J. DIFFERENTIAL GEOMETRY
79 (2008) 197-241

NONLINEAR EVOLUTION BY MEAN CURVATURE AND ISOPERIMETRIC INEQUALITIES

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Abstract

Evolving smooth, compact hypersurfaces in $\mathbb{R}^{n+1}$ with normal speed equal to a positive power k of the mean curvature improves a certain 'isoperimetric difference' for $k \geqslant n-1$. As singularities may develop before the volume goes to zero, we develop a weak level-set formulation for such flows and show that the above monotonicity is still valid. This proves the isoperimetric inequality for $n \leqslant 7$. Extending this to complete, simply connected 3-dimensional manifolds with nonpositive sectional curvature, we give a new proof for the Euclidean isoperimetric inequality on such manifolds.

1. Introduction

Let M^n be a smooth n-dimensional compact manifold without boundary and $F_0 : M^n \to N^{n+1}$ a smooth embedding into an n+1-dimensional Riemannian manifold $(N^{n+1}, \bar{g})$. We assume further that $F_0(M)$ has positive mean curvature in N^{n+1}. Starting from such an initial hypersurface there exists, at least for a short time interval $[0, T)$, an evolution $F(\cdot, t) : M^n \times [0, T) \to N^{n+1}$, which satisfies

$$(\star) \qquad \begin{cases} F(\cdot, 0) = F_0(\cdot) \\ \dfrac{dF}{dt}(\cdot, t) = -H^k(\cdot, t)\nu(\cdot, t) \end{cases}$$

where $k \geqslant 1$, H is the mean curvature and ν is the outer unit normal, such that $-H\nu = \mathbf{H}$ is the mean curvature vector. Let $A(t)$ denote the area of such an evolving hypersurface, $V(t)$ the enclosed volume, and c_{n+1} the Euclidean isoperimetric constant. We aim to exploit the following fact, to which G. Huisken has drawn our attention: the 'isoperimetric difference'

$$(1) \qquad A(t)^{\frac{n+1}{n}} - c_{n+1}V(t)$$

Received 06/01/2006.

is monotonically decreasing under such a flow, provided $k \geqslant n-1$ and the inequality

$$\int_{M_t} |\mathbf{H}|^n \, d\mu \geqslant \Big(\frac{n}{n+1}\, c_{n+1}\Big)^n \tag{2}$$

holds on all of the evolving surfaces for $t \in (0,T)$. In the case that $N = \mathbb{R}^{n+1}$ an easy calculation proves this inequality for an arbitrary closed hypersurface which is at least C^2. If $n=2$ and N^3 has nonpositive sectional curvatures, we can use the monotonicity formula to show that (2) holds on any closed hypersurface.

If we assume that the flow $(\star)$ exists until $V(t)$ decreases to zero, the monotonicity of (1) would prove the Euclidean isoperimetric inequality for this initial configuration. Unfortunately, without special geometric assumptions (see [**10**], [**20**]), singularities may develop even before the volume goes to zero. To cope with this problem, we replace $(\star)$ by the following level-set formulation. Let $\Omega \subset N$ be a bounded, open set with smooth boundary $\partial\Omega$, such that $\partial\Omega$ has positive mean curvature. The evolving surfaces are then given as level-sets of a continuous function $u : \bar{\Omega} \to \mathbb{R}^+,\ u = 0$ on $\partial\Omega$ via

$$\Gamma_t = \partial\{x \in \Omega \mid u(x) > t\},$$

and $(\star)$ is replaced by the degenerate elliptic equation

$$\mathrm{div}_N\Big(\frac{\nabla u}{|\nabla u|}\Big) = -\frac{1}{|\nabla u|^{\frac{1}{k}}}. \tag{$\star\star$}$$

Here the left hand side gives the mean curvature of the level-sets and the right-hand side is the speed, raised to the appropriate power. If u is smooth at $x \in \Omega$ with nonvanishing gradient, then $(\star\star)$ implies that the level-sets Γ_t are evolving smoothly according to $(\star)$ in a neighborhood of x. This formulation is inspired by the work of Evans-Spruck [**6**] and Chen-Giga-Goto [**3**] on mean curvature flow and by the work of Huisken-Ilmanen [**11**] on the inverse mean curvature flow.

We use the method of elliptic regularisation to define a family of approximating problems to $(\star\star)$. We prove the existence of smooth solutions to the approximating problem, which by a uniform a-priori gradient bound subconverge to a lipschitz continuous function u on Ω. We define any such limit function u to be a weak solution to $(\star\star)$ and call it a weak H^k-flow generated by Ω. This is justified since such a weak solution solves $(\star\star)$ in the viscosity sense, and even more we show that as long as the smooth solution to $(\star)$ exists, it coincides with any weak solution. In the case that $n \leqslant 6$ and the ambient space is flat, we can show that this weak solution is unique, i.e., it does not depend on the approximating sequence u^{ε_i}.

Theorem 1.1. *Let Ω be a bounded, open subset of N with smooth boundary, such that $H|_{\partial\Omega} > 0$. If $n = 2$, let $k \geqslant 1$ and N be a complete, simply connected 3-manifold with nonpositive sectional curvatures. If $n \geqslant 3$, let $N = \mathbb{R}^{n+1}$ and $k > n$. If u is a weak H^k-flow generated by Ω, then the isoperimetric difference*

$$I_t := \big(\mathcal{H}^n(\partial^*\{u > t\})\big)^{\frac{n+1}{n}} - c_{n+1}\mathcal{H}^{n+1}(\{u > t\})$$

is a nonnegative, monotonically decreasing function on $[0, T)$, where $T := \sup_\Omega u$.

Here we denote with $\mathcal{H}^l$ the l-dimensional Hausdorff-measure. We can use this monotone quantity to give a proof of the isoperimetric inequality in $\mathbb{R}^{n+1}$ for $n \leqslant 7$. The same technique also works if the ambient manifold is simply connected and complete with nonnegative sectional curvatures, which gives a new proof of the result by B. Kleiner in [**17**].

Corollary 1.2 (Isoperimetric inequality)**.** *Let $U \subset \mathbb{R}^{n+1}$ be a compact domain with smooth boundary and $n + 1 \leqslant 8$, or $U \subset N^3$, where N^3 is as above. Then*

$$\big(\mathcal{H}^n(\partial U)\big)^{\frac{n+1}{n}} \geqslant c_{n+1}\mathcal{H}^{n+1}(U). \tag{3}$$

In the case that N^3 has sectional curvatures bounded above by $-\kappa$, $\kappa \geqslant 0$, define the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$f_\kappa(A) := \int_0^A \frac{a^{\frac{1}{2}}}{(16\pi + 4\kappa a)^{\frac{1}{2}}}\, da. \tag{4}$$

Let $\Omega \subset N^3$ be open and bounded and u be weak H^k-flow with initial condition Ω. We show that under the restriction that all superlevelsets $\{u > t\}$ minimize area from the outside in N, the quantity

$$I_t^\kappa := f_\kappa(\mathcal{H}^2(\partial^*(\{u > t\})) - \mathcal{H}^3(\{u > t\})$$

is a nonnegative, monotonically decreasing function for $t \in [0, T)$. This enables us to give a proof of the following stronger result, which also already appeared in [**17**].

Theorem 1.3. *Let N^3 be a complete, simply connected, 3-dimensional Riemannian manifold with sectional curvatures bounded above by $-\kappa \leqslant 0$. If $U \subset N^3$ is a compact domain with smooth boundary, then*

$$f_\kappa\big(\mathcal{H}^2(\partial U)\big) \geqslant \mathcal{H}^3(U).$$

Moreover, equality holds for geodesic balls in the model space N^3_κ with constant sectional curvature $-\kappa$.

Mean curvature flow in the level set formulation was developed in [**6**] and [**3**], see also [**15**]. G. Huisken and T. Ilmanen developed a weak level set formulation for the inverse mean curvature flow to prove the Riemannian Penrose inequality in [**11**].

As already mentioned before, B. Kleiner proved the Euclidean isoperimetric inequality on a complete, simply connected 3-manifold with nonpositive sectional curvatures in [**17**]. In the case that the ambient manifold is 4-dimensional, the corresponding result was proven by C. Croke, [**4**].

The monotonicity of (1) under mean curvature flow for $n = 1, 2$ was also observed by P. Topping. In [**24**] he uses the monotonicity under curve shortening flow to prove optimal isoperimetric inequalities on 2-surfaces. Utilizing the monotonicity under mean curvature flow of 2-surfaces, he gives sufficient geometric conditions for the formation of singularities under this flow.

Similar monotonicities of area and volume under the affine normal flow were employed by B. Andrews in [**2**] to give new proofs of affine isoperimetric inequalities.

Outline. In §2, we show how to derive the monotonicity of (1) in the case that the flow is smooth. In §3 we define by elliptic regularization the ε-regularized version of ($\star\star$). Using barrier techniques we prove uniform a-priori sup and gradient bounds, which we apply to show existence of solutions u^ε to the regularized problem. Also, by these a-priori bounds the solutions u^ε subconverge as $\varepsilon \to 0$ to a lipschitz-continuous function u on Ω which we define to be a weak solution to ($\star\star$).

In §4 we establish that a weak solution satisfies an avoidance principle w.r.t. smooth H^k-flows. For flat ambient space with $n + 1 \leqslant 7$ we use this to prove uniqueness.

The approximating solutions u^ε have the important geometric property that, scaled appropriately, they constitute a smooth, graphical, translating solution to the H^k-flow in $\Omega \times \mathbb{R}$. This can be applied to obtain an approximation of the weak H^k-flow by smooth flows in one dimension higher.

In §5 we refine our understanding of this approximation and use it to prove properties of the weak limit flow. These properties include that the weak flow is non-fattening, i.e., the sets $\{u = t\} \subset \mathbb{R}^{n+1}$ do not develop positive $\mathcal{H}^{n+1}$-measure. We also show that the sets $\{u > t\}$ minimize area from the outside in Ω. As another consequence, the level sets $\{u = t\}$ are actually quite nice, i.e., for $k > n - 1$ and almost every t they are $C^{1,\alpha}$-hypersurfaces up to a closed set of $\mathcal{H}^n$-measure zero.

In §6, we prove that the estimate (2) holds on Γ_t for a.e. t, provided $k > n$ and the ambient space is flat. To do this, we first replace Γ_t by an outer equidistant hypersurface to the convex hull of $\{u > t\}$. Since such a hypersurface is convex and $C^{1,1}$ we can apply the same proof as in the smooth case to show (2) on this hypersurface. The biggest chunk of work in this chapter is then to translate this estimate back to Γ_t. The main ingredient there is an estimate on the growth of the area of the equidistant hypersurfaces in terms of an integral of the mean curvature of Γ_t. Here again we use the approximation by smooth translating flows in $\Omega \times \mathbb{R}$. In the case that $n = 2$ and the ambient space is not flat, we give a proof of (2) which uses the monotonicity formula and thus needs much less regularity of Γ_t. The stronger estimate needed for Theorem 1.3 is proved by combining the techniques in the flat case with the Gauss-Bonnet formula.

Finally, in §7 we use the approximation by smooth flows in $\Omega \times \mathbb{R}$ together with the estimate from §6 and a lower semicontinuity argument to show that the monotonicity of (1) holds in the limit. This is then applied to yield a proof of the stated isoperimetric inequalities. The restriction on the dimension of the ambient space in the flat case comes from the problem that to start the flow, we have to replace a bounded set $U \subset \mathbb{R}^{n+1}$ with smooth boundary by its outer minimizing hull, which is only known to be smooth, more precisely $C^{1,1}$, for $n \leqslant 6$. For $n > 6$ the outer minimizing hull can have singularities on the part away from the obstacle U. For $n = 7$ these singularities are still isolated and an argument of R. Hardt and L. Simon can be applied to show that we can perturb U slightly such that its outer minimizing hull again is $C^{1,1}$.

We want to especially thank G. Huisken for many stimulating discussions and support. We also want to thank K. Ecker and T. Ilmanen for further discussions and support. Finally we want to thank B. White for bringing to our attention the argument of Hardt-Simon above.

2. The smooth Case

Given a smooth solution to $(\star)$, not necessarily compact, we first compute the evolution equations of geometric quantities like the induced metric g_{ij}, the induced measure $d\mu$, and the mean curvature H.

Lemma 2.1. *The following evolution equations hold.*

i) $\frac{\partial}{\partial t} g_{ij} = -2H^k h_{ij}$,

ii) $\frac{\partial}{\partial t} H = kH^{k-1}\Delta H + k(k-1)H^{k-2}|\nabla H|^2 + |A|^2 H^k + \overline{R}ic(\nu,\nu)H^k$,

iii) $\frac{\partial}{\partial t} d\mu = -H^{k+1} d\mu$.

Proof. All of the above follows from a direct calculation as for example in [**10**].

