define $\mathcal{E}_i : C_i^* \to C_i^*$ to be $(-1)^{\mathrm{degree}}$.

We will think of the small positive number $t$ in the metric (5.1) as a free parameter. One can check that the operators $T_i$, $Y_i$ and $Z_i$ are bounded and have norms which are independent of $t$. In order to apply [16, Lemme 2.4], it suffices to show that if $t$ is made small then $(D_{C,i}^{\mathrm{cone}})^2$, $1 - T_i' T_i - D_{C,i}^{\mathrm{cone}} Y_i - Y_i D_{C,i}^{\mathrm{cone}}$ and $1 - T_{3-i} T_{3-i}' - D_{C,i}^{\mathrm{cone}} Z_i - Z_3 D_{C,i}^{\mathrm{cone}}$ can be made arbitrarily small in norm.

The formula for $(D^{\mathrm{cone}}_{C,i})^2$ was given in (5.4). One computes that

$$1 - T_i'T_i - D^{\mathrm{cone}}_{C,i} Y_i - Y_i D^{\mathrm{cone}}_{C,i} = 1 - T_{3-i}T'_{3-i} - D^{\mathrm{cone}}_{C,i} Z_i - Z_i D^{\mathrm{cone}}_{C,i}$$

$$= \begin{cases} \begin{pmatrix} 0 & \# dx\,(\partial_x \phi) \widehat{\otimes} \widehat{\alpha}_i \\ 0 & 0 \end{pmatrix} & \text{if } * < m - \frac{1}{2}, \\ \\ \begin{pmatrix} 0 & 0 \\ \# dx\,(\partial_x \phi) \widehat{\otimes} \widehat{\beta}_i & 0 \end{pmatrix} & \text{if } * > m - \frac{1}{2}, \end{cases}$$

where $\#$ denotes a power of $\sqrt{-1}$. (To do the calculations, it is convenient to note that for fixed $x \in (0,2)$, the restriction of $D^{\mathrm{cone}}_{C,i}$ to $\{x\} \times \partial M_i$ is of the form (2.5) with $F = \partial M_i$ and $\epsilon = \phi(x)$.)

In each case, the result is of the form $dx \otimes K(x)$ where $K(x)$ is a bounded operator which is only nonzero when $x \in \left(\frac{1}{4}, \frac{1}{2}\right)$. From (5.1), the norm of $dx$ in this region is $t$, while the norm of $K(x)$ is independent of $t$. The theorem follows. q.e.d.

## 7. Equality of the conic and APS-index classes

We first define the generalized APS-index. Fix numbers $t_1, t > 0$ and consider a Riemannian metric on $\mathrm{int}(M)$ whose restriction to $(0,2)\times \partial M$ is

$$g_M = t_1^{-2} dx^2 + g_{\partial M}. \tag{7.1}$$

In what follows, we will think of $t_1$ as a parameter associated to $x = 0$ and $t$ as a parameter associated to $x = 2$. Let $\chi \in C^\infty(0,2)$ be a positive function such that $\chi(x) = t_1$ if $x \in (0, 1/2)$ and $\chi(x) = t$ if $x \in (3/2, 2)$. Let us go through the steps to define the signature operator as in Section 5, with the differences that $\tau$ is now given on $\Omega^*_c(0,2) \widehat{\otimes} \Omega^*(\partial M)$ by

$$\tau(1 \wedge \alpha) = dx \wedge t_1^{-1} \tau_{\partial M} \widehat{\alpha},$$

$$\tau(dx \wedge \alpha) = 1 \wedge i^{-(2m-1)} t_1 \tau_{\partial M} \widehat{\alpha}.$$

and $\tau_{\mathrm{alg}}$ is now given on $\Omega^*_c(0,2) \widehat{\otimes} \widehat{W}^*$ by

$$\tau_{\mathrm{alg}}(1 \wedge \alpha) = dx \wedge \chi(x)^{-1} (\varphi(2-x))^{2m-1-2|\alpha|} \tau_{\widehat{W}} \widehat{\alpha},$$

$$\tau_{\mathrm{alg}}(dx \wedge \alpha) = 1 \wedge i^{-(2m-1)} \chi(x) (\varphi(2-x))^{2m-1-2|\alpha|} \tau_{\widehat{W}} \widehat{\alpha}.$$

That is, metrically speaking, we have a product structure near $x = 0$ and a cone on the algebraic complex near $x = 2$. Consider the corresponding perturbed signature operator $\mathcal{D}_C^{\mathrm{sign}}$. It has an invertible boundary operator $\mathcal{D}_C^{\mathrm{sign}}(1)_0$. We define $(H^1, \Pi_>)$ to be the $C_r^*(\Gamma)$-Hilbert module of Sobolev-$H^1$ elements $(\alpha, v)$ of $\Omega^*_{(2)}(M; \mathcal{V}) \oplus \left(\Omega^*_{(2)}[0,2) \widehat{\otimes} \widehat{W}^*\right)$ such that

$$\Pi_>(\alpha|_{\partial M} \oplus v(0)) = 0,$$

where $\Pi_>$ is the spectral projection onto the positive part of the operator $H$ of Lemma 4.1.

The work of Wu [42] shows that

$$\mathcal{D}_C^{\mathrm{sign},+} : (H^1, \Pi_>)^+ \to L^{2,-}$$

is a $C_r^*(\Gamma)$-Fredholm operator which thus defines a higher index class $\mathrm{Ind}(\mathcal{D}_C^{\mathrm{sign,APS},+}) \in K_0(C_r^*(\Gamma))$.

**Remark 7.1.** The higher index class $\mathrm{Ind}(\mathcal{D}_C^{\mathrm{sign,APS},+})$ is independent of the choices of $t_1$, $t$ and $\chi$. In this section we will take $t = t_1$ and $\chi(x) = t$. In Section 10 we will take $t_1 = 1$. In Section 9 we will consider a $b-$metric, in which case we effectively have $\chi(x) = x$ for $x \in (0, 1/2)$.

**Theorem 7.2.** *The following equality holds in* $K_0(C_r^*(\Gamma))$ :

$$\mathrm{Ind}(\mathcal{D}_C^{\mathrm{sign,cone},+}) = \mathrm{Ind}(\mathcal{D}_C^{\mathrm{sign,APS},+}).$$

*Proof.* This follows from the relative index theorem, as given in [7, Theorem 1.14]. We give an outline of the argument. Let $M_1$ be $M$ with the conic metric. Let $M_2$ be $(0,2) \times \partial M$ with a product metric. Let $M_3$ be $M$ with a metric which is a product near $\partial M$. Let $M_4$ be $(0,2) \times \partial M$ with a warped product metric $dx^2 + f^2(x)\, g_{\partial M}$ which is conic near $\{0\} \times \partial M$ and which is a product near $\{2\} \times \partial M$. If $M_i$ has a boundary component with a product-like metric then impose APS boundary conditions on that boundary component. (We always carry along the complex $\Omega^*(0,2) \widehat{\otimes} \widehat{W}^*$, equipped with a "metric" near $x = 2$ which is always conic and a "metric" near $x = 0$ which is of the same type as that of $M_i$. The complex $\Omega^*(0,2) \widehat{\otimes} \widehat{W}^*$ is coupled to $\Omega^*(M; \mathcal{V})$ as before.) Let $\mathrm{Ind}_i \in K_0(C_r^*(\Gamma))$ be the index of the corresponding signature operator on $M_i$. By the method of proof of the relative index theorem [7, Theorem 1.14], we have $\mathrm{Ind}_1 + \mathrm{Ind}_2 = \mathrm{Ind}_3 + \mathrm{Ind}_4$. However, $\mathrm{Ind}_1 = \mathrm{Ind}(\mathcal{D}_C^{\mathrm{sign,cone},+})$ and $\mathrm{Ind}_3 = \mathrm{Ind}(\mathcal{D}_C^{\mathrm{sign,APS},+})$. Using separation of variables, one easily sees that $\mathrm{Ind}_2 = \mathrm{Ind}_4 = 0$. The theorem follows. q.e.d.

**Remark.** One can also give a proof of Theorem 7.2 along the lines of [3, Theorem 1.5]. Namely, as in the proof of [3, Theorem 3.2], one can perturb the conic and APS operators in the interior of $M$ in order to make their images closed. One can then show that the ensuing operators have isomorphic kernels and cokernels, as in the proof of [3, Theorem 1.5].

## 8. The enlarged $b$-calculus

In order to prove a suitable higher index formula we shall now change the perturbed signature operator introduced above so as to get an element of an appropriate $b$-calculus. Since the complex $C^*$ involves the additional piece $\Omega_c^*(0,2) \widehat{\otimes} \widehat{W}^*$, this step is slightly more complicated than in [34] and [23].

Thus, let $M$ be a manifold with boundary $\partial M$. We denote by $u \in C^\infty(M)$ a boundary defining function and we fix a Riemannian metric $g$ which is product-like in a collar neighbourhood $\mathcal{U}$ of $\partial M$: $g|_{\mathcal{U}} = du^2 + g_{\partial M}$. As in [33], we add to the manifold-with-boundary $M$ a cylindrical end $(-\infty, 0]_u \times \partial M$. Similarly, we add the half-line $(-\infty, 0]_{u'}$ to the interval $[0,2)$ appearing in the definition of $C^*$. The change of variables $u = \log x$, $u' = \log x'$ compactifies these two manifolds and brings us into the framework of the $b$-geometry of [33]. The original manifold is contained in the $b$-manifold so obtained, and the same is true for the interval. We make an abuse of notation and call $(M, g)$ and $[0,2)$ the $b$-manifolds obtained above. Notice that $g$ is now a $b$-metric which is product-like near the boundary: $g = dx^2/x^2 + g_{\partial M}$, for $0 \le x \le 1/2$.

We now define an appropriate *enlarged* $\mathcal{B}^\infty$-$b$-calculus. Besides the usual $\mathcal{B}^\infty$-$b$-calculus on $M$ and the usual $\mathcal{B}^\infty$-$b$-calculus on $[0,2)$, with values in $\widehat{W}^*$, this enlarged version will have to involve operators of the following types:

$$
\begin{aligned}
P &: {}^b\Omega^*([0,2); \widehat{W}^*) \to {}^b\Omega^*(M; \mathcal{V}^\infty) \\
Q &: {}^b\Omega^*(M; \mathcal{V}^\infty) \to {}^b\Omega^*([0,2); \widehat{W}^*).
\end{aligned}
\tag{8.1}
$$

For the definition of the $\mathcal{B}^\infty$-$b$-calculus we refer the reader to [22, Sect.

12] and [24, Appendix].