In the case that we have a smooth solution of closed hypersurfaces in $\mathbb{R}^{n+1}$ to $(\star)$, we first demonstrate how to show the monotonicity of (1) as claimed in the introduction. Here

$$c_{n+1} = \left((n+1)^{n+1}\omega_{n+1}\right)^{\frac{1}{n}}$$

is the Euclidean isoperimetric constant, ω_{n+1} denoting the volume of the unit ball in $\mathbb{R}^{n+1}$. To do this, let us first prove estimate (2) for an arbitrary closed hypersurface $M \subset \mathbb{R}^{n+1}$ which is at least $C^{1,1}$. Let M^+ be the intersection of M with the boundary of its outer convex hull. Since the unit normal map ν (let us always choose the outer unit normal), restricted to M^+, covers $\mathbb{S}^n$ at least once, we can estimate

$$(5)\quad |\mathbb{S}^n| \leqslant \int_{M^+} \nu^* do_{S^n} = \int_{M^+} G\, d\mathcal{H}^n \leqslant \frac{1}{n^n}\int_{M^+} H^n\, d\mathcal{H}^n \leqslant \frac{1}{n^n}\int_M |\mathbf{H}|^n\, d\mathcal{H}^n,$$

which is (2). Here G denotes the Gauss curvature and we used that on M^+ all principal curvatures are nonnegative; thus we can apply the arithmetic-geometric mean inequality in the second estimate. Note that for $k \geqslant n-1$ this implies by Hölder

$$(6)\qquad n^n(n+1)\,\omega_{n+1} \leqslant \Big(\int_M |H|^{k+1} d\mathcal{H}^n\Big)^{\frac{n}{k+1}} |M|^{1-\frac{n}{k+1}}\ .$$

Now use the evolution equations and the above estimate to calculate

$$\begin{aligned}(7)\quad -\frac{d}{dt}V = \int_{M_t} H^k\, d\mathcal{H}^n &\leqslant \Big(\int_{M_t} H^{k+1}\, d\mathcal{H}^n\Big)^{\frac{k}{k+1}} A^{\frac{1}{k+1}} \\ &\quad\cdot \frac{1}{n}\big((n+1)\omega_{n+1}\big)^{-\frac{1}{n}}\Big(\int_{M_t} H^{k+1}\, d\mathcal{H}^n\Big)^{\frac{1}{k+1}} A^{\frac{1}{n}-\frac{1}{k+1}} \\ &\leqslant \frac{1}{n}\big((n+1)\omega_{n+1}\big)^{-\frac{1}{n}}\int_{M_t} H^{k+1}\, d\mathcal{H}^n\cdot A^{\frac{1}{n}} = -\frac{1}{c_{n+1}}\frac{d}{dt}A^{\frac{n+1}{n}}.\end{aligned}$$

Rearranging, this implies $\frac{d}{dt}I(t) \leqslant 0$.

If the ambient manifold N is 3-dimensional and has nonpositive sectional curvatures, it needs some more work to prove (2). If $M \subset N$ is a closed hypersurface which is diffeomorphic to a sphere and at least $C^{1,1}$ one can use the Gauss-Bonnet formula, see (70). In Lemma 6.7 we give a proof which works without any restriction on the topology. The proof uses a variant of the monotonicity formula, and thus needs much less regularity of M. The calculation (7) also applies in this setting to show that $I(t)$ is decreasing in time.

3. Elliptic regularisation

To define a weak solution of ($\star\star$) we apply an approximation scheme known as elliptic regularisation. Similar techniques to show the existence of weak solutions, often in the viscosity sense, have been used by various authors, see [**6**], [**11**], [**15**]. We define the following approximating equation.

$$(\star\star)_\varepsilon \qquad \begin{cases} \operatorname{div}_N\Big(\frac{\nabla u^\varepsilon}{\sqrt{\varepsilon^2+|\nabla u^\varepsilon|^2}}\Big) = -\big(\varepsilon^2+|\nabla u^\varepsilon|^2\big)^{-\frac{1}{2k}} & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega \end{cases}$$

We can give this equation a geometric interpretation. It implies that the downward translating graphs

$$N_t^\varepsilon := \operatorname{graph}\bigg(\frac{u^\varepsilon(x)}{\varepsilon} - \frac{t}{\varepsilon}\bigg), \qquad -\infty < t < \infty$$

solve the H^k-flow $(\star)$ smoothly in the manifold $\Omega \times \mathbb{R}$. To verify this, define the function

$$U^\varepsilon(x,z) := u^\varepsilon(x) - \varepsilon z, \qquad (x,z) \in \Omega \times \mathbb{R}, \tag{8}$$

such that $\{U^\varepsilon = t\} = N_t^\varepsilon$. If we assume smoothness, one can check that U^ε satisfies $(\star\star)$ on $\Omega \times \mathbb{R}$ if and only if u^ε satisfies $(\star\star)_\varepsilon$ on Ω.

Let us now assume that the solutions u^ε converge in a suitable sense to a weak solution u with level sets $\{u = t\}$. The geometric idea in this approximation then is that the possibly singular evolution of $\{u = t\}$ is well approximated by the evolution of N_t^ε, in the sense that $N_t^\varepsilon \approx \{u = t\} \times \mathbb{R}$ for sufficiently small $\varepsilon > 0$.

To show the existence of solutions to $(\star\star)_\varepsilon$ we first have to prove a-priori sup- and gradient-bounds.

Lemma 3.1. *Let u^ε be a smooth solution to $(\star\star)_\varepsilon$. Then for $0 < \varepsilon \leqslant 1$,*

$$\sup_\Omega |u^\varepsilon| \leqslant C(n,k,\operatorname{diam}(\Omega)). \tag{9}$$

Proof. Our aim is to construct a supersolution. So pick a $p_0 \in N$ with $\operatorname{dist}(p_0,\Omega) = 1$. Let $S_r := \partial B(p_0,r)$. Since N has nonpositive sectional curvatures, it is diffeomorphic via the exponential map at p_0 to $\mathbb{R}^{n+1}$ and the hypersurfaces S_r are smooth. The evolution of the mixed second fundamental form is given by

$$\frac{\partial}{\partial r} h^i{}_j = -h^i{}_k h^k{}_j - R_0{}^i{}_{0j} \geqslant -h^i{}_k h^k{}_j.$$

Since S_r is convex for small r, this implies that it remains so for all $r > 0$. Taking the trace, we see that the mean curvature H of S_r satisfies

$$\frac{\partial}{\partial r} H = -|A|^2 - \operatorname{Ric}(\nu,\nu) \geqslant -H^2.$$

Using that $\lim_{r\to 0} H = +\infty$ and integration implies

$$H(p) \geqslant \frac{1}{r} \tag{10}$$

for $p \in S_r$. We make the ansatz $\Phi(p) = \psi(r)$, where $r(x) = \mathrm{dist}(x, p_0)$, and compute

$$\begin{aligned}\mathrm{div}_N\Big(\frac{\nabla\Phi}{\sqrt{\varepsilon^2 + |\nabla\Phi|^2}}\Big) &= \frac{\psi'}{\sqrt{\varepsilon^2 + (\psi')^2}}\,\Delta r + \bar{g}\Big(\nabla\Big(\frac{\psi'}{\sqrt{\varepsilon^2 + (\psi')^2}}\Big), \nabla r\Big)\\ &= \frac{\psi'}{\sqrt{\varepsilon^2 + (\psi')^2}}\,H(p) + \frac{\varepsilon^2\psi''}{\big(\varepsilon^2 + (\psi')^2\big)^{3/2}},\end{aligned}$$

which should be less than $-(\varepsilon^2+(\psi')^2)^{-1/(2k)}$ for Φ to be a supersolution. Let us assume that $\psi' \leqslant 0$. We apply (10) to see that a sufficient condition is that

$$\frac{1}{r} \geqslant -\frac{\varepsilon^2\psi''}{\psi'\big(\varepsilon^2 + (\psi')^2\big)} - \frac{1}{\psi'}\big(\varepsilon^2 + (\psi')^2\big)^{\frac{k-1}{2k}}. \tag{11}$$

Now assume $\Omega \subset B(p_0, R_0)$ for some R_0 large enough. Let $\sigma > 0$ be a constant still to be chosen and take $\psi = \sigma(k+1)^{-1}(R_0^{k+1} - r^{k+1})$, which gives

$$\psi' = -\sigma r^k, \qquad \psi'' = -\sigma k r^{k-1}.$$

The inequality (11) then becomes

$$\frac{1}{r} \geqslant -\frac{\varepsilon^2 k}{r(\varepsilon^2 + \sigma^2 r^{2k})} + \frac{1}{\sigma r^k}(\varepsilon^2 + \sigma^2 r^{2k})^{\frac{k-1}{2k}}.$$

Dropping the first term on the RHS, a sufficient condition again is

$$1 \geqslant \frac{1}{\sigma}\Big(\frac{\varepsilon^2}{r^2k} + \sigma^2\Big)^{\frac{k-1}{2k}} = \frac{1}{\sigma^{\frac{1}{k}}}\Big(\frac{\varepsilon^2}{\sigma^2 r^{2k}} + 1\Big)^{\frac{k-1}{2k}}.$$

Since $r \geqslant 1$ on Ω, we can choose σ large enough such that the above condition is satisfied for all $p \in \Omega$ and $\varepsilon \leqslant 1$. Thus Φ is a positive supersolution on Ω for all $0 < \varepsilon \leqslant 1$. By the maximum principle, we obtain the desired sup-bound. Since every solution to $(\star\star)_\varepsilon$ has to be non-negative this implies also a bound on $|u^\varepsilon|$. q.e.d.

For the gradient estimate we aim to apply a maximum principle for $|\nabla u^\varepsilon|$. To do this let $f : \Omega \to \mathbb{R}$ be a smooth function and consider

$$M = \mathrm{graph}(f)$$

as a hypersurface in $N \times \mathbb{R}$, where $N \times \mathbb{R}$ is equipped with the metric $\tilde{g} = \bar{g} \oplus dz^2$. Let ν be the upward pointing unit normal of M and let us take $\tau = \frac{\partial}{\partial z}$ as the unit vector pointing into the upward $\mathbb{R}$-direction. Then define

$$v = \big(\tilde{g}(\nu, \tau)\big)^{-1}.$$

As in the Euclidean case, $v((p, f(p))) = \sqrt{1 + |\nabla f(p)|^2}$ for all $p \in \Omega$. We now compute $\Delta^M v$ at a point $q \in M$. Let $e_1, \dots, e_{n+1}$ be a local framing of TM around q which is orthonormal at q. We can furthermore assume that $\nabla^M_v e_i = 0$ for all $v \in T_qM$ and $i = 1, \dots, n+1$. Then at q we have

$$\nabla^M v = -v^2 \tilde{g}(\bar{\nabla}_{e_i}\nu, \tau)e_i = -v^2 h_{ij}\tilde{g}(e_j, \tau)e_i \ ,$$

where assume summation on i, j. Thus

$$\begin{aligned}
(12) \qquad \Delta^M v &= \operatorname{div}^M(\nabla^M v) = \tilde{g}(\nabla^M_{e_k}\nabla^M v, e_k) \\
&= \tilde{g}\bigg(\Big(-2v\frac{\partial v}{\partial e_k}h_{ij}\tilde{g}(e_j,\tau) - v^2\nabla^M_{e_k}h_{ij}\tilde{g}(e_j,\tau) \\
&\quad + v^2 h_{ij}h_{kj}\tilde{g}(\nu,\tau)\Big)e_i, e_k\bigg) \\
&= \frac{2}{v}|\nabla^M v|^2 - v^2\tilde{g}(\nabla^M H, \tau) - v^2\widetilde{\operatorname{Ric}}_{\nu k}\tilde{g}(e_k,\tau) + v|A|^2,
\end{aligned}$$

where we used the Codazzi equations from the second to the third line. Note that $\widetilde{\operatorname{Ric}}(\nu, e_k) = \operatorname{Ric}_N(\operatorname{pr}_{T_qN}(\nu), \operatorname{pr}_{T_qN}(e_k))$, and we can further assume that $e_1, \dots, e_n \perp \tau$. Then take

$$\gamma := \frac{\operatorname{pr}_{T_qN}(\nu)}{|\operatorname{pr}_{T_qN}(\nu)|},$$

which is well-defined if $\nu \neq \tau$. Let us for the moment assume that $\nu \neq \tau$. Thus

$$\operatorname{pr}_{T_qN}(\nu) = \sqrt{1 - 1/v^2}\ \gamma \ , \qquad \operatorname{pr}_{T_qN}(e_{n+1}) = \pm\frac{1}{v}\,\gamma,$$

and

$$\widetilde{\operatorname{Ric}}_{\nu k}\tilde{g}(e_k,\tau) = \widetilde{\operatorname{Ric}}(\nu, e_{n+1})\tilde{g}(e_{n+1},\tau) = -\frac{1}{v}\Big(1 - \frac{1}{v^2}\Big)\operatorname{Ric}_N(\gamma,\gamma).$$

This expression vanishes if $\nu = \tau$, which is the right value of the expression in (12). Putting everything together we arrive at

$$(13)\ \ \Delta^M v = \frac{2}{v}|\nabla^M v|^2 - v^2\tilde{g}(\nabla^M H, \tau) + v|A|^2 + v\Big(1 - \frac{1}{v^2}\Big)\operatorname{Ric}_N(\gamma,\gamma).$$

Lemma 3.2. *For any smooth solution u^ε of $(\star\star)_\varepsilon$ the following gradient estimate holds.*

$$\sup_\Omega |\nabla u^\varepsilon| \leqslant \exp(k\, C_R \sup_\Omega u^\varepsilon) \cdot \sup_{\partial\Omega}\big(1 + \sqrt{\varepsilon^2 + |\nabla u^\varepsilon|^2}\big),$$

where $C_R := -\inf\{Ric_N(\zeta,\zeta) \mid \zeta \in T_pN,\ |\zeta| = 1,\ p \in \Omega\}$.

Proof. Examining (13) we see that we can hope to use the maximum principle for $v = \sqrt{1 + |\nabla u^\varepsilon|^2}$ if we can somehow control the last term on the RHS. Recall that for a smooth solution of $(\star\star)$ the gradient bound corresponds to a positive lower bound of the mean curvature of the level sets $\{u = t\} = M_t$. Computing the evolution equation of $\phi(x,t) = \exp(\eta t)H(x,t)$ for $\eta = \sqrt{\eta}\, C_R^{\frac{k+1}{2}}$ we see that

$$H_{\min}(t) \geqslant H_{\min}(0)\exp(-\eta t),$$

which implies a gradient bound, given an a-priori height bound. Following this idea we compute $\Delta^M(wv)$ where

$$w((p,z)) = \exp(-\eta z)$$

on $\Omega \times \mathbb{R}$ and $\eta > 0$ to be chosen later. A direct computation gives

$$\Delta^M w = \eta^2\Big(1 - \frac{1}{v^2}\Big)w + \eta\frac{H}{v}w, \qquad \nabla^M w = -\eta w\Big(\tau - \frac{1}{v}\nu\Big). \tag{14}$$

Combining this with (13):

$$\begin{aligned}\Delta^M(wv) &= w\Delta^M v + v\Delta^M w + \frac{2}{v}\tilde{g}\big(\nabla^M v, \nabla^M(wv)\big)\\ &\quad - \frac{2w}{v}|\nabla^M v|^2\\ &= \frac{2}{v}\tilde{g}\big(\nabla^M v, \nabla^M(wv)\big) + wv\Big(|A|^2 + \eta^2\Big(1 - \frac{1}{v^2}\Big)\\ &\quad + \Big(1 - \frac{1}{v^2}\Big)\mathrm{Ric}_N(\gamma,\gamma) + \eta\frac{H}{v} - v\tilde{g}\big(\nabla^M H, \tau\big)\Big).\end{aligned} \tag{15}$$

Now define

$$C_1 := \sup_{\partial\Omega}\sqrt{\varepsilon^2 + |\nabla u^\varepsilon|}$$

and assume that

$$\sup_{\Omega}\Big(\exp\big(-\frac{\eta}{\varepsilon}u^\varepsilon\big)\sqrt{\varepsilon^2 + |\nabla u^\varepsilon|^2}\Big) > \max\{C_1, 1\} \tag{16}$$

which has to be attained at an interior point. Let us take

$$M = \mathrm{graph}\Big(\frac{u^\varepsilon}{\varepsilon}\Big).$$

Note that equation $(\star\star)_\varepsilon$ implies that

$$H = \frac{1}{\varepsilon^{\frac{1}{k}} v^{\frac{1}{k}}}, \tag{17}$$

where H is the mean curvature of M. Now (16) implies that wv attains an interior maximum at point p_0 on S, which is strictly bigger than $\max\{C_1, 1\}/\varepsilon$. Furthermore, by (14) and (17)

$$-wv^2\tilde{g}\big(\nabla^M H, \tau\big) = \frac{1}{k}\varepsilon^{-\frac{1}{k}}v^{1-\frac{1}{k}}\tilde{g}\big(\nabla^M(wv), \tau\big) + \frac{1}{k}\varepsilon^{-\frac{1}{k}}\eta\, wv^{2-\frac{1}{k}}\Big(1 - \frac{1}{v^2}\Big).$$

Thus at p_0 we have by (15)

$$\begin{aligned}0 &\geqslant |A|^2 + \Big(1 - \frac{1}{v^2}\Big)\mathrm{Ric}_N(\gamma,\gamma) + \eta^2\Big(1 - \frac{1}{v^2}\Big) + \varepsilon^{-\frac{1}{k}}\eta v^{-1-\frac{1}{k}}\\ &\quad + \frac{1}{k}\varepsilon^{-\frac{1}{k}}\eta\, v^{1-\frac{1}{k}}\Big(1 - \frac{1}{v^2}\Big)\\ &\geqslant \Big(1 - \frac{1}{v^2}\Big)\Big(\mathrm{Ric}_N(\gamma,\gamma) + \frac{\eta}{k\varepsilon}(\varepsilon v)^{1-\frac{1}{k}}\Big).\end{aligned} \tag{18}$$

If we now choose $\frac{\eta}{\varepsilon} = k \cdot C_R$ we arrive at a contradiction since at p_0 we have $\varepsilon v > w^{-1}\max\{C_1, 1\} \geqslant 1$. q.e.d.

Lemma 3.3. *Let* $\min_{\partial\Omega} H_{\partial\Omega} := \delta_0 > 0$ *and* $0 < \delta_1 \leqslant \delta_0/(2C_R)$ *be such that* $d(p) := dist(p, \Omega)$ *is smooth on* $\Omega_{\delta_1} = \{p \in \Omega \mid d(p) < \delta_1\}$. *Let* $0 < \varepsilon < \varepsilon_0$ *where* $\varepsilon_0 := \min\{C_2, 1\}$, $C_2 := \sup\{2^{\frac{3k-1}{2}}\delta_0^{-k}, \delta_1^{-1}C_1\}$ *and* C_1 *is the a-priori bound on* $\sup_\Omega |u^\varepsilon|$ *from Lemma* 3.1. *Then any smooth solution* u^ε *of* $(\star\star)_\varepsilon$ *satisfies the estimate*

$$\sup_{\partial\Omega} |\nabla u^\varepsilon| \leqslant C_2. \tag{19}$$

Proof. We construct a barrier at the boundary, and since $u^\varepsilon \geqslant 0$ we only need a barrier from above. To construct a suitable supersolution Φ, we try the ansatz $\Phi(p) = \beta \cdot d(p)$ for a constant $\beta > 0$. Observe that on Ω_{δ_1} we have $\Delta d = -H_{S_r}$, where H_{S_r} is the mean curvature of the hypersurfaces $S_r := \{d = r\}$. Computing as in the proof of Lemma 3.1 we find that a sufficient condition for Φ to be a supersolution on Ω_δ is that

$$H_{S_r} \geqslant \frac{1}{\beta}(\varepsilon^2 + \beta^2)^{\frac{k-1}{2k}}, \tag{20}$$

for all $0 < r < \delta$ and a suitable $0 < \delta < \delta_1$. The evolution equation of H_{S_r} along a geodesic is given by

$$\frac{\partial}{\partial r} H_{S_r} = |A_{S_r}|^2 + \mathrm{Ric}(\nu, \nu) \geqslant -C_R,$$

which implies

$$H_{S_r} \geqslant H_{\partial\Omega} - C_R r \geqslant \delta_0 - C_R r \geqslant \frac{\delta_0}{2}$$

for $0 \leqslant r \leqslant \delta_1$. Assuming that $\varepsilon \leqslant \beta$, a sufficient condition to fulfill (20) is that $\beta \geqslant 2^{\frac{3k-1}{2}}\delta_0^{-k}$. If we further assume that $\beta \geqslant C_1/\delta_1$ we can ensure that

$$\Phi \geqslant u^\varepsilon$$

on S_{δ_1}. Thus, by the maximum principle, $\Phi \geqslant u^\varepsilon$ on Ω_{δ_1}, which gives the desired gradient estimate. q.e.d.

To show the existence of solutions to $(\star\star)_\varepsilon$ we study solutions to the following family of equations

$$(\star\star)_{\varepsilon,\kappa} \qquad \begin{cases} \mathrm{div}_N\Big(\frac{\nabla u^{\varepsilon,\kappa}}{\sqrt{\varepsilon^2 + |\nabla u^{\varepsilon,\kappa}|^2}}\Big) = -\kappa\big(\varepsilon^2 + |\nabla u^{\varepsilon,\kappa}|^2\big)^{-\frac{1}{2k}} & \text{in } \Omega \\ u^{\varepsilon,\kappa} = 0 & \text{on } \partial\Omega \end{cases}$$

for $0 \leqslant \kappa \leqslant 1$ and $0 < \varepsilon < \varepsilon_0$. In the following, we show that for any fixed $0 < \varepsilon < \varepsilon_0$ we have uniform a-priori sup and gradient estimates in κ. Since $\kappa \leqslant 1$ it is easy to check that (9) and (19) also hold for smooth solutions of $(\star\star)_{\varepsilon,\kappa}$:

$$\sup_\Omega |u^{\varepsilon,\kappa}| \leqslant C(n, k, \mathrm{diam}(\Omega)), \qquad \sup_{\partial\Omega} |\nabla u^{\varepsilon,\kappa}| \leqslant C_2, \tag{21}$$

for all $0 \leqslant \kappa \leqslant 1$. Here C_2 is the constant from Lemma 3.3, where we assume the same conditions on $\partial\Omega$. For the interior gradient estimate, fix an $\varepsilon \in (0, \varepsilon_0)$. Let us work on the hypersurface

$$M = \operatorname{graph}\Big(\frac{u^{\varepsilon,\kappa}}{\varepsilon}\Big).$$

Equation $(\star\star)_{\varepsilon,\kappa}$ then implies that the mean curvature H of S is given by

$$H = \frac{\kappa}{\varepsilon^{\frac{1}{k}} v^{\frac{1}{k}}}.$$

As in the proof of Lemma 3.1 we obtain that at an interior maximum of v, we have the inequality (compare (18))

$$\begin{aligned} 0 &\geqslant |A|^2 + \Big(1 - \frac{1}{v^2}\Big)\mathrm{Ric}_N(\gamma,\gamma) + \eta^2\Big(1 - \frac{1}{v^2}\Big) + \kappa\, \varepsilon^{-\frac{1}{k}} \eta v^{-1-\frac{1}{k}} \\ &\quad + \kappa\, \frac{1}{k} \varepsilon^{-\frac{1}{k}} \eta\, v^{1-\frac{1}{k}} \Big(1 - \frac{1}{v^2}\Big) \\ &\geqslant \Big(1 - \frac{1}{v^2}\Big)\big(\mathrm{Ric}_N(\gamma,\gamma) + \eta^2\big), \end{aligned}$$

which gives a contradiction if $\eta > \sqrt{C_R}$ and $v > 1$. This yields the interior estimate

$$\sup_\Omega |\nabla u^{\varepsilon,\kappa}| \leqslant \exp\big(\sqrt{C_R}\, \varepsilon^{-1} \sup_\Omega u^{\varepsilon,\tau}\big) \cdot \sup_{\partial\Omega} \big(\sqrt{\varepsilon^2 + |\nabla u^{\varepsilon,\tau}|^2}\big), \tag{22}$$

for all $0 \leqslant \kappa \leqslant 1$.

Lemma 3.4. *Under the assumptions of Lemma* 3.3*, a smooth solution to* $(\star\star)_\varepsilon$ *exists.*

Proof. We aim to apply the method of continuity to $(\star\star)_{\varepsilon,\kappa}$, $0 \leqslant \kappa \leqslant 1$. Fix an $\varepsilon \in (0, \varepsilon_0)$ and write $(\star\star)_{\varepsilon,\kappa}$ as

$$F^\kappa(w) := \mathrm{div}_N\Bigg(\frac{\nabla w}{\sqrt{\varepsilon^2 + |\nabla w|^2}}\Bigg) + \kappa\big(\varepsilon^2 + |\nabla w|^2\big)^{-\frac{1}{2k}} = 0,$$

with $w = 0$ on $\partial\Omega$. The map

$$F \,:\, C_0^{2,\alpha}(\bar\Omega) \times \mathbb{R} \to C^\alpha(\bar\Omega),$$

defined by $F(w,\kappa) := F^\kappa(w)$ is C^1 and possesses the solution $F(0,0) = 0$. Let

$$I := \{\kappa \in [0,1] \,|\, (\star\star)_{\varepsilon,\kappa} \text{ has a solution } u^{\varepsilon,\kappa} \in C_0^{2,\alpha}(\bar\Omega)\}.$$

Clearly $0 \in I$. We want to show that I is relatively open and closed.

To see that I is closed we note that the a-priori estimates (21) and (22) imply uniform $C^1(\bar\Omega)$-bounds and thus $(\star\star)_{\varepsilon,\kappa}$ is uniformly elliptic. The Nash-Moser-DeGiorgi estimates then yield uniform bounds in $C^{1,\alpha}(\bar\Omega)$. Applying Schauder estimates we obtain uniform bounds in $C^{k,\alpha}(\bar\Omega)$ for any $k \geqslant 2$. Thus by the Arzela-Ascoli theorem I is closed.

To prove that I is also open we linearize the map F^κ at a solution u. This linearization is given by

$$DF^\kappa|_u : C_0^{2,\alpha}(\bar\Omega) \to C^\alpha(\bar\Omega).$$

Now $F^\kappa(w)$ has the form

$$F^\kappa(w) = \nabla_i A^i(\nabla w) + B(\nabla w),$$

which is independent of w. So by the maximum principle the linearization

$$DF^\kappa|_u(v) = \nabla_i(A^i_{p_j}(\nabla u)\nabla_j v) + B_{p_j}(\nabla u)\nabla_j v$$

has only the zero solution. Using linear existence theory and Schauder estimates up to the boundary, we see that $DF^\kappa|_u$ is an isomorphism. So by the implicit function theorem, the set of κ such that $F(u,\kappa) = 0$ has a solution (namely I) is open. Therefore $1 \in I$, which proves the existence of u^ε in $C^{2,\alpha}$. Smoothness follows again by Schauder estimates. q.e.d.

4. Weak H^k-flow and Uniqueness

Given a connected, open and bounded set $\Omega \subset N$ with smooth boundary, s.t. $H|_{\partial\Omega} > 0$, the results of the last section ensure the existence of smooth solutions u^ε to $(\star\star)_\varepsilon$ for sufficiently small $\varepsilon > 0$. The a-priori estimates guarantee uniform bounds in $C^1(\bar\Omega)$, independent of ε. Thus, given any sequence $\varepsilon_i \to 0$, we can extract a subsequence (again denoted by (ε_i)), such that

$$u^{\varepsilon_i} \to u$$

in $C^0(\bar\Omega)$ to a function $u \in C^{0,1}(\bar\Omega)$. This suggests the following definition:

Definition 4.1. Let $\varepsilon_i \to 0$ and corresponding solutions u^{ε_i} to $(\star\star)_{\varepsilon_i}$ be given. Assume that $u^{\varepsilon_i} \to u$ uniformly on $\bar\Omega$, where u, u^{ε_i} are uniformly bounded in $C^{0,1}(\bar\Omega)$. We then call u a weak H^k-flow with initial condition Ω.

By the reasoning above we have the existence of a weak H^k-flow. We now want to show that such a weak solution actually is unique, i.e., the limiting function u is independent of the approximating sequence u^{ε_i}. To do this we first want to show that any weak H^k-flow coincides with the smooth flow, as long as the latter exists.