We shall only need operators of order $-\infty$, i.e., operators which are defined by *smooth* Schwartz kernels on suitable blown-up spaces. The blown-up space (see [33, Sect. 4.2]) corresponding to $P$ in (8.1) is

$$[M \times [0,2)_{x'}\,;\, S] \qquad S = \{x = x' = 0\}.$$

It comes with a blow-down map $\beta : [M \times [0,2); S] \to M \times [0,2)$. There are three boundary hypersurfaces in the manifold-with-corners $[M \times [0,2); S]$; the front face $\mathrm{bf} = \beta^{-1}(S) \equiv S_+N(S)$ and the left and right boundaries

$$\mathrm{lb} = \overline{\beta^{-1}(\partial M \times [0,2))} \qquad \mathrm{rb} = \overline{\beta^{-1}(M \times \{0\})}.$$

We shall require the Schwartz kernel $K_P$ of $P$ to lift to $[M \times [0,2); S]$ as a smooth $C^\infty$ section of the bundle $\beta^*\mathcal{K}$ with

$$\mathcal{K}_{(p,x')} \equiv \mathrm{Hom}_{\mathcal{B}^\infty}(({}^b\Lambda^*[0,2)\widehat{\otimes}\widehat{W})_{x'}; ({}^b\Lambda^*(M) \otimes \mathcal{V}^\infty)_p).$$

The kernel $\beta^* K_P$ is also required to vanish to infinite order at lb and rb. We shall usually employ projective coordinates $(y, s, x')$ in a neighbourhood of $\mathrm{bf} \subset [M \times [0,2); S]$, with $s = x/x'$; see [33, Ch. 4]. Notice that the front face is diffeomorphic to $[-1,1] \times \partial M$. Restriction to the front face followed by Mellin transform along $[-1,1]$ defines the indicial family of $P$ as an entire family of $\mathcal{B}^\infty$-linear maps $I(P,\lambda) : \mathbb{C}^2 \otimes \widehat{W}^* \to \Omega^*(\partial M, \mathcal{V}_0^\infty)$, $\lambda \in \mathbb{C}$, with $\mathbb{C}^2 \cong {}^b\Lambda[0,2)|_{x'=0}$; see [33, Sect. 5.2, formula (5.13)]. Using the Paley-Wiener theorem this construction can be reversed (for smoothing operators) as in [33, Theorem 5.1 and Lemma 5.4].

Operators like $Q$ in (8.1) are defined in a similar way. They are integral operators with a Schwartz kernel on $[0,2) \times M$ which lifts to become smooth on the blown-up space $[[0,2)_{x'} \times M\,;\, \{x = x' = 0\}]$ and vanishes to infinite order at the left and right boundaries.

**Example.** We shall now exhibit two particular operators as in (8.1). These are $b$-versions of the operators $\phi\widehat{f}$ and $\phi\widehat{g}$ already considered in the previous section.

Let $\phi \in C^\infty[0,\infty)$ be a nonincreasing function which is equal to 1 on $[0,1/4)$ and equal to 0 on $[1/2,\infty)$. Let $\mathcal{U} \equiv [0,2) \times \partial M$ be a collar neighbourhood of $\partial M$. As usual, we identify ${}^b\Lambda^{j+1}_{\partial M}(M)$ with $\Lambda^{j+1}(\partial M) \oplus (\Lambda^j(\partial M) \wedge dx/x)$. As already explained, using this identification and a trivialization $\mathcal{V}^\infty|_{\mathcal{U}} \cong [0,2) \times \mathcal{V}_0^\infty$, we can write each element in ${}^b\Omega^{j+1}(\mathcal{U},\mathcal{V}^\infty)$ as $k^0\cdot\gamma^{j+1}(x)+(k^1\wedge\gamma^j)$, with $k^\ell\in{}^b\Omega^\ell[0,2)$ and $\gamma^j\in\Omega^j(\partial M,\mathcal{V}_0^\infty)$. Now let $\rho\in C_c^\infty(\mathbb{R})$ be an even nonnegative test function such that $\int_{\mathbb{R}}\rho(t)dt=1$. Consider, as in [34, Lemma 9], the function $\rho_\delta$ defined by $\rho_\delta(\lambda)=\delta^{-1}\cdot\rho(\lambda/\delta)$ for $\delta>0$. For $j<m$, consider an element of $({}^b\Omega^*[0,2)\widehat{\otimes}\widehat{W})^j$ of the form $\omega=(h^0\otimes w^j)+(h^1\otimes w^{j-1})$. Using the above identification, we define an operator

$$\widehat{g}_b:({}^b\Omega[0,2)\widehat{\otimes}\widehat{W})^j\to{}^b\Omega^{j+1}(M;\mathcal{V}^\infty)$$

by

$$\begin{aligned}\widehat{g}_b(\omega)=\ &\phi\cdot\int_0^\infty\rho_\delta(\log s)\phi(x/s)h^0(x/s)\frac{ds}{s}\wedge\widehat{g}(w^j)\\&-i\,\phi\cdot\int_0^\infty\rho_\delta(\log s)\phi(x/s)h^1(x/s)\frac{ds}{s}\wedge\widehat{g}(w^{j-1})\end{aligned}$$

with $\widehat{g}:\widehat{W}^*\to\Omega^*(\partial M;\mathcal{V}_0^\infty)$ as in Section 2.

Similarly, if $j>m$, we are going to define a $\mathcal{B}^\infty$-$b$-smoothing operator

$$\widehat{f}_b:{}^b\Omega^j(M;\mathcal{V}^\infty)\to({}^b\Omega^*[0,2)\widehat{\otimes}\widehat{W}^*)^{j+1}$$

as follows: If $\omega\in{}^b\Omega^j(M;\mathcal{V}^\infty)$ has $\omega|_{\mathcal{U}}=h^0\omega^j+(h^1\wedge\omega^{j-1})$ with $h^\ell\in{}^b\Omega^\ell[0,2)$ and $\omega^{j-\ell}\in\Omega^{j-\ell}(\partial M;\mathcal{V}_0^\infty)$, define

$$\begin{aligned}\widehat{f}_b(\omega)=\ &\phi\cdot\int_0^\infty\rho_\delta(\log s)\phi(x/s)h^0(x/s)\frac{ds}{s}\otimes\widehat{f}(\omega^j)\\&-i\,\phi\cdot\int_0^\infty\rho_\delta(\log s)\phi(x/s)h^1(x/s)\frac{ds}{s}\otimes\widehat{f}(\omega^{j-1})\end{aligned}$$

where $\widehat{f}:\Omega^*(\partial M;\mathcal{V}_0^\infty)\to\widehat{W}^*$ is as in Section 2.

**Remark.** The operators $\widehat{g}_b$ and $\widehat{f}_b$ also depend on the choice of $\rho_\delta$. They are $\mathcal{B}^\infty$-$b$-smoothing operators of the type described above, namely as in (8.1).

**Definition 8.1.** The enlarged (small) $\mathcal{B}^\infty$-$b$-calculus of order $m$, denoted $\widehat{\Psi}^m_{b,\mathcal{B}^\infty}$, is the space of operators

$$P=\begin{pmatrix}P_{11}&P_{12}\\P_{21}&P_{22}\end{pmatrix}\quad\text{acting on}\quad{}^b\Omega^*(M;\mathcal{V}^\infty)\oplus({}^b\Omega^*[0,2)\widehat{\otimes}\widehat{W}^*)$$

with $P_{11}$ and $P_{22}$ being $\mathcal{B}^\infty$-$b$-pseudodifferential operators as in [22, Sect. 12] and $P_{12}$, $P_{21}$ as in (8.1) above.

By construction, the Schwartz kernels of $P$ will vanish to infinite order at the right and left boundaries. Dropping this last condition but assuming conormal bounds there, one can define, as in [33, Sect. 5.16], [22, Sect. 12], the calculus with bounds $\widehat{\Psi}^{m,\beta}_{b,\mathcal{B}^\infty}$ for $\beta > 0$. Following the previous sections, we shall now consider a new differential $D_C$ on the perturbed complex $C^* = {}^b\Omega^*(M;\mathcal{V}^\infty) \oplus ({}^b\Omega^*[0,2)\widehat{\otimes}\widehat{W})$; on the degree $j$-subspace we put

$$D_C \equiv \begin{pmatrix} D_M & 0 \\ 0 & D_{\mathrm{alg}} \end{pmatrix} + \begin{cases} \begin{pmatrix} 0 & \widehat{g}_b \\ 0 & 0 \end{pmatrix} & \text{if } j < m \\ \begin{pmatrix} 0 & 0 \\ -\widehat{f}_b & 0 \end{pmatrix} & \text{if } j > m \end{cases} \tag{8.2}$$

where $\widehat{g}_b$ and $\widehat{f}_b$ are defined as above. By construction, $D_C \in \widehat{\Psi}^1_{b,\mathcal{B}^\infty}$. Notice that $(D_C)^2$ is nonzero.

Let $\mathcal{D}^{\mathrm{sign},b}_C = D_C + (D_C)^*$ be the $b$-signature operator associated to the $b$-complex $(C^*, D_C)$. Then $\mathcal{D}^{\mathrm{sign},b}_C = D_C - \tau_C D_C \tau_C$ is odd with respect to the $\mathbb{Z}_2$-grading defined by the Hodge duality operator $\tau_C$ on $C^*$ (see Section 4). We shall call $\mathcal{D}^{\mathrm{sign},b}_C$ the *perturbed* $b$-signature operator. More explicitly, on forms of degree $m$, $\mathcal{D}^{\mathrm{sign},b}_C$ is equal to

$$\begin{pmatrix} D_M - \tau_M D_M \tau_M & 0 \\ 0 & D_{\mathrm{alg}} - \tau_{\mathrm{alg}} D_{\mathrm{alg}} \tau_{\mathrm{alg}} \end{pmatrix},$$

whereas on forms of degree $j \neq m$, using Lemma 2.5,

$$\mathcal{D}^{\mathrm{sign},b}_C = \begin{pmatrix} D_M - \tau_M D_M \tau_M & 0 \\ 0 & D_{\mathrm{alg}} - \tau_{\mathrm{alg}} D_{\mathrm{alg}} \tau_{\mathrm{alg}} \end{pmatrix}$$

$$+ \begin{cases} \begin{pmatrix} 0 & \widehat{g}_b \\ \tau_{\widehat{W}} \widehat{f}_b \tau_{\partial M} & 0 \end{pmatrix} & \text{if } j < m \\ \begin{pmatrix} 0 & -\tau_{\partial M} \widehat{g}_b \tau_{\widehat{W}} \\ -\widehat{f}_b & 0 \end{pmatrix} & \text{if } j > m \end{cases}.$$

The perturbed $b$-signature operator $\mathcal{D}^{\mathrm{sign},b}_C$ is an element of the enlarged $\mathcal{B}^\infty$-$b$-calculus defined above.

**Notation.** The perturbed signature operator depends both on the choice of the functions $\rho$, $\phi$ and on the real number $\delta$. For brevity, we shall write

$$\mathcal{D}_C^{\mathrm{sign},b} = \begin{pmatrix} D_M - \tau_M D_M \tau_M & 0 \\ 0 & D_{\mathrm{alg}} - \tau_{\mathrm{alg}} D_{\mathrm{alg}} \tau_{\mathrm{alg}} \end{pmatrix} + \begin{pmatrix} 0 & S(\delta) \\ T(\delta) & 0 \end{pmatrix}.$$

Notice that by employing the bundles $\mathcal{V}$, $\mathcal{V}_0$ and by requiring the maps in the definition of $P_{12}$, $P_{21}$ to be $C_r^*(\Gamma)$-linear and the operators $P_{11}$, $P_{22}$ to be in $\Psi^m_{b,C_r^*(\Gamma)}$ (see [22, Sect. 11]), one can define in a similar way an enlarged $C_r^*(\Gamma)$-$b$-calculus, denoted $\widehat{\Psi}^*_{b,C_r^*(\Gamma)}$.

## 9. The $b$-index class

We shall now show that under the present assumptions, the operator $\mathcal{D}_C^{\mathrm{sign},b}$ defines an index class $\mathrm{Ind}(\mathcal{D}_C^{\mathrm{sign},b,+}) \in K_0(\mathcal{B}^\infty)$.

We first construct a parametrix for $\mathcal{D}_C^{\mathrm{sign},b}$ with $C_r^*(\Gamma)$-compact remainder.