Let $F(\cdot,t) : \partial\Omega \times [0,T) \to N$ be the unique solution to $(\star)$ with initial condition $F(\cdot,0) = \mathrm{Id}_{\partial\Omega\to\partial\Omega}$. We may further assume that $T > 0$ is maximal. Comparing with shrinking distance-spheres as in the proof of Lemma 3.1 we see that T is finite. Let us write $M_t = F(\partial\Omega, t)$. As in the beginning of the proof of Lemma 3.2, we furthermore obtain that the mean curvature of the hypersurfaces remains strictly positive. Thus we have that $M_{t_1} \cap M_{t_2} = \emptyset$ if $t_1 \neq t_2$. For $0 < \tau \leqslant T$ define

$$\Omega_\tau = \bigcup_{0<t<\tau} M_t \subset \Omega$$

and $u^* : \Omega_\tau \to \mathbb{R}^+$ by $u^*(p) = t$, if $x \in M_t$.

Lemma 4.2. *Let u be a weak solution with initial condition Ω. Then*

$$u = u^*$$

in Ω_T.

Proof. Choose any $0 < \tau < T$. Then u^* is a smooth solution of $(\star\star)$ on $\overline{\Omega}_\tau$. Our aim is to twist u^* somewhat to construct upper and lower barriers for $(\star\star)_\varepsilon$ for ε small enough. To do this let $\psi : [0, \tau) \to \mathbb{R}^+$ be a smooth, increasing function with $\psi(0) = 0$ and define

$$v(x) = \psi(u^*(x))$$

for $x \in \Omega_\tau$. Since u^* is a solution of $(\star\star)$ we see by a direct calculation that in Ω_τ

$$\operatorname{div}_N\left(\frac{\nabla v}{|\nabla v|}\right) = \frac{1}{|\nabla v|}\left(\delta^{ij} - \frac{\nabla^i v \nabla^j v}{|\nabla v|^2}\right)\nabla_i \nabla_j v = -\frac{(\psi')^{\frac{1}{k}}}{|\nabla v|^{\frac{1}{k}}}\ , \tag{23}$$

where in the second equality we use normal coordinates at $p \in \Omega_\tau$. This implies that v is a subsolution to $(\star\star)$ if $\psi' \leqslant 1$ and a supersolution if $\psi' \geqslant 1$. Using (23) we furthermore have in normal coordinates at a point $p \in \Omega_\tau$ that

$$\begin{aligned}
&\left(\delta^{ij} - \frac{\nabla^i v \nabla^j v}{\varepsilon^2 + |\nabla v|^2}\right)\nabla_i \nabla_j v + \left(\varepsilon^2 + |\nabla v|^2\right)^{\frac{k-1}{2k}} \\
&= \left(\left(\varepsilon^2 + |\nabla v|^2\right)^{\frac{k-1}{2k}} - |\nabla v|^{\frac{k-1}{k}}\right) \\
&\quad + \left(\frac{\nabla^i v \nabla^j v}{|\nabla v|^2} - \frac{\nabla^i v \nabla^j v}{\varepsilon^2 + |\nabla v|^2}\right)\nabla_i \nabla_j v + \left(1 - (\psi')^{\frac{1}{k}}\right)|\nabla v|^{\frac{k-1}{k}}.
\end{aligned} \tag{24}$$

Now on Ω_τ there exists a positive constant $C_1 < \infty$ such that

$$\frac{1}{C_1} \leqslant |\nabla u^*| \leqslant C_1\ .$$

The positive lower bound on the gradient is due to the fact that the flow $F(\cdot, t)$ is smooth up to τ. The upper bound comes from the uniform positive lower bound on the mean curvature of the surfaces M_t.

We first want to construct an upper barrier for $(\star\star)_\varepsilon$. Pick any sufficiently small $\delta > 0$ and take $\psi(r) = (1 + \delta)r$ for $r \in [0, \tau - \delta]$ and continue ψ smoothly in a convex way on $(\tau - \delta, \tau]$, such that

$$\frac{\psi'(\tau)}{C_1} \geqslant \sup_{0\leqslant\varepsilon\leqslant\varepsilon_0}\ \sup_{\Omega} |\nabla u^\varepsilon| + 1 \tag{25}$$

where ε_0 is the constant from Lemma 3.3. Observe that $|\nabla v| = \psi'|\nabla u^*|$. Thus (25) implies that the solutions u^ε cannot touch v in a neighborhood of the inner boundary of Ω_τ. Having fixed ψ, we have the bounds

$$\frac{1}{C_2} \leqslant |\nabla v| \leqslant C_2, \quad |\nabla^2 v| \leqslant C_3 \text{ on } \Omega_\tau$$

for some positive constants C_2, C_3. Since $\psi' \geqslant 1+\delta$, we see from equation (24) that v is a supersolution on Ω_τ of $(\star\star)_\varepsilon$ for sufficiently small ε. As explained above, the solutions u^{ε_i} which converge to u cannot touch v on the inner boundary of Ω_τ, so v acts as an upper barrier for sufficiently small ε_i. Taking the limit $u^{\varepsilon_i} \to u$ we obtain that $u \leqslant v$ on Ω_τ. Letting $\delta \searrow 0$ and $\tau \nearrow T$ we arrive at $u \leqslant u^*$ on Ω_T.

For the lower barrier take again τ, δ as above and let $\psi(r) = (1-\delta)r$ for $r \in [0, \tau - 2\delta]$. The aim is then to continue ψ on $(\tau - 2\delta, \tau]$ in a concave way, such that we can extend v by a constant on $\Omega\backslash\Omega_\tau$ to obtain a C^2-subsolution to $(\star\star)_\varepsilon$ on the whole of Ω. To estimate the RHS of (24) from below we drop the first term and use that $\nabla v = \psi'\nabla u^*$, $\nabla_i\nabla_j v = \psi''\nabla_i u^*\nabla_j u^* + \psi'\nabla_i\nabla_j u^*$. Thus we get a lower estimate of the RHS by

$$\begin{aligned}(26)\quad &\left(1 - \frac{(\psi')^2|\nabla u^*|^2}{\varepsilon^2 + (\psi')^2|\nabla u^*|^2}\right)\Big(\psi''|\nabla u^*|^2 - \psi'|\nabla^2 u^*|\Big)\\ &\quad + (\psi')^{\frac{k-1}{k}}\Big((1-(\psi')^{\frac{1}{k}})|\nabla u^*|^{\frac{k-1}{k}}\Big) \geqslant C_1^2\psi'' - C_3\psi' + \gamma(\psi')^{\frac{k-1}{k}},\end{aligned}$$

where $\gamma > 0$ is a constant depending only on δ and C_1. Thus the RHS is nonnegative on $[\tau-\delta, \tau]$, if

$$\psi' \leqslant \left(\frac{\gamma}{2C_3}\right)^k \quad \text{and} \quad -\psi'' \leqslant \frac{1}{2C_1^2}\gamma(\psi')^{\frac{k-1}{k}}.$$

It is then a direct calculation that the choice $\psi(r) = -\alpha(\tau - r)^{k+1} + b$ satisfies the above constraints for a constant $\alpha > 0$, depending on $k, \delta, \gamma, C_1, C_3$, and b still free to choose. We then adjust b such that we can continue ψ on $[\tau - 2\delta, \tau - \delta]$ smoothly in a concave way. On $\Omega_{\tau-\delta}$ we can again use (24) to see that for small enough ε the function v is a subsolution. Equation (26) then guarantees that this also works on $\Omega_\tau \setminus \Omega_{\tau-\delta}$. We extend v to the whole of Ω by setting $v(p) = b$ for $p \in \Omega\backslash\Omega_\tau$. Thus for $k > 1$, v is a C^2 subsolution of $(\star\star)_\varepsilon$ for sufficiently small ε and thus a lower barrier for u^{ε_i}. Using the a-priori estimates we can take the limit $k \searrow 1$ to see that v also acts as a lower barrier in this case. Then arguing as in the case of the upper barrier we obtain finally that $u \geqslant u^*$ on Ω_T. q.e.d.

Corollary 4.3 (Avoidance of smooth flows)**.** *Let u be a weak H^k-flow with initial condition Ω and $(M_t)_{t_0 \leqslant t \leqslant t_1}$, $t_0 \geqslant 0$, be a smooth, compact H^k-flow with positive mean curvature. Assume that M_{t_0} and u*

are disjoint at t_0, i.e., $M_{t_0} \cap \{u = t_0\} = \emptyset$; then they remain so for all future times, i.e., $M_t \cap \{u = t\} = \emptyset$, $\forall\ t_0 \leqslant t \leqslant t_1$.

Proof. Let Ω' be the bounded and open set in $\mathbb{R}^{n+1}$ such that $M_{t_0} = \partial\Omega'$. We can assume that $\Omega' \subset \Omega$ or $\Omega \subset \Omega'$; otherwise there is nothing to prove. We treat the case that $\Omega' \subset \Omega$, and the other case follows similarly. Let u^{ε_i} be the sequence of solutions to $(\star\star)_{\varepsilon_i}$ converging to u. Then take $\tilde{u}^{\varepsilon_i}$ to be the solutions to $(\star\star)_{\varepsilon_i}$ on Ω', where we take the same sequence $\{\varepsilon_i\}$. We can furthermore assume that $\tilde{u}^{\varepsilon_i} \to \tilde{u}$ uniformly, s.t. $\tilde{u}$ is a weak solution with initial condition Ω'. By Lemma 4.2 we have that

$$M_t = \{\tilde{u} = (t - t_0)\} \tag{27}$$

for $t_0 \leqslant t \leqslant t_1$. Since $u > t_0$ on Ω' also $u^{\varepsilon_i} > t_0$ for sufficiently large i, and thus by the maximum principle $u^{\varepsilon_i} \geqslant \tilde{u}^{\varepsilon_i} + t_0$ which gives

$$u \geqslant \tilde{u} + t_0$$

on Ω'. We can now shift M_{t_0} a little bit forward in time such that M_{t_0} remains disjoint from u and repeat the above argument. This yields

$$u > \tilde{u} + t_0$$

on Ω', which proves the claim, using (27). q.e.d.

By interposing a $C^{1,1}$-hypersurface between two weak H^k-flows, we can argue as it is done in [**14**] for set-theoretic subsolutions to mean curvature flow, that also two weak H^k-flows satisfy the avoidance principle. Since the proof depends on the translational invariance of the flow, it works only if the surrounding space is Euclidean. Furthermore, we can start a smooth H^k-flow only from a $C^{1,1}$-hypersurface if it has nonnegative mean curvature, so we have to restrict ourselves to low dimensions.

Theorem 4.4. *Let $N = \mathbb{R}^{n+1}$, $n \leqslant 6$, and $u, \tilde{u}$ be two weak H^k-flows generated by two open sets $\Omega, \tilde{\Omega} \subset \mathbb{R}^{n+1}$ where at least one of them is bounded. Assume that for $t_1, t_2 \geqslant 0$ we have $\{u = t_1\} \cap \{\tilde{u} = t_2\} = \emptyset$. Then $\{u = (t_1 + \tau)\} \cap \{\tilde{u} = (t_2 + \tau)\} = \emptyset$ for all $\tau \geqslant 0$.*

Proof. We can assume that $\{\tilde{u} \geqslant t_2\} \Subset \{u > t_1\}$. We want to show that $\mathrm{dist}\big(\{\mathrm{u} = (\mathrm{t}_1 + \tau)\}, \{\tilde{\mathrm{u}} = (\mathrm{t}_2 + \tau)\}\big)$ is non-decreasing in τ. Observe that by translational invariance of the H^k-flow Corollary 4.3 proves this if one of the two flows is actually smooth. By [**14**] we know that there exists a closed $C^{1,1}$-hypersurface $S \subset \{u > t_1\} \setminus \{\tilde{u} \geqslant t_2\}$ which separates $\{u = t_1\}$ and $\{\tilde{u} = t_2\}$ such that

$$\mathrm{dist}\big(\{u = t_1\}, \{\tilde{u} = t_2\}\big) = \mathrm{dist}\big(\{u = t_1\}, S\big) + \mathrm{dist}\big(\{\tilde{u} = t_2\}, S\big). \tag{28}$$

Let E be the open set, bounded by S, such that $\{\tilde{u} \geqslant t_2\} \subset E$. Now take $E' \subset \{u > t_1\}$ to be the outer minimizing hull of E in $\{u > t_1\}$, i.e.,

E' is the intersection of all minimizing hulls in $\{u > t_1\}$, which contain E. Here we call a set $F \subset G$, where $G \subset \mathbb{R}^{n+1}$ is open, a minimizing hull in G, if it minimizes area from the outside in G, that is, if

$$|\partial^* F \cap K| \leqslant |\partial^* H \cap K| \tag{29}$$

for any H containing F such that $H \setminus F \Subset G$, and any compact set K containing $H \setminus F$. For details on minimizing hulls, see [**11**], Chapter 1. Since u is weak H^k-flow, all the sets $\{u > t\}$ are minimizing hulls in Ω; see Corollary 5.7, which is proved independently of this uniqueness result. By Corollary 4.3 the sets $\{u = t\}$ cannot develop an interior, thus there is a $\tau > t_1$ such that $E \Subset \{u > \tau\}$. This implies that E' cannot touch $\partial\{u > t_1\}$. Since $\partial E = S$ is $C^{1,1}$ and $n \leqslant 6$, a result of Sternberg, Williams and Ziemer [**23**] implies that $S' := \partial E'$ is also a $C^{1,1}$-hypersurface. Since E' locally minimizes area from the outside, S' carries a nonnegative weak mean curvature which is in L^∞. By an argument of Huisken and Ilmanen, see [**11**], Lemma 2.5, S' can be approximated by a sequence S'_i of smooth hypersurfaces from the inside, which are uniformly controlled in C^2 and with strictly positive mean curvature. Let M^i_t be the smooth evolution along the H^k-flow of the S'_i's. By Proposition 3.9 in [**20**] these flows exist on a uniform time interval $[0, \varepsilon)$, for some $\varepsilon > 0$. We first want to show that

$$\operatorname{dist}\big(\{u = t_1\}, S\}\big) = \operatorname{dist}\big(\{u = t_1\}, S'\}\big).$$

Assume to the contrary that

$$\operatorname{dist}\big(\{u = t_1\}, S\}\big) > \operatorname{dist}\big(\{u = t_1\}, S'\}\big);$$

then the shortest distance has to be attained at a point $p \in S'$ such that S' is a smooth minimal surface around p. Thus the M^i_t move initially near p as slowly as we wish, and note that u has to avoid the flows M^i_t. In addition, the shortest distance from $\{u = t_1\}$ to M^i_0 has to be attained at a sequence of points $p^i \in M^i_0$, which we can assume to converge to p. Translating the M^i_0's such that they touch $\{u = t_1\}$ in p^i, we get by the avoidance principle w.r.t. smooth flows a contradiction to the $C^{0,1}$-bound of u. Thus we can replace S by S' in (28). Now u and $\tilde{u}$ avoid the evolution of all the M^i_t which proves the statement of the theorem in the limit $i \to \infty$. q.e.d.

By shifting the initial condition a little bit in time this implies uniqueness.

Corollary 4.5. *Let $N = \mathbb{R}^{n+1}$, $n \leqslant 6$ and $\Omega \subset \mathbb{R}^{n+1}$ be a bounded, open set with smooth boundary such that $H_{\partial\Omega} > 0$. Then the weak H^k-flow generated by Ω is unique.*

Note that in the case $k = 1$, i.e., mean curvature flow, any weak flow as above, without any restriction on the dimension and on the ambient space, coincides with the level-set flow of $\partial\Omega$ and is thus unique, see [**6**].

5. Further Properties and Regularity

In the following section, let $\Omega \subset N$ be a fixed open and bounded set with smooth boundary such that $H_{\partial\Omega} > 0$. Let $u \in C^{0,1}(\overline{\Omega};\mathbb{R}^+)$ be a weak H^k-flow generated by Ω, i.e., there exists a sequence $\varepsilon_i \searrow 0$ and solutions u^{ε_i} to $(\star\star)_{\varepsilon_i}$ which are uniformly bounded in $C^{0,1}(\overline{\Omega};\mathbb{R}^+)$ converging to u in $C^0(\overline{\Omega})$. Then the hypersurfaces $N_t^i \subset N \times \mathbb{R}$, defined by

$$N_t^i := N_t^{\varepsilon_i} = \operatorname{graph}\left(\frac{u^{\varepsilon_i}}{\varepsilon_i} - \frac{t}{\varepsilon_i}\right),$$

which are level sets $\{U^{\varepsilon_i} = t\}$ of the function $U^{\varepsilon_i}\big((x,z)\big) = u^{\varepsilon_i}(x) - \varepsilon_i z$ on $\Omega \times \mathbb{R}$, are smooth translating solutions of the H^k-flow $(\star)$, see (8). Equation $(\star\star)_{\varepsilon_i}$ implies that the mean curvature H_t^i of N_t^i is given by

$$H_t^i = \frac{1}{\big(\varepsilon_i^2 + |\nabla u^{\varepsilon_i}|^2\big)^{\frac{1}{2k}}}. \tag{30}$$

To fix some further notation define the following subsets of $\Omega \times \mathbb{R}$:

$$E_t^i := \{U^{\varepsilon_i} > t\}, \qquad E_t' := \{U > t\},$$

where $U\big((x,z)\big) = u(x)$ on $\Omega \times \mathbb{R}$. The sets E_t' can be written as $E_t' = E_t \times \mathbb{R}$, where $E_t := \{u > t\} \subset \Omega$. A first observation is that the sets E_t^i are minimizing hulls in $\Omega \times \mathbb{R}$, see (29).

Lemma 5.1. *The sets E_t^i are minimizing area from outside in $\Omega \times \mathbb{R}$, that is*

$$|\partial^* E_t^i \cap K| \leqslant |\partial^* F \cap K|$$

for F with $E_t^i \subset F$, $F \setminus E_t^i \subset K \subset \Omega \times \mathbb{R}$, where K is compact. Here we take the right hand side to be $+\infty$ if F is not a Caccioppoli-set.

Proof. The outward unit normal to the surfaces N_t^i, which is given by $\nu = -\nabla U^{\varepsilon_i}/|\nabla U^{\varepsilon_i}|$ is a smooth vectorfield on $\Omega \times \mathbb{R}$ with $\operatorname{div}(\nu) = |DU^{\varepsilon_i}|^{-1/k} > 0$. Thus we get by the divergence theorem for Caccioppoli-sets, using ν as a calibration:

$$\begin{aligned} 0 \leqslant \int_{F\setminus E_t^i} \operatorname{div}(\nu)\, dx &= -\int_{\partial^* E_t^i \cap K} \nu \cdot \nu_{\partial^* E_t^i}\, d\mathcal{H}^n + \int_{\partial^* F \cap K} \nu \cdot \nu_{\partial^* F}\, d\mathcal{H}^n \\ &\leqslant -|\partial^* E_t^i \cap K| \ + \ |\partial^* F \cap K|, \end{aligned} \tag{31}$$

where we take $\nu_{\partial^* E_t^i}$, $\nu_{\partial^* F}$ to be the outward unit normals to $\partial^* E_t^i$, $\partial^* F$. q.e.d.

Corollary 5.2 (Mass Bound)**.** *Let $\Omega' := \Omega \times [a,b]$. Then*

$$|N_t^i \cap \Omega'| \leqslant (b-a)\ \mathcal{H}^n(\partial\Omega) + 2\mathcal{H}^{n+1}(\Omega) \tag{32}$$

for all $-\infty < t < \infty$.

Proof. Let $\Omega_j \Subset \Omega$ be a sequence of open sets, such that $\partial\Omega_j \to \partial\Omega$ in C^1. Then $F_j := \big(\Omega_j \times (b-a)\big) \cup E_t^i$ are valid comparison sets, and the above lemma gives the estimate for $j \to \infty$. q.e.d.

We can use this a-priori mass bound and the lower bound on the mean curvature together with evolution equations to deduce space-time bounds, independent of ε_i.

Lemma 5.3. *Let $I = [a,b] \subset \mathbb{R}$ be a bounded interval. Then*

$$\int_{-\infty}^{+\infty}\int_{N_t^i \cap (\Omega\times I)} (H_t^i)^{k+1}\, d\mathcal{H}^{n+1}\, dt \leqslant (b-a)\,\mathcal{H}^n(\partial\Omega) + 2\mathcal{H}^{n+1}(\Omega). \tag{33}$$

Proof. Observe that by the Coarea formula

$$\int_{\Omega\times I} |DU^{\varepsilon_i}|^{-\frac{1}{k}}\, dx = \int_{-\infty}^{+\infty}\int_{N_t^i \cap (\Omega\times I)} (H_t^i)^{k+1}\, d\mathcal{H}^{n+1}\, dt.$$

We can then use (31) and argue as in in the proof of Corollary 5.2. q.e.d.

Lemma 5.4. *Let $k > 3$, $I = [a,b] \subset \mathbb{R}$ be a bounded interval and $\Omega' \Subset \Omega$. Then*

$$\int_{-\infty}^{+\infty}\int_{N_t^i \cap (\Omega'\times I)} |\nabla H_t^i|^2\, d\mathcal{H}^{n+1}\, dt \leqslant C(\Omega,\Omega',I,k).$$

Proof. Choose $\phi \in C_c^2(\Omega\times\mathbb{R}), 0 \leqslant \phi \leqslant 1$, such that $\phi = 1$ on $\Omega' \times I$ and let $\alpha > 0$. By the evolution equation for the mean curvature and integration by parts we can compute

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{N_t^i} \phi (H_t^i)^{-\alpha}\, d\mathcal{H}^{n+1} \\
&= \int_{N_t^i} (-\alpha)(1+\alpha)k\phi(H_t^i)^{k-\alpha-3}|\nabla H_t^i|^2 \\
&\quad + \alpha k (H_t^i)^{k-\alpha-2}\langle \nabla H_t^i, \nabla\phi\rangle - \alpha\phi(H_t^i)^{k-\alpha-1}(|A|^2 + \overline{\mathrm{R}}\mathrm{ic}(\nu,\nu)) \\
&\quad - \langle \nabla\phi, \nu\rangle (H_t^i)^{k-\alpha} - \phi(H_t^i)^{k+1-\alpha}\, d\mathcal{H}^{n+1}.
\end{aligned}$$

We can estimate the first term in the second line as follows:

$$\langle \nabla H_t^i, \nabla\phi\rangle (H_t^i)^{k-\alpha-2} \leqslant \frac{1}{2}\frac{|\nabla\phi|^2}{\phi}(H_t^i)^{k-\alpha-1} + \frac{1}{2}\phi(H_t^i)^{k-\alpha-3}|\nabla(H_t^i)|^2.$$

Since $|\nabla\phi|^2/\phi \leqslant C(\phi)$ we can apply this estimate and integrate from t_1 to t_2 to arrive at

$$\begin{aligned}
&\int_{t_1}^{t_2} k\phi(H_t^i)^{k-\alpha-3}|\nabla H_t^i|^2 \, d\mathcal{H}^{n+1}\, dt \\
&\leqslant \frac{2}{\alpha}\int_{N_{t_1}^i} \phi(H_t^i)^{-\alpha}\, d\mathcal{H}^{n+1} - \frac{2}{\alpha}\int_{N_{t_2}^i} \phi(H_t^i)^{-\alpha}\, d\mathcal{H}^{n+1} \\
&\quad + C\int_{t_1}^{t_2}\int_{N_t^i} (H_t^i)^{k-\alpha-1} + (H_t^i)^{k-\alpha}\, d\mathcal{H}^{n+1}\, dt,
\end{aligned}$$

where $C = C(\phi, \alpha^{-1}, k, \sup_\Omega|\overline{\mathrm{R}}\mathrm{ic}|)$. Now for $t_1 \ll -1,\ t_2 \gg 1$ we have $N_{t_1}^i \cap \operatorname{supp}(\phi) = N_{t_2}^i \cap \operatorname{supp}(\phi) = \emptyset$ such that the first two terms on the RHS drop out. Furthermore, the a-priori gradient bound from Lemmata 3.1, 3.2 and 3.3 together with (30) give a uniform positive lower bound on H_t^i. Now choose $\alpha = (k-3)/2$ and use Lemma 5.3 to prove the claimed estimate. q.e.d.

The space-time estimate (33) implies that the measure of the sets E_t is Hölder-continuous in time, which also excludes that the level sets of u can "fatten up":

Lemma 5.5. *The weak H^k-flow u is non-fattening, i.e., $\mathcal{H}^{n+1}(\{u = t\}) = 0$ for all $t \in [0, T]$, where $T = \sup_\Omega u$.*

Proof. Let $\Omega' := \Omega \times (0,1)$. Then for any $t_1, t_2 \in \mathbb{R},\ t_1 < t_2$ we have by the Coarea formula together with (32), (33) and Hölder's inequality

$$\begin{aligned}
&|\mathcal{H}^{n+2}(E_{t_1}^i \cap \Omega') - \mathcal{H}^{n+2}(E_{t_2}^i \cap \Omega')| \\
&= \int_{t_1}^{t_2}\int_{N_t^i\cap\Omega'} (H_t^i)^k \, d\mathcal{H}^{n+1}\, dt \\
&\leqslant |t_2 - t_1|^{\frac{1}{k+1}} \left(\int_{t_1}^{t_2} \left(\int_{N_t^i\cap\Omega'} (H_t^i)^k \, d\mathcal{H}^{n+1} \right)^{\frac{k+1}{k}} dt \right)^{\frac{k}{k+1}} \\
&\leqslant C\,|t_2 - t_1|^{\frac{1}{k+1}} \left(\int_{t_1}^{t_2} \int_{N_t^i\cap\Omega'} (H_t^i)^{k+1} \, d\mathcal{H}^{n+1}\, dt \right)^{\frac{k}{k+1}} \\
&\leqslant C|t_2 - t_1|^{\frac{1}{k+1}},
\end{aligned}$$

where the constant C does not depend on i. Observe that since $U^{\varepsilon_i} \to U$ locally uniformly on $\Omega \times \mathbb{R}$, we have that

$$E_t^i \to E_t' \tag{34}$$

in L^1_{loc}, provided that $\mathcal{H}^{n+2}\{U=t\}=0$. Thus (34) holds for all t up to a countable set $S=\{t\in[0,T]\mid \mathcal{H}^{n+2}\{U=t\}>0\}$. Taking the limit we have

$$|\mathcal{H}^{n+2}(E'_{t_1}\cap\Omega')-\mathcal{H}^{n+2}(E'_{t_2}\cap\Omega')|\leqslant C|t_2-t_1|^{\frac{1}{k+1}}$$

for all $t_1,t_2\in\mathbb{R}\backslash S$. Now let $t_0\in S$ and pick sequences $t_j^-\nearrow t_0$, $t_j^+\searrow t_0$ where $t_j^-,t_j^+\in\mathbb{R}\setminus S$. Since

$$E'_{t_j^-}\to\{U\geqslant t_0\},\quad E'_{t_j^+}\to\{U>t_0\}$$

this implies that $\mathcal{H}^{n+2}\{U=t_0\}=0$, and thus $S=\emptyset$. q.e.d.

We have seen before that the sets E^i_t are minimizing area from outside in $\Omega\times\mathbb{R}$. We now want to show that this property passes to the limit; moreover, we show that this is always preserved under L^1_{loc}-convergence.