**Remark.** It does not seem to be mentioned in the literature that the definition of a $C_r^*(\Gamma)$-compact operator in [36, Section 2] differs from that in [21, Definition 4]. The $C_r^*(\Gamma)$-compact operators of [36, Section 2] form a left-ideal in the $C_r^*(\Gamma)$-bounded operators, whereas the $C_r^*(\Gamma)$-compact operators of [21, Definition 4] form a 2-sided ideal. In fact, the $C_r^*(\Gamma)$-compact operators of [21, Definition 4] consist of the adjointable operators $K$ for which both $K$ and $K^*$ are compact in the sense of [36, Section 2]. In [36, Theorem 2.4] it is implicitly assumed that the $C_r^*(\Gamma)$-compact operators, as defined in [36, Section 2], form a 2-sided ideal. (The mistake is in the sentence "Without loss of generality...") Hence, there is a gap in the proof of [36, Theorem 2.4]. However, it is easy to correct this by using the definition of $C_r^*(\Gamma)$-compact operators from [21, Definition 4] throughout the paper [36]. Then the results of [36, Section 2] go through automatically and one can check that the claims of [36, Section 3] remain valid. This definition of $C_r^*(\Gamma)$-compact operators should also be used in [22]–[25].

The boundary behaviour of an element in the enlarged $b$-calculus can be analyzed by looking separately at the $b$-boundary $x = 0 = x'$ and the boundary $x' = 2$. Let $\mathcal{U} \cong [0, 7/4) \times \partial M$ be a collar neighbourhhod of $\partial M$. We shall consider the restriction of our operators to ${}^b\Omega^*(\mathcal{U}; \mathcal{V}|_{\mathcal{U}}) \oplus {}^b\Omega^*([0, 7/4)) \widehat{\otimes} \widehat{W}^*$ and to $\Omega^*([3/2, 2)) \widehat{\otimes} \widehat{W}^*$ separately. In this section, we consider a "conic metric" at $x' = 2$ on $\Omega^*([3/2, 2)) \widehat{\otimes} \widehat{W}^*$ and we put a $b-$metric at $x' = 0$. That is, we will use a formula as in Section 7 to define $\tau_{\mathrm{alg}}$, but now with a function $\chi \in C^\infty(0,2)$ such that $\chi(x) = x$ for $x \in (0,1/2]$ and $\chi(x) = t$ for $x \in (3/2,2)$.

We shall construct a parametrix by patching a $b$-boundary parametrix and a parametrix for the conic-signature operator on $\Omega^*([3/2,2))\widehat{\otimes}\widehat{W}^*$. We pass to the implementation of this program, concentrating first and foremost on the $b$-parametrix near the $b$-boundary.

First, notice that elements in the enlarged $b$-calculus form an algebra. The proof of the appropriate composition formulae proceeds as in [33]. Next, we recall that each $P \in \widehat{\Psi}^m_{b,C^*_r(\Gamma)}$ has a well defined indicial family

$$I(P,\lambda) : \Omega^*(\partial M;\mathcal{V}_0) \oplus \widehat{W} \to \Omega^*(\partial M;\mathcal{V}_0) \oplus \widehat{W},$$

where we have implicitly used suitable identifications in a neighbourhood of the $b$-boundary. If $P_{11}$ and $P_{22}$ are $b$-elliptic in the usual sense (i.e., symbolically) and if $I(P,\lambda)$ is uniformly invertible for each $\lambda \in \mathbb{R}$ then, by inverse Mellin transform, we can construct a parametrix $G \in \widehat{\Psi}^{-m,\beta}_{b,C^*_r(\Gamma)}$, $\beta > 0$, with remainders $R_1, R_2 \in \widehat{\Psi}^{-\infty,\beta}_{b,C^*_r(\Gamma)}$ having vanishing indicial family or, equivalently, a vanishing restriction to the front face. (For this construction; see [33, Proposition 5.28] and [22, Theorem 11.1].)

These remainders define bounded maps between $C^*_r(\Gamma)$-Hilbert modules, from

$$L^2_b(M;{}^b\Lambda^*M \otimes \mathcal{V}) \oplus L^2_{b,\mathrm{comp}}([0,7/4);{}^b\Lambda^*[0,7/4)\widehat{\otimes}\widehat{W}^*)$$

to

$$x^\beta H^1_b(M;{}^b\Lambda^*M \otimes \mathcal{V}) \oplus (x')^\beta H^1_{b,\mathrm{loc}}([0,7/4);{}^b\Lambda^*[0,7/4)\widehat{\otimes}\widehat{W}^*).$$

(Recall that in order to have a *compact* Sobolev embedding in the framework of $b$-Sobolev spaces it is necessary to have a gain both in the order of the Sobolev space and in the weighting; see [22, Lemma 11.2]. For the definition of the $C^*_r(\Gamma)$-Hermitian scalar product on $L^2_b(M,{}^b\Lambda^*M \otimes \mathcal{V})$ we refer to [22, Sect. 11]).

Going back to our perturbed $b$-signature operator $\mathcal{D}^{\mathrm{sign},b}_C$, we can compute its indicial family as follows. First, using Lemma 4.1 and a harmless abuse of notation, we can fix the identifications

$$
\begin{aligned}
\Psi^{+} \equiv \Phi^{-1} &: \left({}^{b}\Omega^{*}(M;\mathcal{V})|_{\partial M} \oplus ({}^{b}\Omega^{*}[0,2)\widehat{\otimes}\widehat{W}^{*})|_{x'=0}\right)^{+} \\
&\to \Omega^{*}(\partial M;\mathcal{V}_0) \oplus \widehat{W}^{*} \\
\Psi^{-} \equiv \Phi^{-1}\circ\Theta &: \left({}^{b}\Omega^{*}(M;\mathcal{V})|_{\partial M} \oplus ({}^{b}\Omega^{*}[0,2)\widehat{\otimes}\widehat{W}^{*})|_{x'=0}\right)^{-} \\
&\to \Omega^{*}(\partial M;\mathcal{V}_0) \oplus \widehat{W}^{*} \; .
\end{aligned}
$$

We thus obtain an isomorphism

$$
\begin{aligned}
\Psi &= \Psi^{+}\oplus\Psi^{-} : {}^{b}\Omega^{*}(M;\mathcal{V})|_{\partial M} \oplus ({}^{b}\Omega^{*}[0,2)\widehat{\otimes}\widehat{W}^{*})|_{x'=0} \\
&\to (\Omega(\partial M;\mathcal{V}_0)\oplus\widehat{W})\otimes\mathbb{C}^2 \; .
\end{aligned}
$$

Using *this* isomorphism, the matrices

$$
\gamma = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
$$

and proceeding as in [34, Sect. 1], we obtain the indicial family

$$
I(\mathcal{D}_C^{\mathrm{sign},b},\lambda) : (\Omega(\partial M;\mathcal{V}_0)\oplus\widehat{W})\otimes\mathbb{C}^2 \to (\Omega(\partial M;\mathcal{V}_0)\oplus\widehat{W})\otimes\mathbb{C}^2
$$

to be

$$
\begin{aligned}
I(\mathcal{D}_C^{\mathrm{sign},b},\lambda) = &\begin{pmatrix} \gamma\lambda + \sigma\mathcal{D}_{\mathrm{sign},\partial M} & 0 \\ 0 & \gamma\lambda + \sigma\mathcal{D}^{\mathrm{sign}}_{\widehat{W}} \end{pmatrix} \\
&+ (-i)\cdot \begin{cases} \begin{pmatrix} 0 & \sigma\widehat{\rho}_\delta(\lambda)\tau_{\partial M}\widehat{g} \\ -\sigma\widehat{\rho}_\delta(\lambda)\widehat{f}\tau_{\partial M} & 0 \end{pmatrix} & \text{if } j<m \\ \begin{pmatrix} 0 & \sigma\widehat{\rho}_\delta(\lambda)\widehat{g}\tau_{\widehat{W}} \\ -\sigma\widehat{\rho}_\delta(\lambda)\tau_{\widehat{W}}\widehat{f} & 0 \end{pmatrix} & \text{if } j>m \end{cases} .
\end{aligned} \tag{9.1}
$$

Following [34, Lemma 9], we shall now show that $I(\mathcal{D}_C^{\mathrm{sign},b},\lambda)$ is invertible for all $\lambda\in\mathbb{R}$. Since $\widehat{\rho}_\delta(\lambda)\in\mathbb{R}$ for $\lambda\in\mathbb{R}$ we see that $I(\mathcal{D}_C^{\mathrm{sign},b},\lambda) = \gamma\lambda + A(\lambda)$ with $A(\lambda)$ self-adjoint. From the definition of $\gamma$ we only need to check the invertibility of $I(\mathcal{D}_C^{\mathrm{sign},b},\lambda)$ at $\lambda=0$. Since $\widehat{\rho}_\delta(0)=1$, we obtain immediately that $I(\mathcal{D}_C^{\mathrm{sign},b},0) = \mathcal{D}_C^{\mathrm{sign}}(1)_0$ and the invertibility thus follows from the very definition of the perturbed differential complex given in Section 2. In summary, the perturbed signature operator $\mathcal{D}_C^{\mathrm{sign},b}\in\widehat{\Psi}^1_{b,C_r^*(\Gamma)}$ has an invertible indicial family for $\lambda\in\mathbb{R}$.

We can therefore apply the above $b$-parametrix construction to $P = \mathcal{D}_C^{\mathrm{sign},b}$, obtaining a $G \in \widehat{\Psi}^{-m,\beta}_{b,C_r^*(\Gamma)}$. If we now patch this $b$-parametrix $G$ with a parametrix for the signature operator on $\Omega^*([3/2,2))\widehat{\otimes}\widehat{W}^*$, we obtain a parametrix $\widetilde{G}$ for $\mathcal{D}_C^{\mathrm{sign},b}$ with $C_r^*(\Gamma)$-compact remainders. We omit the standard details.

Thanks to the work of Mischenko and Fomenko, we infer that the operator $\mathcal{D}_C^{\mathrm{sign},b}$ has a well defined index class in $K_0(C_r^*(\Gamma))$. Proceeding as in [26, Section VI] and [24, Appendix], this index class can be sharpened into a $K_0(\mathcal{B}^\infty)$-class, using an appropriate $\mathcal{B}^\infty$-$b$-Mischenko-Fomenko decomposition theorem [22, Sect. 15].

**Proposition 9.1.** *The index class* $\mathrm{Ind}(\mathcal{D}_C^{\mathrm{sign},b,+}) \in K_0(\mathcal{B}^\infty)$ *only depends on the signature operator on* $M$ *and not on the choice of the finitely generated Hermitian complex* $W^*$*, the homotopy equivalence* $f$ *or the functions* $\rho_\delta$ *or* $\phi$*.*

*Proof.* The independence of the choices of $\rho_\delta$ and $\phi$ is proved as in [23, Prop. 6.4]. The fact that different choices of $W^*$ and $f$ do not affect the index class is proved using the idea of the proof of [30, Proposition 15]. q.e.d.

**Definition 9.2.** We shall call the higher index class $\mathrm{Ind}(\mathcal{D}_C^{\mathrm{sign},b,+}) \in K_0(\mathcal{B}^\infty)$ constructed above the $b$-signature-index class associated to a manifold-with-boundary $M$ satisfying Assumption 1.

The $b$-signature-index class depends neither on the choice of Riemannian metric nor on the choice of the Hodge duality operator $\tau_M$, as different choices give operators that can be connected by a suitable 1-parameter family of operators. In Section 11 we shall compute the Chern character of $\mathrm{Ind}(\mathcal{D}_C^{\mathrm{sign},b,+}) \in K_0(\mathcal{B}^\infty)$, with values in the noncommutative de Rham homology of $\mathcal{B}^\infty$, in terms of the usual local integral and the higher eta invariant defined in Section 2.