Lemma 5.6. *Let $U\subset\mathbb{R}^{n+1}$ be open and $E_h\subset U$ a sequence of Caccioppoli-sets in U, which converge in $L^1_{loc}(U)$ to $E\subset U$ such that $|\partial^*E_h\cap K|\leqslant C(K)$ for all $K\subset U$, K compact, independently of h. If all the E_h are minimizing area from outside in U then so does E.*

Proof. Let $E\subset F$ with $F\backslash E\subset K\subset U, K$ compact. Since $F\cup E_h\to E$ and $E_h\to E$ in $L^1_{\mathrm{loc}}(U\setminus K)$ there exists a compact set $K'\subset U$ with $K\subset\mathrm{int}(K')$ with smooth boundary $\partial K'$ such that

$$|\partial^*(F\cup E_h)\cap\partial K'|=|\partial^*(F\cap E_h)\cap\partial K'|=|\partial^*E_h\cap\partial K'|=0 \tag{35}$$
$$\text{for all } h \text{ and } \int_{\partial K'}|\varphi^-_{F\cup E_h}-\varphi^+_{E_h}|\,d\mathcal{H}^n\to 0.$$

Here $\varphi^-_{F\cup E_h}$, $\varphi^+_{E_h}$ denote the inner, resp. outer, trace of $F\cup E_h$ and E_h on $\partial K'$. We can assume w.l.o.g that $|\partial^*E_h\cap U|\leqslant C$ for all h, and we obtain for $F_h:=E_h\cup(F\cap K')$, compare [**7**], Prop. 2.8:

$$|\partial^*F_h\cap U|=|\partial^*E_h\cap(U\backslash K')|+\int_{\partial K'}|\varphi^-_{F\cup E_h}-\varphi^+_{E_h}|\,d\mathcal{H}^n+|\partial^*(F\cup E_h)\cap K'|.$$

The set F_h is a valid comparison set for E_h, thus $|\partial^*E_h\cap U|\leqslant|\partial^*F_h\cap U|$, which yields

$$|\partial^*(F\cup E_h)\cap K'|\geqslant|\partial^*E_h\cap K'|-\int_{\partial K'}|\varphi^-_{F\cup E_h}-\varphi^+_{E_h}|\,d\mathcal{H}^n.$$

Recall the general inequality

$$|\partial^*(E_1\cup E_2)\cap A|+|\partial^*(E_1\cap E_2)\cap A|\leqslant|\partial^*E_1\cap A|+|\partial^*E_2\cap A|, \tag{36}$$

which holds for any two Caccioppoli-sets E_1,E_2 in U, and A any open subset of U. By (35) we can apply this with $E_1=E_h$, $F=E_2$ and

$A = K'$ to get

$$|\partial^* F \cap K'| \geqslant |\partial^*(E_h \cap F) \cap K'| - \int_{\partial K'} |\varphi^-_{F\cup E_h} - \varphi^+_{E_h}|\, d\mathcal{H}^n.$$

Since $E_h \cap F \to E$ in L^1_{loc} we can use lower semicontinuity and (35) to pass to limits:

$$|\partial^* F \cap K'| \geqslant |\partial^* E \cap K'|.$$

q.e.d.

Corollary 5.7. *The sets E'_t are minimizing area from the outside in $\Omega \times \mathbb{R}$ for all $t \in (0, T)$. As well, the sets E_t are minimizing area from the outside in Ω for all $t \in (0, T)$.*

Proof. The proof of Lemma 5.6 also works if we replace $\mathbb{R}^{n+1}$ by $N \times \mathbb{R}$. The first statement follows from Lemma 5.5. For the second statement let F be a valid comparison set for E_t in Ω, i.e., $E_t \subset F$, $F \setminus E_t \subset K \subset \Omega$, K compact. Define $F' := (F \times (-l, l)) \cup E'_t$, which is a valid comparison set for E'_t. Thus, for $K' = K \times [-l + 1, l + 1]$ we have $|\partial^* E'_t \cap K'| \leqslant |\partial^* F' \cap K'|$, i.e.,

$$2l|\partial^* E_t \cap K| \leqslant 2l|\partial^* F \cap K| + 2\mathcal{H}^{n+1}(F \setminus E_t).$$

Taking the limit $l \to \infty$ proves the second statement. q.e.d.

Corollary 5.8. *The function $t \mapsto |\partial^* E_t|,\ t \in [0, T)$, is monotonically decreasing.*

In the following we want to show the convergence of the hypersurfaces

$$N^i_t \to \Gamma_t \times \mathbb{R}$$

for a.e. t in the sense of measures, where

$$\Gamma_t := \partial\{u > t\} \subset \Omega \subset N,$$

i.e., $\Gamma_t \times \mathbb{R} = \partial\{U > t\}$. So define Radon measures on $\Omega \times \mathbb{R}$ by

$$\mu^i_t := \mathcal{H}^{n+1} \llcorner N^i_t\ , \qquad \mu_t := \mathcal{H}^{n+1} \llcorner \partial^* E'_t.$$

To prove the convergence $\mu^i_t \to \mu_t$ we exploit the property that the sets E^i_t minimize area from the outside. We first want to define a set $B \subset [0, T]$ of times where we can expect that such a convergence holds true. Observe that we have $\partial^* E_t \subset \Gamma_t \subset \{u = t\}$ for all t.

Lemma 5.9. *There is a set $B \subset [0, T]$ of full measure, s.t.*

$$\mathcal{H}^n(\{u = t\} \setminus \partial^* E_t) = 0$$

for all $t \in B$.

Proof. Since u is in $C^{0,1}(\bar{\Omega}) \subset BV(\Omega)$ we can compare the Coarea-formula for BV-functions and Lipschitz-functions to get

$$\int_0^T \mathcal{H}^n(\partial^* E_t)\, dt = \int_\Omega |Du|\, d\mathcal{H}^{n+1} = \int_0^T \mathcal{H}^n(\{u = t\})\, dt.$$

Since the integrals are finite, this yields

$$\int_0^T \mathcal{H}^n(\{u = t\} \setminus \partial^* E_t)\, dt = 0,$$

which implies the statement. q.e.d.

Thus $\mu_t = \mathcal{H}^{n+1} \llcorner (\Gamma_t \times \mathbb{R})$ for all $t \in B$. Moreover, μ_t is $(n+1)$-rectifiable for all $t \in B$.

Proposition 5.10. *For all $t \in B$, $\mu_t^i \to \mu_t$ in the sense of Radon measures.*

Proof. We give the proof only in the case $N = \mathbb{R}^{n+1}$. Since it uses only local techniques, it is straightforward to see that the same proof, with some minor modifications, works also for a general N.
Fix a $t \in B$. By the mass bound (32) we can extract a subsequence $\mu_t^{i_j}$ such that $\mu_t^{i_j} \to \mu$, where μ is a Radon measure on $\Omega \times \mathbb{R}$.

Claim 1: $\operatorname{supp}(\mu) \subset \{u = t\} \times \mathbb{R}$.
Let $x \in \Omega \times \mathbb{R}$, $x \notin \{u = t\} \times \mathbb{R}$, i.e., $U(x) \neq t$. Let us assume $U(x) > t$. Thus there is a $\delta > 0$ such that $B_\delta(x) \Subset \Omega \times \mathbb{R}$ and $U(y) > t$ for all $t \in \bar{B}_\delta(x)$. So for i sufficiently large $U^{\varepsilon_i} > t$ on $B_\delta(x)$, i.e., $N_t^i \cap B_\delta(x) = \emptyset$ which implies $\mu_t^i(B_\delta(x)) = 0$ for large enough i. So $x \notin \operatorname{supp}(\mu)$.

Claim 2: Let $B_\rho(x) \Subset \Omega \times \mathbb{R}$. Then

$$\frac{\mu\big(\bar{B}_\rho(x)\big)}{\omega_{n+1}\rho^{n+1}} \leqslant (n+2)\frac{\omega_{n+2}}{\omega_{n+1}}.$$

We have $\mu(B_\rho(x)) \leqslant \liminf_{j\to\infty} \mu_t^{i_j}(B_\rho(x))$. Using that $\mu_t^{i_j}(B_\rho(x)) = |\partial^* E_t^{i_j} \cap B_\rho(x)|$ and the $E_t^{i_j}$ minimize area from the outside, we obtain by comparison with $E_t^{i_j} \cup B_\rho(x)$:

$$\mu_t^{i_j}(B_\rho(x)) \leqslant (n+2)\, \omega_{n+2}\rho^{n+1}.$$

Thus for $\varepsilon > 0$:

$$\mu\big(\bar{B}_\rho(x)\big) \leqslant \mu(B_{\rho+\varepsilon}(x)) \leqslant (n+2)\, \omega_{n+2}(\rho+\varepsilon)^{n+1},$$

which proves the claim for $\varepsilon \to 0$.

The second claim establishes that μ is absolutely continuous w.r.t to $\mathcal{H}^{n+1}$-measure. By Claim 1, Lemma 5.9, and the differentiation theorem for Radon measures there is a function $\theta \in L^\infty(\Gamma_t \times \mathbb{R}, \mathcal{H}^{n+1})$ such that we can write

$$\mu = \mu_t \llcorner \theta. \tag{37}$$

Claim 3: $\theta \geqslant 1$ $\mathcal{H}^{n+1}$-a.e. on $\Gamma_t \times \mathbb{R}$.
By the differentiation theorem

$$\theta(x) = \lim_{\rho\to 0} \frac{\mu(B_\rho(x))}{\mu_t(B_\rho(x))}, \tag{38}$$

$\mathcal{H}^{n+1}$-a.e. Let $x \in \Gamma_t \times \mathbb{R}$ such that this holds. Now for all but at most countably many ρ, provided $B_\rho(x) \Subset \Omega \times \mathbb{R}$, we have

$$\mu(B_\rho(x)) = \lim_{j\to\infty} \mu_t^{i_j}(B_\rho(x)).$$

On the other hand $\mu_t^{i_j} = \mathcal{H}^{n+1} \llcorner \partial^* E_t^{i_j}$ and by lower semicontinuity

$$\mu_t(B_\rho(x)) = |\partial^* E'_t \cap B_\rho(x)| \leqslant \liminf_{j\to\infty} |\partial^* E_t^{i_j} \cap B_\rho(x)|.$$

Thus $\theta \geqslant 1$ $\mathcal{H}^{n+1}$-a.e. on $\Gamma_t \times \mathbb{R}$.

Claim 4: $\theta(x) \leqslant 1$ for almost all $x \in \partial^* E'_t$.

Here we use again the property that the sets $E_t^{i_j}$ are minimizing area from the outside. Let $x \in \partial^* E'_t$ such that (38) holds. By a translation we can assume that $x = 0$. Since $x \in \partial^* E'_t$, we know that as $\lambda \to 0$, the rescalings $\lambda^{-1}\partial^* E'_t \to T_x\partial^* E'_t$ in the sense of Radon measures, and $\lambda^{-1}E'_t \to H$ in L^1_{loc} where H is one of the halfspaces bounded by $T_x\partial^* E'_t$. By a rotation we can assume that $T_x\partial^* E'_t = \{x^{n+2} = 0\}$ and $H = \{x^{n+2} < 0\}$. Let $\varepsilon > 0$ be given, and choose a $\lambda > 0$ such that

$$\left|\theta(x) - \frac{\mu(B_\lambda(x))}{\omega_{n+1}\lambda^{n+1}}\right| \leqslant \varepsilon. \tag{39}$$

We can do this since $\Theta^{n+1}(\partial^* E'_t, x) = 1$ and by (38). Then define the δ-slab

$$S_\delta := \{x \in \mathbb{R}^{n+2} \mid |x_{n+2}| \leqslant \delta\}.$$

By adjusting λ maybe even further, we can assume that

$$\mathcal{H}^{n+1}(\lambda^{-1}\partial^* E'_t \cap (B_2 \setminus S_\varepsilon)) \leqslant \varepsilon \frac{\omega_{n+1}}{(n+2)\omega_{n+2}} \tag{40}$$

and

$$\mathcal{H}^{n+2}\big((\lambda^{-1}E'_t \setminus H) \cup (H \setminus \lambda^{-1}E'_t)\big) \leqslant \varepsilon^2. \tag{41}$$

By Claim 1 we have $\theta \leqslant (n+2)\omega_{n+2}/\omega_{n+1}$, $\mathcal{H}^{n+1}$-a.e. and thus (40) implies

$$\mu_\lambda(B_2 \setminus S_\varepsilon) \leqslant \varepsilon \tag{42}$$

where μ_λ is the rescaling of μ by the factor λ^{-1}, defined by $\mu_\lambda(A) = \lambda^{-(n+1)}\mu(\lambda A)$. We then choose i (dropping the subscript t) big enough such that

$$\left|\mu_\lambda^{i_j}(B_2 \setminus S_\varepsilon) - \mu_\lambda(B_2 \setminus S_\varepsilon)\right| \leqslant \varepsilon\ , \qquad \left|\mu_\lambda^{i_j}(B_2) - \mu_\lambda(B_2)\right| \leqslant \varepsilon \tag{43}$$

and

$$\mathcal{H}^{n+2}\big((\lambda^{-1}E'_t \setminus \lambda^{-1}E_t^{i_j}) \cup (\lambda^{-1}E_t^{i_j} \setminus \lambda^{-1}E'_t)\big) \leqslant \varepsilon^2. \tag{44}$$

Combining this with (42) and (41) we get

$$\mu_\lambda^{i_j}(B_2 \setminus S_\varepsilon) \leqslant 2\varepsilon \tag{45}$$

and

$$\mathcal{H}^{n+2}\big((\lambda^{-1}E_t^{i_j} \setminus H) \cup (H \setminus \lambda^{-1}E_t^{i_j})\big) \leqslant 2\varepsilon^2. \tag{46}$$

In other words, up to a set of small measure, $\lambda^{-1}E_t^{i_j}$ looks like the halfspace H on B_2. We now employ that $\lambda^{-1}E_t^{i_j}$ is minimizing area from the outside to get an upper bound on the area contained in the slab $S_\varepsilon \cap B_1$. Now by (46) there is a $\delta \in [\varepsilon, 2\varepsilon]$ such that

$$\mathcal{H}^{n+1}\Big(\big(\{x_{n+2} = -\delta\} \cap (\lambda^{-1}E_t^{i_j})^C\big) \cap B_2\Big) \leqslant 2\varepsilon. \tag{47}$$

We take as comparison set F,

$$F := \lambda^{-1}E_t^{i_j} \cup \big(S_\delta \cap B_1\big).$$

Since $\lambda^{-1}E_t^{i_j}$ is minimizing area from the outside, together with (47), (45), this implies

$$\mu_\lambda^{i_j}(B_1) \leqslant \omega_{n+1} + 4\varepsilon n\omega_{n+1} + 4\varepsilon,$$

which in turn gives with (43) that

$$\mu_\lambda(B_1) \leqslant \omega_{n+1} + 4\varepsilon n\omega_{n+1} + 5\varepsilon.$$

Finally, applying this to (39), we arrive at

$$\theta(x) \leqslant 1 + (4n + 5/\omega_{n+1} + 1)\,\varepsilon,$$

which proves the claim.