## 10. Equality of the APS and $b$-index classes

We recall Remark 7.1 and that we assume $t_1 = 1$ in the definition of the index class $\mathrm{Ind}(\mathcal{D}_C^{\mathrm{sign},\mathrm{APS},+})$.

**Theorem 10.1.** *The following equality holds in* $K_0(C_r^*(\Gamma))$ :

$$\mathrm{Ind}(\mathcal{D}_C^{\mathrm{sign},\mathrm{APS},+}) = \mathrm{Ind}(\mathcal{D}_C^{\mathrm{sign},b,+}).$$

*Proof.* The signature operator $\mathcal{D}_C^{\mathrm{sign},+}$, associated to the odd operator $\mathcal{D}_C^{\mathrm{sign}}(1)$ of Section 4, induces an operator, denoted $\mathcal{D}_C^{\mathrm{sign,cyl},+}$, acting on the same $C_r^*(\Gamma)$–Hilbert modules as $\mathcal{D}_C^{\mathrm{sign},b,+}$. This amounts to adding a cylindrical end to the manifold with boundary $M$ of Section 4 and a half-line to the interval $[0,2)$. The extended operator $\mathcal{D}_C^{\mathrm{sign,cyl},+}$ is *not* in the $b$-calculus; the role of the function $\rho_\delta$ in Section 8 was precisely that of providing a perturbation belonging to the $b$-calculus. This will be crucial in order to prove the higher APS-index *formula* in Section 11. However as far as *Fredholm properties* are concerned, the operator $\mathcal{D}_C^{\mathrm{sign,cyl},+}$, i.e., the operator $\mathcal{D}_C^{\mathrm{sign},+}$ of Section 2 extended with cut-off functions to the cylindrical ends, can be proven directly to be $C_r^*(\Gamma)$-Fredholm. In order to show this fact we shall still employ ideas from [33, Sect. 5.4, Sect. 5.5]. The Schwartz kernel of the perturbation in $\mathcal{D}_C^{\mathrm{sign,cyl}}$, constructed from $f$ and $g$ in Section 1, lifts to a distribution on the $b$-stretched product which will be smooth outside the $b$-diagonal and vanishing to infinite order at the left and right boundaries. The operator $\mathcal{D}_C^{\mathrm{sign,cyl}}$ still admits an indicial family:

$$I(\mathcal{D}_C^{\mathrm{sign,cyl}},\lambda) = \begin{pmatrix} \gamma\lambda + \sigma\mathcal{D}_{\mathrm{sign}}(\partial M) & 0 \\ 0 & \gamma\lambda + \sigma\mathcal{D}^{\mathrm{sign}}_{\widehat{W}} \end{pmatrix} + (-i)\begin{cases} \begin{pmatrix} 0 & \sigma\tau_{\partial M}\widehat{g} \\ -\sigma\widehat{f}\tau_{\partial M} & 0 \end{pmatrix} & \text{if } j < m \\ \begin{pmatrix} 0 & \sigma\widehat{g}\tau_{\widehat{W}} \\ -\sigma\tau_{\widehat{W}}\widehat{f} & 0 \end{pmatrix} & \text{if } j > m \end{cases}.$$

Let us prove that there exists a bounded operator $G : H_b^{-1} \oplus H_b^{-1} \to L_b^2 \oplus L_b^2$ and a positive real number $s$ such that $G \circ \mathcal{D}_C^{\mathrm{sign,cyl}}(1) - \mathrm{Id}$ is bounded from $L_b^2 \oplus L_b^2$ into $x^s H_b^1 \oplus (x')^s H_b^1$; the two $L^2$ spaces here refer to $M$ and $[0,2)$, respectively. $G$ will then be an inverse modulo $C_r^*(\Gamma)$-compacts; this will prove the $C_r^*(\Gamma)$-Fredholm-property for $\mathcal{D}_C^{\mathrm{sign,cyl}}$. $G$ will be obtained as in Section 5 by patching a "$b$-parametrix" and a parametrix for the (conic-)signature operator on $\Omega([3/2,2))\widehat{\otimes}\widehat{W}^*$. We shall only concentrate on the "$b$-parametrix". Working symbolically first, we can find a $G_\sigma \in \widehat{\Psi}^{-1}_{b,\mathcal{B}^\infty}$ such that $G_\sigma \circ \mathcal{D}_C^{\mathrm{sign,cyl}} = \mathrm{Id} + R$, with $R$ sending $H_b^{p-1} \oplus H_b^{p-1}$ into $H_b^p \oplus H_b^p$ for any $p \in \mathbb{Z}$ and with a Schwartz kernel which lifts to the stretched product as a distribution, smooth outside $\Delta_b$ and vanishing to infinite order at the left and right boundaries.

If $N >> 1$ then there exists $A >> 1$ such that

$$(\sum_{k\geq 0}^{A}(-1)^k R^k)\circ G_\sigma \circ \mathcal{D}_C^{\text{sign,cyl}} = \mathrm{Id} + K_A,$$

with the Schwartz kernel of $K_A$ lifting to a $C^N$-function on the $b$-stretched product, smooth near lb and rb and vanishing to infinite order there. If $N$ is big enough then, proceeding as in [33] (see Section 5.13), we can find $T : H_b^{-1}\oplus H_b^{-1} \to L_b^2 \oplus L_b^2$ such that the lift of the Schwartz kernel of $T$ is $C^k$ in the interior, has conormal bounds at the right and left boundaries and satisfies

$$I(T,\lambda)\circ I(\mathcal{D}_C^{\text{sign,cyl}},\lambda) = -I(K_A,\lambda),$$

we simply take the inverse Mellin transform of

$$-I(K_A,\lambda)\circ (I(\mathcal{D}_C^{\text{sign,cyl}}),\lambda)^{-1}.$$

The operator $G = T + \sum_{k\geq 0}^{A}(-1)^k R^k \circ G_\sigma$ provides a left $b$-parametrix with an (adjointable) remainder which continuously maps $L_b^2 \oplus L_b^2$ into $x^s H_b^1 \oplus (x')^s H_b^1$ for a suitable $s > 0$.

Given $t \in [0,1]$, each $F(t) = t\mathcal{D}_C^{\text{sign,cyl},+} + (1-t)\mathcal{D}_C^{\text{sign},b,+}$ has an invertible indicial family. Once again, $F(t)$ is not a $b-$pseudo-differential operator but its Schwartz kernel lifted to the $b-$stretched product vanishes to infinite order at lb and rb. One can then construct as above a parametrix $G$ sending the $C_r^*(\Gamma)-$Hibert module $H_b^{-1}\oplus H_b^{-1}$ into $L_b^2\oplus L_b^2$ such that $F(t)\circ G - \mathrm{Id}$ and $G\circ F(t) - \mathrm{Id}$ are $C_r^*(\Gamma)-$compact. Since the family $\{F(t)\}_{0\leq t\leq 1}$ is obviously continuous, we have $\mathrm{Ind}(\mathcal{D}_C^{\text{sign},b,+}) = \mathrm{Ind}(\mathcal{D}_C^{\text{sign,cyl},+})$. Now we can find a finite number of elements $u_1,\ldots,u_N$ of $({}^b\Omega^*(M,\mathcal{V}^\infty)\oplus {}^b\Omega^*([0,2))\widehat{\otimes}\widehat{W}^*)^-$ which vanish in a neighborhood of the $b$-boundary ($x=0$, $x'=0$) (i.e., at $-\infty$ on the cylinders) such that if one denotes by $K$ the operator defined by $K(s_1,\ldots,s_N) = \sum_{j=1}^N s_j u_j$ for any $(s_1,\ldots,s_N) \in (C_r^*(\Gamma))^N$ then the operator $\mathcal{D}_C^{\text{sign,cyl},+}\oplus K$ is surjective from $(H_b^1)^+ \oplus (C_r^*(\Gamma))^N$ to $(L_b^2)^-$. Then, as is well known, $\mathrm{Ind}(\mathcal{D}_C^{\text{sign,cyl},+}) = [\mathrm{Ker}(\mathcal{D}_C^{\text{sign,cyl},+}\oplus K) - (\mathcal{B}^\infty)^N]$. Of course, we can assume that $u_1,\ldots,u_N$ are such that

$$\mathrm{Ind}(\mathcal{D}_C^{\text{sign,APS},+}) = [\mathrm{Ker}_{\mathrm{APS}}(\mathcal{D}_C^{\text{sign},+}\oplus K) - (\mathcal{B}^\infty)^N]$$

Since the identification $\mathrm{Ker}_{\mathrm{APS}}(\mathcal{D}_C^{\text{sign},+}\oplus K) \equiv \mathrm{Ker}(\mathcal{D}_C^{\text{sign,cyl},+}\oplus K)$ follows by standard arguments, the theorem is proved.

## 11. The higher index formula for the $b$-signature operator

We want to adapt the proof of the higher APS-index formula given in [23] to the present situation. First, we need to comment about the existence of a heat kernel for the perturbed signature Laplacian $(\mathcal{D}_C^{\mathrm{sign},b})^2$. We shall concentrate on the $b$-boundary, since the heat kernel near $x'=2$ is a consequence of [8]. We can write $(\mathcal{D}_C^{\mathrm{sign},b})^2$ as

$$(\mathcal{D}_C^{\mathrm{sign},b})^2 = \Delta + P \quad \text{with} \quad P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix},$$

where $P$ is a smoothing operator in the enlarged $\mathcal{B}^\infty$-$b$-calculus. Moreover, on ${}^b\Omega^*(\mathcal{U};\mathcal{V}|_{\mathcal{U}}) \oplus ({}^b\Omega^*([0,1/2]) \otimes \widehat{W}^*)$, we have

$$\begin{aligned}\Delta &= \begin{pmatrix} D_M - \tau_M D_M \tau_M & 0 \\ 0 & D_{\mathrm{alg}} - \tau_{\mathrm{alg}} D_{\mathrm{alg}} \tau_{\mathrm{alg}} \end{pmatrix}^2 \\ &= \begin{pmatrix} \Delta_M & 0 \\ 0 & \Delta_{[0,2)} \otimes \mathrm{Id} + \mathrm{Id} \otimes \Delta_{\widehat{W}} \end{pmatrix}.\end{aligned}$$

The heat kernel of $\Delta$ is certainly well-defined as an element of a $\mathcal{B}^\infty$-$b$-heat calculus; see [33, Sect. 7] and [22, Sect. 10]. Using exactly the same technique as in the proof of [34, Proposition 8], we can construct the heat kernel of $(\mathcal{D}_C^{\mathrm{sign},b})^2$ as follows. We set $H^{(0)} = \exp(-t\Delta)$ and consider

$$(\frac{d}{dt} + (\mathcal{D}_C^{\mathrm{sign},b})^2)H^{(0)} = R^{(0)}, \qquad R^{(0)} = P\exp(-t\Delta).$$

Using the indicial family of $(\mathcal{D}_C^{\mathrm{sign},b})^2$ and the heat equation on the boundary, we can inductively remove the whole Taylor series of $R^{(0)}$ at the front faces where it is not zero and thus define an $H$ in an enlarged $\mathcal{B}^\infty$-$b$-heat calculus such that

$$(\frac{d}{dt} + (\mathcal{D}_C^{\mathrm{sign},b})^2)H = R \quad \text{with} \quad R \in C^\infty((0,\infty)_t; \rho_{\mathrm{bf}}^\infty \cdot \widehat{\Psi}^{-\infty}_{b,\mathcal{B}^\infty}).$$

At this point, the heat kernel $\exp(-t(\mathcal{D}_C^{\mathrm{sign},b})^2)$ is obtained by summing the usual Duhamel's series:

$$e^{-t(\mathcal{D}_C^{\mathrm{sign},b})^2} = H + \sum_{k>1} \int_{tS^k} H(t-t_k)R(t_k - t_{k-1}) \cdots R(t_1)dt_k \dots dt_1 \\ \in \widehat{\Psi}^{-\infty}_{b,\mathcal{B}^\infty} \quad \forall t > 0,$$

with $tS^k = \{(t_1, t_2, \ldots, t_k); 0 \leq t_1 \leq t_2 \cdots \leq t_k \leq t\}$. We refer the reader to [34, Proposition 8] for the details.