To finally prove that $\mu = \mu_t$ we combine Claim 3 and Claim 4 to see that $\theta = 1\ \mathcal{H}^{n+1}$-a.e. on $\Gamma_t \times \mathbb{R}$, and use (37). Thus the limit measure μ does not depend on the subsequence, so the whole sequence converges, i.e., $\mu_t^i \to \mu_t$. q.e.d.

Recall that we have the uniform space-time bound, see (33),

$$\int_0^T\!\!\int_{\Omega'} (H_t^i)^{k+1}\, d\mu_t^i\, dt \leqslant (b-a)\mathcal{H}^n(\partial\Omega) + 2\mathcal{H}^{n+1}(\Omega),$$

where $\Omega' := \Omega \times (a,b)$. Thus by Fatou's lemma

$$\liminf_{i\to\infty} \int_{\Omega'} (H^i_t)^{k+1}\, d\mu^i_t < \infty \tag{48}$$

for almost all $t \in [0,T]$. Then we let

$$\tilde{B} := \{t \in B \mid (48) \text{ holds for any bounded interval } (a,b)\}.$$

Note that $\tilde{B}$ again has full measure. Thus, for every $t \in \tilde{B}$ there is subsequence (i_j) such that by the mass bound and Hölder's inequality

$$\sup_{j\geqslant 0} \int_A |H^{i_j}_t|\, d\mu^{i_j}_t \leqslant C\big(\mu^{i_j}_t(A)\big)^{\frac{k}{k+1}} < C(\Omega) \tag{49}$$

for every $A \subset \Omega'$. By the compactness theorem for $(n+1)$-rectifiable varifolds of Allard, there is a further subsequence (which we again denote by (i_j)) such that $N^{i_j}_t$ converges in the sense of varifolds to a limit, which again is $(n+1)$-rectifiable. Since $\mu^i_t \to \mu_t$ this implies

$$N^{i_j}_t \to \Gamma_t \times \mathbb{R}$$

in the sense of varifolds, where we see $\Gamma_t \times \mathbb{R}$ as a $(n+1)$-rectifiable, unit density varifold. The estimate (49) then implies that the total variation $\delta(\Gamma_t \times \mathbb{R})$ is absolutely continuous w.r.t. μ_t, i.e., $\Gamma_t \times \mathbb{R}$ carries a weak mean curvature $\mathbf{H}$, and thus by the product structure also Γ_t. The varifold convergence now implies that

$$\mu^i_t \llcorner (-H^i_t \nu^i_t) \to \mu_t \llcorner \mathbf{H}$$

in the sense of vector valued Radon measures. So by lower semicontinuity results for convex functionals of Hutchinson, see [**13**],

$$\int_{\Omega'} |\mathbf{H}|^{k+1}\, d\mu_t \leqslant \liminf_{i\to\infty} \int_{\Omega'} (H^i_t)^{k+1}\, d\mu^i_t.$$

So again by Fatou's lemma

$$\begin{aligned}\int_0^T\!\!\int_{\Omega\times(a,b)} |\mathbf{H}|^{k+1}\, d\mu_t\, dt &\leqslant \liminf_{i\to\infty} \int_0^T\!\!\int_{\Omega'} (H^i_t)^{k+1}\, d\mu^i_t\, dt \\ &\leqslant (b-a)\mathcal{H}^n(\partial\Omega) + 2\mathcal{H}^{n+1}(\Omega).\end{aligned}$$

This implies

$$\int_0^T\!\!\int_{\Omega\cap\Gamma_t} |\mathbf{H}|^{k+1}\, d\mathcal{H}^n\, dt \leqslant \mathcal{H}^n(\partial\Omega). \tag{50}$$

Applying Allard's regularity theorem, we can summarize the above in the following regularity and approximation result.

Theorem 5.11. *There is a set $\tilde{B} \subset [0,T]$ of full measure, such that for all $t \in \tilde{B}$ the following is true.*

i) $\mathcal{H}^n(\Gamma_t \setminus \partial^* E_t) = 0$.
ii) *For* $k > n-1$ *the surfaces* Γ_t *are up to a closed set* $A_t \subset \Gamma_t$, *with* $\mathcal{H}^n(A_t) = 0$, *in* $C^{1,1-\frac{k+1}{n}}$.
iii) *There is a subsequence* (i_j), *depending on* t, *such that* $N_t^{i_j} \to \Gamma_t \times \mathbb{R}$ *in the sense of varifolds. If* $k > n$, *then away from the set* $A_t \times \mathbb{R}$, *this convergence is in* $C^{1,\alpha}$ *for any* $0 < \alpha < 1 - \frac{k+1}{n+1}$.

6. The main estimate

In this section we show that the estimate (2) is valid for almost all $t \in [0,T]$ if $N = \mathbb{R}^{n+1}$. The estimate in case N is a complete, simply-connected 3-manifold with nonpositive sectional curvatures will be given at the end of the section.

Our aim is to transfer the computation presented in (5) for the smooth case to the setting of lower regularity of solutions of the weak flow. Assume that $k > n$. By Theorem 5.11 and Lemma 5.4 we know that there is a set $\tilde{B} \subset [0,T]$ of full measure, such that for $t \in \tilde{B}$ the following statements are true: Up to a closed set of $\mathcal{H}^n$-measure zero, the set $\Gamma_t = \partial\{u > t\}$ is a $C^{1,\alpha}$ hypersurface, which carries a weak mean curvature in $L^{k+1}(\Gamma_t)$. We can also assume that there is a sequence $\varepsilon_i \to 0$ such that

$$N_t^i \to \Gamma_t \times \mathbb{R}$$

in the sense of varifolds, and

$$\limsup_{i\to\infty} \int_{N_t^i \cap (\Omega\times I)} |H_i|^{k+1} + |\nabla H_i|^2 \, d\mu_t^{\varepsilon_i} \; < \; \infty, \tag{51}$$

for I a bounded interval. We will in the following always abbreviate $H_i := H_t^i$ where there is no danger of ambiguity. Our aim is to do all the computations on equidistant hypersurfaces to Γ_t and then pass to limits. For this purpose, define for $s > 0$

$$L_s(\Gamma_t) = \partial\{x \in \mathbb{R}^n \mid \operatorname{dist}(x, E_t) \leqslant s\}.$$

Lemma 6.1. *Let* $\phi \in C_c^0(\mathbb{R}^{n+2})$. *Then for all* $p < k+1$

$$\int_{N_t^i} \phi H_i^p \, d\mathcal{H}^{n+1} \; \to \int_{\Gamma_t\times\mathbb{R}} \phi |\mathbf{H}|^p \, d\mathcal{H}^{n+1}. \tag{52}$$

Proof. By the results stated above we know that

$$N_t^i \to \Gamma_t \times \mathbb{R}$$

not only in the sense of varifolds, but also by Allard's theorem away from the singular set $\operatorname{sing}(\Gamma_t) \times \mathbb{R}$ locally uniformly in $C^{1,\alpha}$. Furthermore,

$\mathrm{sing}(\Gamma_t) \times \mathbb{R}$ is closed and has $\mathcal{H}^{n+1}$-measure zero. So given any $\delta > 0$ there is a neighborhood S of $\mathrm{sing}(\Gamma_t) \times \mathbb{R}$ such that

$$\limsup_{i\to\infty} \left| \int_{N_t^i \cap S} \phi H^p \, d\mathcal{H}^{n+1} \right| \leqslant C \sup |\phi| \ \delta. \tag{53}$$

Here we used the uniform L^{k+1}-estimate on H from (51). Now outside S the convergence of the hypersurfaces is locally uniform in $C^{1,\alpha}$. So it suffices to check (52) locally, such that $\Gamma_t \times \mathbb{R}$ and N_t^i can be written as converging graphs (with bounded gradient) over a fixed hyperplane. Since the hypersurfaces converge as varifolds, and we have local convergence in C^1, the mean curvature H_i converges weakly to H. The uniform L^2-estimate on ∇H_i from (51) gives that $H_i \to H$ in L^2, and by interpolation in L^p for every $p < k+1$. Using a suitable partition of unity and (53) we get the claimed convergence. q.e.d.

To be able later to control the convergence as $s \to 0$ we need to control the local area growth of the hypersurfaces L_s in the parameter s.

Lemma 6.2. *For almost all $s > 0$ the following statement is true: Let $K \subset L_s(\Gamma_t)$ be $\mathcal{H}^n$-measurable and define $P(K) := \{x \in \Gamma_t \mid \exists\ y \in K \text{ with } |y - x| = s\}$. Then*

$$\mathcal{H}^n(K) \leqslant \int\limits_{P(K)\cap\Gamma_t} \Big(1 + \frac{1}{n+1} \, |\mathbf{H}(x)| \, s\Big)^{n+1} \, d\mathcal{H}^n(x).$$

Remark 6.3. Note that the above estimate is in principle the Heintze-Karcher estimate [**9**]. This estimate gives the above inequality when Γ_t is a smooth hypersurface, where in the integrand $(n+1)$ is everywhere replaced by n. Since in our case the hypersurfaces Γ_t may be singular, we do the estimate on the approximating hypersurfaces N_t^i in one dimension higher and then show that we can pass to limits. Since we have to work with $(n+1)$-dimensional hypersurfaces in $\Omega\times\mathbb{R}$, we only get the estimate with $(n+1)$ replacing n in the integrand. Furthermore, if it would be possible to obtain the original Heintze-Karcher estimate for our possibly singular hypersurfaces Γ_t, then the estimate (62) would follow immediately by comparing with the area of big spheres.

Proof. We can assume that K is compact, and so $P(K)$ is compact as well. Define for $\gamma, \eta > 0$

$$\begin{aligned} K_\gamma &:= \{x \in \mathbb{R}^{n+1} \mid \mathrm{dist}(x, K) < \gamma\}, \\ P_\eta &:= \{x \in \mathbb{R}^{n+1} \mid \mathrm{dist}(x, P(K)) < \eta\}. \end{aligned}$$

Claim 1: For all $\eta > 0$ there exists a $\gamma_0 > 0$ such that $P(K_\gamma \cap L_s(\Gamma_t)) \subset P_\eta$ for all $\gamma \leqslant \gamma_0$.

Assume to the contrary that there are points $y_i \in L_s(\Gamma_t) \setminus K$ with $y_i \to y_\infty \in K$ and points $x_i \in \Gamma_t$ with $|y_i - x_i| = s$, but $\mathrm{dist}(x_i, P(K)) \geqslant \eta$. We can assume that $x_i \to x_\infty \in \Gamma_t$. Thus $|y_\infty - x_\infty| = s$, but $x_\infty \notin P(K)$. This proves Claim 1.

Since the distance function to E_t is lipschitz, we can argue as in Lemma 5.9 that for a.e. s the set $\{x \in \Omega \mid \mathrm{dist}(x, E_t) < s\}$ is a Caccioppoli-set and

$$\partial^*\{x \in \Omega \mid \mathrm{dist}(x, E_t) < s\} = L_s(\Gamma_t) \tag{54}$$

up to $\mathcal{H}^n$-measure zero. A further thing to note is that by (51) the mean curvature H^{ε_i} is uniformly bounded in $L^p(N_t^i)$ for some $p > n+1$. This gives that also

$$N_t^i \to \Gamma_t \times \mathbb{R}$$

locally in Hausdorff-distance, which in turn implies that

$$\{z \in \Omega \times \mathbb{R} | \ \mathrm{dist}(z, \tilde{E}_t^i) < s\} \to \{x \in \Omega \mid \mathrm{dist}(x, E_t) < s\} \times \mathbb{R}$$

in L^1_{loc}. Then using the lower semicontinuity of the BV-norm we deduce, note (54),

$$\mathcal{H}^{n+1}((K_\gamma \times I) \cap (L_s(\Gamma_t \times \mathbb{R}))) \leqslant \liminf_{i\to\infty} \mathcal{H}^{n+1}((K_\gamma \times I) \cap L_s(N_t^i)), \tag{55}$$

for any bounded interval I. We now want to apply a result of Li and Nirenberg [**18**], or equivalently Itoh and Tanaka [**16**], which says: Given a bounded open set $S \subset \mathbb{R}^{n+2}$ with smooth boundary, define G to be the largest open subset of S such that every point x in G has a unique closest point on ∂S. Then the set $\Sigma(S) := S \setminus G$ has finite $\mathcal{H}^{n+1}$-measure. Furthermore, for every $x \in G$ the distance function to the boundary is smooth.