Next, we need to introduce a superconnection $\mathbb{A}_s$ as in [26, (51)] and [22], and define the associated superconnection heat kernel. We fix, as in (2.8), a $\tau_W$-invariant connection

$$\nabla^{\widehat{W}} : \widehat{W}^* \to \Omega_1(\mathcal{B}^\infty) \otimes_{\mathcal{B}^\infty} \widehat{W}^* ,$$

define the superconnection

$$\begin{aligned} s(D_{\mathrm{alg}} - \tau_{\mathrm{alg}} D_{\mathrm{alg}} \tau_{\mathrm{alg}}) + \nabla^{\widehat{W}} &: C^\infty([0,2); {}^b\Lambda^*[0,2)) \widehat{\otimes} \widehat{W}^* \\ &\to C^\infty([0,2); {}^b\Lambda^*[0,2)) \widehat{\otimes} (\Omega_*(\mathcal{B}^\infty) \otimes_{\mathcal{B}^\infty} \widehat{W}^*) \end{aligned}$$

and consider the total superconnection

$$\begin{aligned} \mathbb{A}_s = \begin{pmatrix} s(D_M - \tau_M D_M \tau_M) + \nabla & 0 \\ 0 & s(D_{\mathrm{alg}} - \tau_{\mathrm{alg}} D_{\mathrm{alg}} \tau_{\mathrm{alg}}) + \nabla^{\widehat{W}} \end{pmatrix} &: C^* \\ &\to C^* \oplus (\Omega_1(\mathcal{B}^\infty) \otimes_{\mathcal{B}^\infty} C^*), \end{aligned}$$

with

$$\nabla : C^\infty(M; {}^b\Lambda^* M \otimes \mathcal{V}^\infty) \to \Omega_1(\mathcal{B}^\infty) \otimes_{\mathcal{B}^\infty} C^\infty(M; {}^b\Lambda^* M \otimes \mathcal{V}^\infty) \tag{11.1}$$

as in [27, Proposition 9]. $\mathbb{A}_s$ extends to a map $\Omega_*(\mathcal{B}^\infty) \otimes_{\mathcal{B}^\infty} C^* \to \Omega_*(\mathcal{B}^\infty) \otimes_{\mathcal{B}^\infty} C^*$ which is odd with respect to the total $\mathbb{Z}_2$-grading and satisfies Leibniz' rule.

This is the unperturbed superconnection; we shall need

$$\widetilde{\mathbb{A}}_s = s\mathcal{D}_C^{\mathrm{sign},b,s} + \begin{pmatrix} \nabla & 0 \\ 0 & \nabla^{\widehat{W}} \end{pmatrix}$$

with

$$\begin{aligned} \mathcal{D}_C^{\mathrm{sign},b,s} = &\begin{pmatrix} D_M - \tau_M D_M \tau_M & 0 \\ 0 & D_{\mathrm{alg}} - \tau_{\mathrm{alg}} D_{\mathrm{alg}} \tau_{\mathrm{alg}} \end{pmatrix} \\ &+ \epsilon(s) \begin{pmatrix} 0 & S(\delta) \\ T(\delta) & 0 \end{pmatrix} \end{aligned}$$

where we have used the notation at the end of the previous section and $\epsilon \in C^\infty(0,\infty)$ is a nondecreasing function such that $\epsilon(s) = 0$ for $s \in (0,2)$ and $\epsilon(s) = 1$ for $s > 4$.

**Remark.** The operator $\mathcal{D}_C^{\mathrm{sign},b,s}$, and thus the superconnection $\widetilde{\mathbb{A}}_s$, depend on $\delta$. Using Duhamel's expansion and the existence of the heat kernel $\exp(-(s\mathcal{D}_C^{\mathrm{sign},b,s})^2)$ we can define the superconnection heat kernel $\exp(-\widetilde{\mathbb{A}}_s^2)$. For each $s > 0$, it is a smoothing operator in the enlarged $b$-calculus with coefficients in $\Omega_*(\mathcal{B}^\infty)$. We denote the latter space by $\widehat{\Psi}^{-\infty}_{b,\Omega_*(\mathcal{B}^\infty)}$; thus $\exp(-\widetilde{\mathbb{A}}_s^2) \in \widehat{\Psi}^{-\infty}_{b,\Omega_*(\mathcal{B}^\infty)}$. Then

$$\exp(-\widetilde{\mathbb{A}}_s^2) = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix},$$

with

$$E_{11} \in \Psi^{-\infty}_{b,\Omega_*(\mathcal{B}^\infty)}(M; \Omega_*(\mathcal{B}^\infty) \otimes_{\mathcal{B}^\infty} {}^b\Lambda^* M \otimes \mathcal{V}^\infty),$$
$$E_{22} \in \Psi^{-\infty}_{b,\Omega_*(\mathcal{B}^\infty)}([0,2); \Omega_*(\mathcal{B}^\infty) \otimes_{\mathcal{B}^\infty} {}^b\Lambda^*[0,2)\widehat{\otimes}\widehat{W}^*).$$

Each operator such as $E_{11}$ has a well defined $b$-supertrace, with values in $\overline{\Omega}_*(\mathcal{B}^\infty)$; see [22, Section 13]. Similarly each element such as $E_{22}$ will have a well-defined $b$-supertrace, also with values in $\overline{\Omega}_*(\mathcal{B}^\infty)$. Notice that the heat kernel in a neighbourhood of $x' = 2$ has a well-defined supertrace - there is no need for regularization there.

We define the $b$-supertrace of $\exp(-\widetilde{\mathbb{A}}_s^2)$ as

$$b\,\text{-STR}(\exp(-\mathbb{A}_s^2)) = b\,\text{-STR}(E_{11}) + b\,\text{-STR}(E_{22}) \in \overline{\Omega}_*(\mathcal{B}^\infty).$$

The same definition applies to any element

$$R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \in \widehat{\Psi}^{-\infty}_{b,\Omega_*(\mathcal{B}^\infty)}.$$

This $b$-supertrace is not necessarily zero on supercommutators. As in [33] (and then [34], [22]), one can write a formula for the $b$-supertrace of a supercommutator of two elements $R, S \in \widehat{\Psi}^{-\infty}_{b,\Omega_*(\mathcal{B}^\infty)}$.

**Proposition 11.1.** *Given*

$$R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}, \quad S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \in \widehat{\Psi}^{-\infty}_{b,\Omega_*(\mathcal{B}^\infty)},$$

*the following formula holds in* $\overline{\Omega}_*(\mathcal{B}^\infty)$ *:*

$$b-\mathrm{STR}[R,S] = \frac{\sqrt{-1}}{2\pi}\int_{-\infty}^{+\infty} \mathrm{STR}(\frac{\partial I(R,\lambda)}{\partial\lambda}\circ I(S,\lambda))d\lambda.$$

*Moreover, the same formula holds if* $R_{11}$ *and* $R_{22}$ *are b-differential.*

*Proof.* On applying some straightforward linear algebra, the proof can be eventually reduced to the one in [33, Prop. 5.9]; the details are exactly as in [34, Prop. 9], [22, Sect. 13], but with the additional (harmless) complication coming from the fact that we are dealing with the *enlarged* $b$-calculus. Since the details are elementary but tedious, we omit them. q.e.d.

Using the $b$-supercommutator formula and proceeding as in [34, Proposition 11], we can now compute the $s$-derivative of $b-\mathrm{STR}(\exp(-\widetilde{\mathbb{A}}_s^2))$. To this end, we first need to analyze the boundary behaviour of $\widetilde{\mathbb{A}}_s$.

Let

$$\begin{aligned}\nabla^{\partial M} &: C^\infty(\partial M;\Lambda^*(\partial M)\otimes\mathcal{V}_0^\infty\otimes \mathrm{Cl}(1)) \\ &\to \Omega_1(\mathcal{B}^\infty)\otimes_{\mathcal{B}^\infty} C^\infty(\partial M;\Lambda^*(\partial M)\otimes\mathcal{V}_0^\infty\otimes\mathrm{Cl}(1))\end{aligned}$$

be the $\mathrm{Cl}(1)$ analog of the connection in (11.1). We consider

$$\begin{aligned}\widetilde{\mathbb{B}}_s(\lambda) = s &\begin{pmatrix}\sigma\mathcal{D}_{\mathrm{sign}}(\partial M) & 0\\ 0 & \sigma\mathcal{D}^{\mathrm{sign}}_{\widehat{W}}\end{pmatrix}\\ &+ s\epsilon(s)\begin{pmatrix}0 & I(S(\delta),\lambda)\\ I(T(\delta),\lambda) & 0\end{pmatrix}\\ &+\begin{pmatrix}\nabla^{\partial B} & 0\\ 0 & \nabla^{\widehat{W}}\end{pmatrix}\end{aligned}$$

where we recall, see (9.1), that using our identifications at the boundary,

$$\begin{aligned}&\begin{pmatrix}0 & I(S(\delta),\lambda)\\ I(T(\delta),\lambda) & 0\end{pmatrix}\\ &= (-i)\cdot\begin{cases}\begin{pmatrix}0 & \sigma\widehat{\rho}_\delta(\lambda)\tau_{\partial M}\widehat{g}\\ -\sigma\widehat{\rho}_\delta(\lambda)\widehat{f}\tau_{\partial M} & 0\end{pmatrix} & \text{if } j<m\\ \begin{pmatrix}0 & \sigma\widehat{\rho}_\delta(\lambda)\widehat{g}\tau_{\widehat{W}}\\ -\sigma\widehat{\rho}_\delta(\lambda)\tau_{\widehat{W}}\widehat{f} & 0\end{pmatrix} & \text{if } j>m\end{cases}.\end{aligned}\tag{11.2}$$

For each fixed $\lambda \in \mathbb{R}$, $\widetilde{\mathbb{B}}_s(\lambda)$ is a Cl(1)-superconnection, mapping

$$\begin{aligned}&(C^\infty(\partial M; \Lambda^*(\partial M) \otimes \mathcal{V}_0^\infty) \oplus \widehat{W}^*) \otimes \mathrm{Cl}(1)\\ &\qquad \to \Omega_*(\mathcal{B}^\infty) \otimes_{\mathcal{B}^\infty} (C^\infty(\partial M; \Lambda^*(\partial M) \otimes \mathcal{V}_0^\infty) \oplus \widehat{W}^*) \otimes \mathrm{Cl}(1).\end{aligned}$$

It depends on $\delta$ through the indicial families (11.2).

**Proposition 11.2.** *The following formula holds in $\overline{\Omega}_*(\mathcal{B}^\infty)$:*

$$\frac{d}{ds}\left(b - \mathrm{STR}\exp(-\widetilde{\mathbb{A}}_s^2)\right) = -d\left(\mathrm{b} - \mathrm{STR}\left(\frac{d\widetilde{\mathbb{A}}_s}{ds}e^{-\widetilde{\mathbb{A}}_s^2}\right)\right) - \widehat{\eta}_\delta(s) \tag{11.3}$$

*with*

$$\begin{aligned}\widehat{\eta}_\delta(s) = &\frac{i}{2\pi}\int_{\mathbb{R}} \mathrm{STR}\left(s\begin{pmatrix}\gamma & 0\\ 0 & \gamma\end{pmatrix} \cdot \frac{d\widetilde{\mathbb{B}}_s(\lambda)}{ds}e^{-\widetilde{\mathbb{B}}_s^2}\right)e^{-(s\lambda)^2}d\lambda\\ &+ \frac{i}{2\pi}\int_{\mathbb{R}} \mathrm{STR}\left(s\epsilon(s)\begin{pmatrix}\gamma & 0\\ 0 & \gamma\end{pmatrix}\right.\\ &\qquad \left.\cdot\, \lambda\left(\frac{d}{d\lambda}\begin{pmatrix}0 & I(S(\delta),\lambda)\\ I(T(\delta),\lambda) & 0\end{pmatrix}\right) \cdot e^{-\widetilde{\mathbb{B}}_s^2}\right)e^{-(s\lambda)^2}d\lambda\end{aligned}$$

*Proof.* Using Proposition 11.1, the proof given in [34, Prop. 11], [23, Prop. 7.2] can be easily adapted to the present situation. q.e.d.

We define $\widetilde{\eta}_\delta(s) = \mathcal{R}\widehat{\eta}_\delta(s)$, with $\mathcal{R}$ as in Section 2, and we put $\widetilde{\eta}(\delta) = \int_0^\infty \widetilde{\eta}_\delta(s)ds$. The convergence of the integral can be proven as in Proposition 7.4 in [23]. Regarding the relationship of $\widetilde{\eta}(\delta)$ to the $\widetilde{\eta}_{\partial M}$ of Section 2 (see equation (2.10 with $F = \partial M$), we have the following proposition.

**Proposition 11.3.** *The following equality holds for any $\delta > 0$:*

$$\widetilde{\eta}(\delta) = \widetilde{\eta}_{\partial M} \text{ in } \overline{\Omega}_*(\mathcal{B}^\infty)/d\overline{\Omega}_*(\mathcal{B}^\infty). \tag{11.4}$$

*Proof.* The proof of Proposition 7.4 (2) in [23] applies *mutatis mutandis* to the more general case treated here. q.e.d.

These two propositions are crucial in the proof of the following higher index theorem for the $b$-signature operator :

**Theorem 11.4.** *Let $M$ be an even-dimensional manifold-with-boundary. Let $\Gamma$ be a finitely-generated discrete group and let $\nu : M \to B\Gamma$ be a continuous map, defined up to homotopy. We make Assumption 1 on*

*$\partial M$. Let $g$ be a $b$-metric on $M$ which is product-like near the boundary and let $R^M$ be the associated curvature 2-form. The following formula holds for the Chern character of the higher $b$-signature-index class:*

$$\mathrm{ch}(\mathrm{Ind}(\mathcal{D}_C^{\mathrm{sign},b,+})) = \int_M L(R^M/2\pi) \wedge \omega - \widetilde{\eta}_{\partial M} \ \text{ in } \ \overline{\mathrm{H}}_*(\mathcal{B}^\infty) \tag{11.5}$$

*with $\widetilde{\eta}_{\partial M}$ equal to the higher eta-invariant of Section 2.*

*Proof.* Integrating formula (11.3), we obtain that for $u > t > 0$ and modulo exact forms,

$$\mathcal{R}\, b - \mathrm{STR}(e^{-\widetilde{\mathbb{A}}_u^2}) = \mathcal{R}\, b - \mathrm{STR}(e^{-\widetilde{\mathbb{A}}_t^2}) - \int_t^u \widetilde{\eta}_\delta(s) ds \,. \tag{11.6}$$

The limit of the right-hand-side as $t \to 0^+$ can be computed as in [26, Proposition 12] and [22] (we also use [8, Section 4] near $x' = 2$). Let us consider the asymptotic expansion near $t = 0$ of the first summand on the right-hand-side of (11.6). Using [26, Proposition 12], [22] and [8, Section 4], one sees that the coefficient of $t^0$ will be the sum of three noncommutative differential forms. The first term is

$$\int_M L(R^M/2\pi) \wedge \omega \ .$$

The second term is the $t^0$ term of the $b$-integral over $[0, 2)$ of the pointwise supertrace of the heat kernel associated to the superconnection

$$t(D_{\mathrm{alg}} - \tau_{\mathrm{alg}} D_{\mathrm{alg}} \tau_{\mathrm{alg}}) + \nabla^{\widehat{W}} \ .$$

As we are effectively computing a heat kernel on the real line, the local heat trace asymptotics will be of the form $t^{-1} \sum_{j=0}^\infty c_j t^j$. (Note that the usual $t$ of the heat kernel expansion is, in our case, $t^2$.) Using the Duhamel formula, we see that the $t^0$ term is proportionate to the local supertrace of $[(D_{\mathrm{alg}} - \tau_{\mathrm{alg}} D_{\mathrm{alg}} \tau_{\mathrm{alg}}), \nabla^{\widehat{W}}]$. As $\nabla^{\widehat{W}}$ is independent of $x$, this equals $\mathrm{Str}[\mathcal{D}^{\mathrm{sign}}_{\widehat{W}}, \nabla^{\widehat{W}}]$. However, being the supertrace of a supercommutator involving $\nabla^{\widehat{W}}$, this is exact as an element of $\overline{\Omega}_*(\mathcal{B}^\infty)$. As we are working modulo exact forms, it thus vanishes.

The third term is a eta-contribution coming from $x' = 2$:

$$- \int_0^\infty \widetilde{\eta}_{\widehat{W}}(s) ds \ ,$$

where we recall that

$$\widetilde{\eta}_{\widehat{W}}(s) = \frac{1}{\sqrt{\pi}} \mathcal{R}\,\mathrm{STR}_{\mathrm{Cl}(1)} \sigma \mathcal{D}^{\mathrm{sign}}_{\widehat{W}} \exp(-(s\sigma\mathcal{D}^{\mathrm{sign}}_{\widehat{W}} + \nabla^{\widehat{W}})^2) \ .$$

Thus, modulo exact forms, we obtain:

$$\lim_{t\to 0^+} \mathcal{R}\, b-\mathrm{STR}(e^{-\widetilde{\mathbb{A}}_t^2}) = \int_M L(R^M/2\pi)\wedge\omega - \int_0^\infty \widetilde{\eta}_{\widehat{W}}(s)ds,$$

However, as in [27, p. 227], a duality argument shows that for all $s > 0$, modulo exact forms,

$$\widetilde{\eta}_{\widehat{W}}(s) = 0 \ . \tag{11.7}$$

Summarizing,

$$\mathcal{R}\, b-\mathrm{STR}(e^{-\widetilde{\mathbb{A}}_u^2}) = \int_M L(R^M/2\pi)\wedge\omega - \int_0^u \widetilde{\eta}_\delta(s)ds \ .$$

Proceeding as in [22, Theorem 14.1] and [24, Appendix], one can now complete the proof of the theorem. We omit the details as they are very similar to those explained at length in the above references. (The Appendix of [24], which is based on results in [29, Section 6], extends the higher APS-index theorem proven in [22, Section 14] to any finitely-generated discrete group $\Gamma$, under a gap hypothesis for the boundary operator.) This proves the theorem. q.e.d.

## 12. Proofs of Theorem 0.1 and Corollaries 0.2–0.4

Before proving the main results, let us comment about our normalization of eta-invariants. In the case $\mathcal{B}^\infty = \mathbb{C}$, our definition of $\widetilde{\eta}_F$ in (3.7) gives *half* of the eta-invariant as defined in [1, Theorem 3.10(iii)] for Dirac-type operators. This normalization is more convenient for our purposes, albeit unconventional, and is also used in [3]. In [1, Theorem 4.14(iii)], the eta-invariant of the (tangential) signature operator is defined in terms of an operator on *even* forms, and so gives half of the APS eta-invariant of the corresponding Dirac-type operator. The upshot is that when considering the signature operator, our eta-invariant coincides with that considered in [1, Theorem 4.14(iii)].

*Proof of Theorem 0.1.* Suppose first that $M$ is even-dimensional. By Theorem 6.1, the conic index in $K_0(C_r^*(\Gamma))$ is an oriented-homotopy invariant. By Theorems 7.2 and 10.1, the conic index equals the $b$-index. As $\mathcal{B}^\infty$ is a dense subalgebra of $C_r^*(\Gamma)$ which is closed under the holomorphic functional calculus in $C_r^*(\Gamma)$, there is an isomorphism $K_0(\mathcal{B}^\infty) \cong K_0(C_r^*(\Gamma))$ [9, Section IIIC]. By Theorem 11.4, the Chern character of the index, as an element of $\overline{\mathrm{H}}_*(\mathcal{B}^\infty)$, equals $\sigma_M$. The theorem follows in this case.

If $M$ is odd-dimensional, say of dimension $n = 2m-1$, then we can reduce to the even-dimensional case by a standard trick, replacing $M$ by $M \times S^1$ and replacing $\Gamma$ by $\Gamma \times \mathbb{Z}$. Observe that by Fourier transform, $C_r^*(\Gamma \times \mathbb{Z}) \cong C_r^*(\Gamma) \otimes C^0(S^1)$ and $\mathcal{B}^\infty_{\Gamma \times \mathbb{Z}^1} \cong \mathcal{B}^\infty_\Gamma \otimes C^\infty(S^1)$, where $\otimes$ denotes a projective tensor product. Under these identifications, the signature operator of $M \times S^1$ can be identified with the suspension of the signature operator on $M$, as in [32, p. 250] and [4, p. 124]. Moreover, instead of the universal graded algebra $\Omega_*(\mathcal{B}^\infty_{\Gamma\times\mathbb{Z}})$, it suffices to deal with the smaller differential graded algebra $\Omega_*(\mathcal{B}^\infty_\Gamma) \widehat{\otimes} \Omega^*(S^1)$. Let $\tau'$ be the standard generator for $\mathrm{H}^1(\mathbb{Z};\mathbb{Z}) \subset \mathrm{H}^1(\mathbb{Z};\mathbb{C})$. Put $\tau = \sqrt{-1}\,\tau'/2\pi \in \mathrm{H}^1(\mathbb{Z};\mathbb{C})$. There is a natural desuspension map

$$\langle \cdot\,,\, \tau \rangle : \overline{\mathrm{H}}_*(\mathcal{B}^\infty_{\Gamma\times\mathbb{Z}}) \to \overline{\mathrm{H}}_*(\mathcal{B}^\infty_\Gamma)$$

and one can check, as in [35, Lemma 6], that

$$\Big\langle \int_{M\times S^1} L(T(M\times S^1)) \wedge \omega_{\Gamma\times\mathbb{Z}} - \widetilde{\eta}_{\partial M\times S^1}\,,\, \tau \Big\rangle$$
$$= \int_M L(TM) \wedge \omega_\Gamma - \widetilde{\eta}_{\partial M} \text{ in } \overline{H}_*(\mathcal{B}^\infty_\Gamma).$$

In order to directly apply the even-dimensional results to $M \times S^1$, we would have to know that if $\partial M$ satisfies Assumption 1, then $\partial M \times S^1$ satisfies Assumption 1. This is not quite true. However, if we consider the complex $\widehat{W}^*$ for $\partial M$, from (3.5), and take its graded tensor product with $\Omega^*(S^1)$, then the terms in degrees $m-1$ and $m$ are $L \widehat{\otimes} \Omega^1(S^1)$ and $L' \widehat{\otimes} \Omega^0(S^1)$, respectively, with the differential between them being the zero map. We can then go through all of the arguments in Sections 5-11 for $M \times S^1$, using $\widehat{W}^* \widehat{\otimes} \Omega^*(S^1)$ as the perturbing complex. The important point is that by duality, we again have $\widehat{\eta}_{\widehat{W}} = 0$ [27, p. 227]. This implies Theorem 0.1 in the odd-dimensional case. q.e.d.

*Proof of Corollary 0.2.* This is an immediate consequence of Theorem 0.1. q.e.d.

**Example.** Given $N > 0$ and $0 \leq j \leq 4N-2$, put $M_1 = (\mathbb{C}P^{2N} \# \mathbb{C}P^{2N} \# D^{4N}) \times T^j$ and $M_2 = (\mathbb{C}P^{2N} \# \overline{\mathbb{C}P}^{2N} \# D^{4N}) \times T^j$. Then

$M_1$ and $M_2$ are homotopy equivalent as topological spaces, as they are both homotopy equivalent to $(\mathbb{C}P^{2N-1} \vee \mathbb{C}P^{2N-1}) \times T^j$. Put $\Gamma = \mathbb{Z}^j$ and let $\nu_i : M_i \to B\mathbb{Z}^j$ be the classifying maps for the universal covers. Then Assumption 1 is satisfied. Take $\tau = [B\mathbb{Z}^j] \in \mathrm{H}^j(B\mathbb{Z}^j; \mathbb{C})$. Then

$$\langle \sigma_{M_1}, \tau \rangle = \sigma(\mathbb{C}P^{2N} \# \mathbb{C}P^{2N} \# D^{4N}) = 2,$$

while

$$\langle \sigma_{M_2}, \tau \rangle = \sigma(\mathbb{C}P^{2N} \# \overline{\mathbb{C}P}^{2N} \# D^{4N}) = 0.$$

Thus by Corollary 0.2, $M_1$ and $M_2$ are not homotopy equivalent as manifolds-with-boundary.

*Proof of Corollary 0.3.* From [26, Corollary 2], the higher signature of $M$ corresponding to $\tau \in \mathrm{H}^*(\Gamma; \mathbb{C})$ is a nonzero constant (which only depends on the degree of $\tau$) times $\langle \int_M L(TM) \wedge \omega, Z_\tau >$. Let us choose a Riemannian metric on $M$ so that a tubular neighborhood of $F$ is isometrically a product. Then

$$\begin{aligned}\int_M L(TM) \wedge \omega &= \int_A L(TM) \wedge \omega + \int_B L(TM) \wedge \omega \\ &= \left( \int_A L(TM) \wedge \omega \right) - \widetilde{\eta}_{\partial A} + \left( \int_B L(TM) \wedge \omega \right) - \widetilde{\eta}_{\partial B},\end{aligned}$$

as $\partial A$ and $\partial B$ differ in their orientations and $\widetilde{\eta}$ is odd under a change of orientation. Here we have chosen a (stable) Lagrangian subspace $L$ for $\partial A$, if necessary, and then taken the (stable) Lagrangian subspace $-L$ for $\partial B$. Thus

$$\int_M L(TM) \wedge \omega = \sigma_A + \sigma_B. \tag{12.1}$$

The corollary follows from pairing both sides of (12.1) with the cyclic cocycle $Z_\tau$. q.e.d.

Corollary 0.4 contains a condition in the odd-dimensional case concerning the preservation of a stable Lagrangian subspace. In order to state this condition precisely, let us first discuss some generalities. Let $F$ be a connected manifold with universal cover $\widetilde{F}$ and corresponding covering group $G$. Let $\phi$ be a diffeomorphism of $F$. Then we can lift $\phi$ to a diffeomorphism $\widetilde{\phi}$ of $\widetilde{F}$ such that $\pi \circ \widetilde{\phi} = \phi \circ \pi$. Given such a lift, there is an automorphism $\alpha$ of $G$ so that for all $g \in G$ and $\widetilde{f} \in \widetilde{F}$,

$$\widetilde{\phi}(g\widetilde{f}) \;=\; \alpha(g)\, \widetilde{\phi}(\widetilde{f}). \tag{12.2}$$

Let $\Gamma$ be a discrete group and let $\rho : G \to \Gamma$ be a homomorphism. There is a corresponding normal $\Gamma$-cover $\widehat{F}$ of $F$ given by pairs $[\gamma, \widetilde{f}] \in \Gamma \times \widetilde{F}$, with the equivalence relation

$$[\gamma, g\widetilde{f}] \equiv [\gamma\rho(g), \widetilde{f}]. \tag{12.3}$$

Suppose that some fixed $\gamma' \in \Gamma$ satisfies

$$\rho(\alpha(g)) = \gamma' \rho(g) (\gamma')^{-1} \tag{12.4}$$

for all $g \in G$. Then there is a corresponding normal $\Gamma$-cover of the mapping torus $\frac{[0,1]\times F}{(0,f)\equiv(1,\phi(f))}$ of $\phi$. Namely, take $[0,1] \times \widehat{F}$ and put on it the equivalence relation generated by

$$(0, \gamma, \widetilde{f}) \equiv (1, \gamma(\gamma')^{-1}, \widetilde{\phi}(\widetilde{f})) \tag{12.5}$$

for $\gamma \in \Gamma$ and $\widetilde{f} \in \widetilde{F}$. It is easy to check that this equation gives a well-defined diffeomorphism from $\{0\} \times \widehat{F}$ to $\{1\} \times \widehat{F}$.

Conversely, let $\mathcal{T}$ be a mapping torus with fiber $F$. Then $\pi_1(\mathcal{T})$ is the semidirect product $G\widetilde{\times}_\alpha\mathbb{Z}$ for some automorphism $\alpha$ of $G$. Explicitly, the product in $G\widetilde{\times}_\alpha\mathbb{Z}$ is

$$(g, n) \cdot (g', n') = (g\alpha^n(g'), n + n'). \tag{12.6}$$

Let $\mathcal{T}'$ be a normal $\Gamma$-cover of $\mathcal{T}$. It is classified by a homomorphism $r : G\widetilde{\times}_\alpha\mathbb{Z} \to \Gamma$. Let $\rho$ be the composite homomorphism $G \to G\widetilde{\times}_\alpha\mathbb{Z} \to \Gamma$ and put $\gamma' = r(e, 1)$. Then $\mathcal{T}'$ arises from the above construction, using $\alpha$, $\rho$ and $\gamma'$.

Let $C_r^*(\Gamma) \times_\Gamma \widetilde{F}$ be the vector bundle generated by pairs $[\sum_\gamma c_\gamma \gamma, \widetilde{f}]$ with $\sum_\gamma c_\gamma \gamma \in C_r^*(\Gamma)$, and with the equivalence relation

$$[\sum_\gamma c_\gamma \gamma, g\widetilde{f}] \equiv [\sum_\gamma c_\gamma \gamma\rho(g), \widetilde{f}]. \tag{12.7}$$

Given $\widetilde{\phi}$ and $\gamma'$, we obtain an automorphism of the $C_r^*(\Gamma)$-bundle $C_r^*(\Gamma) \times_\Gamma \widetilde{F}$ coming from

$$[\sum_\gamma c_\gamma \gamma, \widetilde{f}] \to [\sum_\gamma c_\gamma \gamma(\gamma')^{-1}, \widetilde{\phi}(\widetilde{f})], \tag{12.8}$$

which covers $\phi$.

Now with reference to Corollary 0.4, suppose that $\dim(M_i) = 2k+1$. For simplicity, we assume that the hypersurface $F$ is connected; the general case is similar. There are two diffeomorphisms of $F$ and two homotopies $[0,1] \times F \to B\Gamma$. Gluing together the two homotopies at the ends, using the two diffeomorphisms of $F$, we construct a mapping torus $T$ with fiber $F$ and a well-defined map from $T$ to $B\Gamma$. Hence there is a corresponding $\Gamma$-cover of $T$. Then let $\widetilde{\phi}$ and $\gamma'$ be as above. The automorphism of (12.8) induces an automorphism $\mathcal{A}$ of $\overline{\mathrm{H}}^*(F; \mathcal{V}_0)$. The precise statement of the condition in Corollary 0.4 is that $\mathcal{A}$ should preserve a (stable) Lagrangian subspace $L$ of $\overline{\mathrm{H}}^k(F; \mathcal{V}_0)$.

*Proof of Corollary 0.4.* Suppose first that $M_1$ and $M_2$ are even-dimensional. Then Corollary 0.4 is an immediate consequence of Corollary 0.3, along with the fact that $\sigma_A$ and $\sigma_B$ are smooth topological invariants, i.e., independent of the choices of Riemannian metric and function $h$, and only depend on $\nu$ through its homotopy class. That $\sigma_A$ and $\sigma_B$ are smooth topological invariants follows from Theorem 0.1 or more directly from Theorem 11.4, using the fact that the index class is a smooth topological invariant. (Without using the index class, the smooth topological invariance also follows from the variational formulae in [27, Proposition 27] and [30, Theorem 6]. The papers [27] and [30] deal with a slightly stronger assumption than Assumption 1, but their proofs can be extended to the present case, too.)

Now suppose that $M_1$ and $M_2$ are odd-dimensional. Let $L$ be a (stable) Lagrangian subspace of $\overline{\mathrm{H}}^k(F; \mathcal{V}_0)$ which is preserved by the automorphism $\mathcal{A}$ described above. We first think of $L$ as living on $\partial A$, where $A$ comes from the decomposition $M_1 = A \cup_F B$. Equivalently, in terms of the homotopy $\widetilde{\nu}_A : [1,2] \times A \to B\Gamma$ between $\nu_1\big|_A$ and $\nu_2\big|_A$, we think of $L$ as a (stable) Lagrangian subspace $L_{1,A}$ on $\{1\} \times \partial A$. Using $\widetilde{\nu}_A$, we transfer $L_{1,A}$ to a (stable) Lagrangian subspace $L_{2,A}$ on $\{2\} \times \partial A$.

Let $L_{2,B}$ be the (stable) Lagrangian subspace on $\{2\} \times \partial B$ corresponding to $L_{2,A}$ through the diffeomorphism $\phi_2$ of the decomposition $M_2 = A \cup_F B$. Using the homotopy $\widetilde{\nu}_B : [1,2] \times B \to B\Gamma$ between $\nu_1\big|_B$ and $\nu_2\big|_B$, we transfer $L_{2,B}$ to a (stable) Lagrangian subspace $L_{1,B}$ on $\{1\} \times \partial B$. The fact that $L$ is preserved by $\mathcal{A}$ ensures that $L_{1,B}$ is identical to the (stable) Lagrangian subspace on $\{1\} \times \partial B$ corresponding to $L_{1,A}$ through the diffeomorphism $\phi_1$ of the decomposition $M_1 = A \cup_F B$.

From Corollary 0.3, the higher signature of $M_1$ equals $\langle \sigma_A(L_{1,A}), \tau \rangle + \langle \sigma_B(-L_{1,B}), \tau \rangle$, where we have indicated the dependence on the (stable) Lagrangian subspace. Similarly, the higher signature of $M_2$ equals $\langle \sigma_A(L_{2,A}), \tau \rangle + \langle \sigma_B(-L_{2,B}), \tau \rangle$ Using the homotopy $\widetilde{\nu}_A$, we have $\langle \sigma_A(L_{1,A}), \tau \rangle = \langle \sigma_A(L_{2,A}), \tau \rangle$. Using the homotopy $\widetilde{\nu}_B$, we have

$\langle \sigma_B(-L_{1,B}), \tau \rangle = \langle \sigma_B(-L_{2,B}), \tau \rangle$. The corollary follows. q.e.d.

## 13. Appendix

Let $M$ be a compact oriented manifold-with-boundary and let $\nu : M \to B\Gamma$ be a continuous map. Let $M'$ be the associated normal $\Gamma$-cover of $M$. If $\dim M = 2m$ (resp. $\dim M = 2m+1$) then, for the purposes of this Appendix, we assume that the differential form Laplacian has a strictly positive spectrum on $\Omega^m(\partial M')$ (resp. $\Omega^m(\partial M')$). This is a slightly stronger assumption than Assumption 1; see Lemma 2.1 and Lemma 3.1.

Under this assumption, a higher $b$-signature-index class for manifolds with boundary was introduced in [25]. A higher signature index formula was then proven in the virtually nilpotent case using the higher APS index theorem proved in [22], [23]. The regularization proposed there followed [23] and employed the notion of a *symmetric* spectral section $\mathcal{P}$ for the boundary signature operator of $\mathcal{D}_{\mathrm{sign,b}}$. The index class in [25] was denoted by $\mathrm{Ind}(\mathcal{D}^+_{\mathrm{sign,b}}, \mathcal{P})$ and was proven to be independent of the particular choice of symmetric spectral section $\mathcal{P}$. We shall now indicate how to prove that $\mathrm{ch}(\mathrm{Ind}(\mathcal{D}^+_{\mathrm{sign,b}}, \mathcal{P})) = \sigma_M$, with $\sigma_M$ as (0.1). This will imply (for virtually nilpotent groups) that the higher signatures considered in [30] and in [25] are in fact the same. We shall only sketch the argument.

According to Theorem 11.4, it suffices to show that

$$\mathrm{ch}(\mathrm{Ind}(\mathcal{D}_C^{\mathrm{sign,b},+})) = \mathrm{ch}(\mathrm{Ind}(\mathcal{D}^+_{\mathrm{sign,b}}, \mathcal{P})). \tag{13.1}$$

Recall [23, Definition 6.3] that $\mathrm{Ind}(\mathcal{D}^+_{\mathrm{sign,b}}, \mathcal{P})$ is equal to $\mathrm{Ind}((\mathcal{D}_{\mathrm{sign,b}} + A_{\mathcal{P}})^+)$, with $A_{\mathcal{P}}$ a regularizing operator associated to $\mathcal{P}$. $A_{\mathcal{P}}$ is constructed as in Section 8 starting from a perturbation $A^0_{\mathcal{P}}$ on $\partial M$ which makes the boundary operator $(\mathcal{D}_{\mathrm{sign,b}})_0$ invertible. The symmetry of $\mathcal{P}$ corresponds to a vanishing of $A^0_{\mathcal{P}}$ in middle degree plus a suitable $\mathbb{Z}_2$-grading of $A^0_{\mathcal{P}}$ outside the middle degree, see [25, Definition 4.2].

We can extend the operator $\mathcal{D}_{\mathrm{sign,b}} + A_{\mathcal{P}}$ and make it act on ${}^b\Omega^*(M; \mathcal{V}^\infty) \oplus {}^b\Omega^*[0,2) \widehat{\otimes} \widehat{W}^*$ without changing the Chern character of the corresponding index class. For this, it suffices to first consider the operator

$$\mathcal{D}^{\oplus}_{\mathrm{sign},b} = \begin{pmatrix} \mathcal{D}_{\mathrm{sign},b} & 0 \\ 0 & D_{\mathrm{alg}} - \tau_{\mathrm{alg}} D_{\mathrm{alg}} \tau_{\mathrm{alg}} \end{pmatrix} \tag{13.2}$$

and then the operator in the enlarged $b$-calculus given by

$$\mathcal{D}^{\oplus}_{\text{sign,b}} + \begin{pmatrix} A_{\mathcal{P}} & 0 \\ 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} \mathcal{D}_{\text{sign},b} + A_{\mathcal{P}} & 0 \\ 0 & D_{\text{alg}} - \tau_{\text{alg}} D_{\text{alg}} \tau_{\text{alg}} \end{pmatrix}. \tag{13.3}$$

Notice that the boundary operator of (13.3) is not invertible; however we can add a perturbation $A^0_{\widehat{W}}$ to $\mathcal{D}_{\text{sign},\widehat{\text{W}}}$ and make it invertible on all of $\Omega^*(\partial M; \mathcal{V}^\infty) \oplus \widehat{W}$. The resulting operator is thus

$$\begin{pmatrix} (\mathcal{D}_{\text{sign},b})_0 + A^0_{\mathcal{P}} & 0 \\ 0 & \mathcal{D}_{\text{sign},\widehat{\text{W}}} + A^0_{\widehat{W}} \end{pmatrix}. \tag{13.4}$$

Let $A_{\widehat{W}}$ be the corresponding perturbation for $D_{\text{alg}} - \tau_{\text{alg}} D_{\text{alg}} \tau_{\text{alg}}$, constructed as in Section 8. It is clear that the index class associated to the operator

$$\begin{aligned} &\mathcal{D}^{\oplus}_{\text{sign,b}} + \begin{pmatrix} A_{\mathcal{P}} & 0 \\ 0 & A_{\widehat{W}} \end{pmatrix} \\ &\qquad \equiv \begin{pmatrix} \mathcal{D}_{\text{sign},b} + A_{\mathcal{P}} & 0 \\ 0 & D_{\text{alg}} - \tau_{\text{alg}} D_{\text{alg}} \tau_{\text{alg}} + A_{\widehat{W}} \end{pmatrix} \end{aligned} \tag{13.5}$$

has the same Chern character as $\text{Ind}(\mathcal{D}^{+}_{\text{sign},b}, \mathcal{P})$. We denote by $\mathcal{D}^{\oplus}_{\text{sign},b} + A_{\mathcal{P},\widehat{W}}$ the operator in (13.5). Thus

$$\text{ch}(\text{Ind}(\mathcal{D}^{+}_{\text{sign},b}, \mathcal{P})) = \text{ch}(\text{Ind}(\mathcal{D}^{\oplus}_{\text{sign},b} + A_{\mathcal{P},\widehat{W}})^{+}). \tag{13.6}$$

It remains to show that $\text{Ind}((\mathcal{D}^{\oplus}_{\text{sign},b} + A_{\mathcal{P},\widehat{W}})^{+}) = \text{Ind}(\mathcal{D}^{\text{sign,b},+}_{C})$ in $K_0(C^*_r(\Gamma)) \otimes \mathbb{Q}$. To this end, we remark that the boundary operator of $\mathcal{D}^{\oplus,+}_{\text{sign,b}}$, denoted as usual by $(\mathcal{D}^{\oplus}_{\text{sign,b}})_0$, is *invertible* in the middle degree of the Hermitian complex $\Omega(\partial M; \mathcal{V}^\infty) \oplus \widehat{W}$. The notion of spectral section and of symmetric spectral section for $(\mathcal{D}^{\oplus}_{\text{sign,b}})_0$ can be extended to the more general situation considered in Sections 8 and 9. The APS-spectral projections $\Pi_{>}$ for the boundary operator of (13.5) and for the boundary operator of $(\mathcal{D}^{C,+}_{\text{sign,b}})$ in Section 8 are both examples of spectral sections for $(\mathcal{D}^{\oplus}_{\text{sign,b}})_0$. Here we have used the correspondence, explained in detail in [34] and [42], between APS-spectral projections for perturbed Dirac-type operators and spectral sections for unperturbed Dirac-type operators. Let us denote by $\mathcal{P}^{\oplus}$ and $\mathcal{Q}^{C}$ these particular spectral sections. It turns out that $\mathcal{P}^{\oplus}$ and $\mathcal{Q}^{C}$ are in fact *symmetric* spectral sections; this follows from the structure of the above perturbation (see (13.4)) and of the mapping-cone-perturbation of Section 4 (see

(2.7)). More precisely, it is implied by the vanishing in middle degree plus a $\mathbb{Z}_2$-grading outside the middle degree for

$$\begin{pmatrix} A^0_{\mathcal{P}} & 0 \\ 0 & A^0_{\widehat{W}} \end{pmatrix}$$

and for

$$\begin{cases} \begin{pmatrix} 0 & \epsilon\tau_{\partial M}\widehat{g} \\ -\epsilon\widehat{f}\tau_{\partial M} & 0 \end{pmatrix} & \text{if } j < m - \frac{1}{2} \\ \begin{pmatrix} 0 & \epsilon\widehat{g}\tau_{\widehat{W}} \\ -\epsilon\tau_{\widehat{W}}\widehat{f} & 0 \end{pmatrix} & \text{if } j > m - \frac{1}{2}, \end{cases}$$

respectively.

Thus, by definition,

$$\begin{aligned} \operatorname{Ind}((\mathcal{D}^{\oplus}_{\mathrm{sign},b} + A_{\mathcal{P},\widehat{W}})^+) &= \operatorname{Ind}(\mathcal{D}^{\oplus,+}_{\mathrm{sign},b}, \mathcal{P}^{\oplus}), \\ \operatorname{Ind}(\mathcal{D}^{\mathrm{sign,b},+}_{C}) &= \operatorname{Ind}(\mathcal{D}^{\oplus,+}_{\mathrm{sign},b}, \mathcal{Q}^{C})\,. \end{aligned} \tag{13.7}$$

By the relative index theorem of [25, Prop. 6.2], suitably extended to this more general setting, we obtain

$$\operatorname{Ind}(\mathcal{D}^{\oplus,+}_{\mathrm{sign},b}, \mathcal{P}^{\oplus}) - \operatorname{Ind}(\mathcal{D}^{\oplus,+}_{\mathrm{sign},b}, \mathcal{Q}^{C}) = [\mathcal{Q}^{C} - \mathcal{P}^{\oplus}]. \tag{13.8}$$

However, both $\mathcal{P}^{\oplus}$ and $\mathcal{Q}^{C}$ are symmetric spectral sections. Thus by the symmetry argument in [25], see in particular [25, Prop. 4.4], we have $[\mathcal{Q}^{C} - \mathcal{P}^{\oplus}] = 0$ in $K_0(C^*_r(\Gamma)) \otimes \mathbb{Q}$. The claim then follows from (13.6), (13.7) and (13.8). The odd dimensional case is similar, using Cl(1)-symmetric spectral sections.

This proof shows that in the particular case $F = \partial M$ (and $\Gamma$ virtually nilpotent), the two regularizations of the higher eta invariant proposed in [30, Definition 8] and [25, Definition 5.2] coincide. The regularization using symmetric spectral sections can also be given for any closed oriented manifold $F$ and any normal $\Gamma$-cover $F'$ satisfying the assumption that $\Delta_{F'}$ is $L^2$-invertible in middle degree (but with $\Gamma$ still virtually nilpotent). Using the above arguments and an extension of the jump-formula for higher eta-forms [34, Proposition 17], [23, Theorem 5.1], one can show that in this general situation, the two definitions of the higher eta form given in [25, Definition 5.2] and [30, Definition 8] coincide.

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