Since the sets $\mathbb{R}^{n+2} \setminus \tilde{E}_t^i$ have a smooth boundary and converge locally in Hausdorff-distance to $\mathbb{R}^{n+2} \setminus (E_t \times \mathbb{R})$ we can apply the above result to deduce that the sets $\Sigma_i \subset \mathbb{R}^{n+2} \setminus \tilde{E}_t^i$, defined as above have locally finite $\mathcal{H}^{n+1}$-measure. Thus for almost all $s \in (0,1)$

$$\mathcal{H}^{n+1}(L_s(N_t^i) \cap \Sigma_i) = 0 \ \text{ for all } i \in \mathbb{N}. \tag{56}$$

Now pick an $s_0 \in (0,1)$ such that (55) and (56) hold. Let $x_0 \in L_{s_0}(N_t^i) \setminus \Sigma_i$. Then there is neighborhood of x_0 such that every point has a unique closest point on N_t^i and the distance to N_t^i is smooth. Even more, the same is true for a neighborhood of the line l connecting x_0 to its closest point on N_t^i. We compute along l

$$\frac{\partial}{\partial s} H = -|A|^2 \leqslant -\frac{1}{(n+1)} H^2 \tag{57}$$

where H is the mean curvature of $L_s(N_t^i)$ along l. Comparing with the solution of the ODE yields

$$H(s) \leqslant \max\Big(\frac{(n+1)H(0)}{H(0)\,s+(n+1)}, 0\Big) = \frac{(n+1)H(0)}{H(0)\,s+(n+1)}, \tag{58}$$

where the last equality holds since $H(0) > 0$. The evolution of the measure along l is given by

$$\frac{\partial}{\partial s} d\mu = H(s)\, d\mu,$$

which can be integrated to

$$d\mu(s) = \exp\bigg(\int_0^s H(\tau)\, d\tau\bigg)\, d\mu(0). \tag{59}$$

Inserting the estimate (58)

$$\int_0^s H(\tau)\, d\tau \leqslant \int_0^s \frac{(n+1)H(0)}{H(0)\,\tau+(n+1)}\, d\tau = \log\bigg(\Big(1+\frac{s}{(n+1)}\,H(0)\Big)^{n+1}\bigg),$$

we arrive at

$$d\mu(s) \leqslant \Big(1+\frac{s}{(n+1)}\,H(0)\Big)^{n+1} d\mu(0). \tag{60}$$

If we denote with $W_\gamma^i = \{z \in N_t^i \mid \exists\, w \in (K_\gamma \times (0,1)) \cap L_s(N_t^i) \text{ with } |z - w| = s\}$ we estimate, using (56) and (60),

$$\mathcal{H}^{n+1}((K_\gamma \times I) \cap L_s(N_t^i)) \leqslant \int_{W_\gamma^i \cap N_t^i} \Big(1+\frac{s}{(n+1)}\,H\Big)^{n+1} d\mathcal{H}^{n+1} \tag{61}$$

for all $i \in \mathbb{N}$. Given an $\eta > 0$ assume that $\gamma > 0$ is small enough such that also $2\gamma < \gamma_0$.

Claim 2: For i large enough $W_\gamma^i \subset P_{2\eta} \times (-\eta, 1+\eta)$.

Assume this would not be the case. Then there exist points $w_i \in (K_\gamma \times (0,1)) \cap L_s(N_t^i)$ and points $z_i \in N_t^i$ with $|w_i - z_i| = s$ and $\operatorname{dist}(z_i, P_\eta \times (0,1)) \geqslant \eta$. We can further assume that $w_i \to w_\infty$ and $z_i \to z_\infty$ with $|w_\infty - z_\infty| = s$. Since $N_t^i \to \Gamma_t \times \mathbb{R}$ in Hausdorff-distance we have that

$$w_\infty \in (K_{2\gamma} \times [0,1]) \cap (L_s(\Gamma_t) \times [0,1]) \ \text{ and } \ z_\infty \in \Gamma_t$$

but

$$z_\infty \notin (P_\eta \times [0,1]) \cap (\Gamma_t \times \mathbb{R}).$$

This contradicts Claim 1.

We now combine (55), (61), Claim 2, and Lemma 6.1 to arrive at

$$\mathcal{H}^{n+1}((K_\gamma \times (0,1)) \cap (L_s(\Gamma_t \times \mathbb{R}))) \leqslant \int\limits_{(P_{2\eta} \times (-\eta, 1+\eta)) \cap (\Gamma_t \times \mathbb{R})} \Big(1+ (n+1)^{-1}\, H(0)\, s\Big)^{n+1} d\mathcal{H}^{n+1},$$

for almost all η, and γ chosen appropriately. Then let $\eta, \gamma \to 0$. q.e.d.

The next lemma shows that there can't be two different points $p_1, p_2 \in L_s(\Gamma_t)$ and a point $x_0 \in \Gamma_t$ with $|p_1 - x_0| = |p_2 - x_0| = s$.

Lemma 6.4. *Assume that in a point $x_0 \in \Gamma_t$ the set E_t can be touched from outside by two balls $B_s(p_1), B_s(p_2)$ for $s > 0$. Then $p_1 = p_2$.*

Proof. Assume to the contrary that two different balls $B_s(p_1), B_s(p_2)$ touch E_t in x_0. Since $\mathbf{H} \in L^p(\Gamma_t)$ for some $p > n$ the density Θ^n exists at x_0. Since E_t minimizes area from the outside, we can argue as in the proof of Proposition 5.10 to show that Θ^n satisfies the bound $\Theta^n(x_0) \leqslant C(n)$. By upper semicontinuity $\Theta^n(x_0) \geqslant 1$. Thus the blow-ups

$$\frac{1}{\lambda_i}(\Gamma_t - x_0)$$

converge for some sequence $\lambda_i \to 0$ to a stationary cone C with $\Theta^n_C(0) = \Theta^n_{\Gamma_t}(x_0)$. Since $p_1 \neq p_2$ the cone C has to be a subset of two different closed halfspaces T_1, T_2 where 0 is contained in either of the boundaries of T_1, T_2. This implies that $\operatorname{supp}(C)$ has to be a subset of ∂T_1 (see for example [**21**], Thm. 36.5). Since the same holds also for ∂T_2, we have $C \subset \partial T_1 \cap \partial T_2$. This yields $\mathcal{H}^n(C) = 0$, which contradicts $\Theta^n_C(0) > 0$. q.e.d.

Given a $\mathcal{H}^n$-measurable function f on $L_s(\Gamma_t)$, this enables us to define the "pull-back" $\tilde{f}$ to Γ_t by

$$\tilde{f}(x) = \begin{cases} f(y) & \text{if } \exists\ y \in L_s(\Gamma_t) \text{ with } |y - x| = s, \\ 0 & \text{else,} \end{cases}$$

for any $x \in \Gamma_t$. By the regularity of Γ_t, $\tilde{f}$ is also $\mathcal{H}^n$-measurable and we get the following corollary:

Corollary 6.5. *For almost all $s > 0$ the following statement is true: Let f be a $\mathcal{H}^n$-measurable, nonnegative function on $L_s(\Gamma_t)$ and define $\tilde{f}$ as above. Then*

$$\int\limits_{L_s(\Gamma_t)} f(y)\, d\mathcal{H}^n(y) \leqslant \int\limits_{\Gamma_t} \tilde{f}(x) \left(1 + \frac{1}{(n+1)} |\mathbf{H}(x)|\, s\right)^{n+1} d\mathcal{H}^n(x).$$

In the next proposition we use the previous estimate to prove (2) for almost every t.

Proposition 6.6. *Let $k > n$. Then for all $t \in \tilde{B}$ the estimate*

$$\int_{\Gamma_t} |\mathbf{H}|^n \, d\mu \geqslant \left(\frac{n}{n+1}\, c_{n+1}\right)^n \tag{62}$$

holds.

Proof. We first need to investigate further the regularity of the hypersurfaces Γ_t.

Claim 1: Let $t \in \tilde{B}$. Then Γ_t is twice differentiable $\mathcal{H}^n$-a.e.

Let $S \subset \Gamma_t$ be the singular part of Γ_t, such that away from S, Γ_t can be written locally as the graph of a $C^{1,\alpha}$-function u. By (51) we know that the mean curvature H of Γ_t is in $L^p(\Gamma_t)$ for some $p > n$. This implies that u is a weak solution of the equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = H.$$

Since we can assume that $p \geqslant 2$, we can deduce as for example in [**7**], Theorem C.1, that $u \in W^{2,2}$ and u is a strong solution of the equation

$$a_{ij}(Du)\, D_{ij}u = \sqrt{1+|Du|^2}H, \tag{63}$$

where

$$a_{ij}(Du) = \delta_{ij} - \frac{D_i u\, D_j u}{1+|Du|^2}.$$

Since u is in C^1 the coefficients a_{ij} are uniformly elliptic and continuous. Thus we can apply the estimates of Calderon-Zygmund to deduce that $u \in W^{2,p}$. Since $p > n, u$ is twice differentiable $\mathcal{H}^n$-a.e. (see for example [**5**], section 6.4, Theorem 1). Note that at a point x_0 where u is twice differentiable, we can write

$$\begin{aligned} u(x) = u(x_0) + D_i u(x_0)(x-x_0)_i + \frac{1}{2} D_{ij}u(x_0)(x-x_0)_i(x-x_0)_j \\ + o(|x-x_0|^2), \end{aligned}$$

and equation (63) holds point wise a.e.

We now argue in a similar spirit as [**17**]. Let us denote with $C(\Gamma_t)$ the outer convex hull of Γ_t and let $C_s = L_s(C(\Gamma_t))$. Observe that forming the convex hull and taking an outer equidistant surface commutes, i.e.,

$$C_s = L_s(C(\Gamma_t)) = C(L_s(\Gamma_t)).$$

C_s is convex and $C^{1,1}$ (see Appendix B in [**1**]). The nearest point projection $\pi_s : C_s \to C_0$ is well defined, distance nonincreasing and is a bijection between $C_s \cap L_s(\Gamma_t)$ and $\Gamma_t \cap C_0$.

Claim 2: Let $s > 0$. For $\mathcal{H}^n$-a.e. $p \in \Gamma_t \cap C_0$ we can estimate

$$\frac{nH_{\Gamma_t}(p)}{H_{\Gamma_t}(p)\, s + n} \geqslant H_{C_s}(\pi_s^{-1}(p)).$$

We can assume that p is not in the singular part of Γ_t and, by Claim 1, that Γ_t is twice differentiable at p. Since $\mathcal{H}^n$-zero sets on C_s are mapped under π_s to $\mathcal{H}^n$-zero sets on $\Gamma_t \cup C_0$, we can as well assume that C_s is twice differentiable at $q := \pi_s^{-1}(p)$. Now let Σ be a supporting hypersurface of C_s at q, i.e., Σ is locally a smooth hypersurface, which touches C_s in q from the outside. For a given $\delta > 0$ we can assume that

$$H_{C_s}(q) - \delta \leqslant H_\Sigma(q) \leqslant H_{C_s}(q). \tag{64}$$

Furthermore, we may assume that Σ is a convex hypersurface, with principal curvatures local bounded around q by $1/s$. We translate Σ such that it touches Γ_t in p from the outside. By Claim 1, this implies that

$$H_{\Gamma_t}(p) \geqslant H_\Sigma(q).$$

We now evolve Σ (translated back to its original position) equidistantly towards Γ_t. Let us denote with Σ_τ the so obtained hypersurface with $\mathrm{dist}(\Sigma, \Sigma_\tau) = \tau$ for $0 < \tau < s$. Since we have assumed that the principal curvatures of Σ are locally bounded by $1/s$, Σ_τ remains smooth, and it touches $C_{s-\tau}$ in q' from the outside, with $\pi_{s-\tau}(q') = p$. Now, as above, this yields

$$H_{\Gamma_t}(p) \geqslant H_{\Sigma_\tau}(q'). \tag{65}$$

We use the evolution of the mean curvature under the equidistant flow, as in the proof of Lemma 6.2, see (57) and (58), to obtain

$$H_\Sigma(q) \leqslant \frac{nH_{\Sigma_\tau}(q')}{H_{\Sigma_\tau}(q')\tau + n}. \tag{66}$$

Note that here the hypersurfaces are n-dimensional, and thus $n+1$ is replaced by n in (57) and (58). Now

$$\frac{\partial}{\partial H}\left(\frac{nH}{H\tau + n}\right) \geqslant 0.$$

So we can combine (64), (65) and (66) to conclude that

$$H_{C_s}(q) - \delta \leqslant \frac{nH_{\Gamma_t}(p)}{H_{\Gamma_t}(p)\tau + n}.$$

Taking the limits $\tau \to s$ and $\delta \to 0$ proves Claim 2.

We finally prove the proposition. Since the Weingarten map on C_s, restricted to $C_s \cap L_s(\Gamma_t)$, covers $\mathbb{S}^n$ at least once we can estimate as in the smooth case

$$|\mathbb{S}^n| \leqslant \int\limits_{C_s \cap L_s(\Gamma_t)} G \, d\mathcal{H}^n \leqslant \frac{1}{n^n} \int\limits_{C_s \cap L_s(\Gamma_t)} \left(H_{C_s}\right)^n d\mathcal{H}^n.$$

We estimate further, applying Corollary 6.5 and Claim 2, to arrive at

$$
\begin{aligned}
n^n|\mathbb{S}^n| &\leqslant \int\limits_{\Gamma_t\cap C_0} \big(H_{C_s}(\pi_s^{-1}(x))\big)^n \Big(1+\frac{1}{(n+1)}\,H_{\Gamma_t}(x)\,s\Big)^{n+1}\,d\mathcal{H}^n(x)\\
&\leqslant \int\limits_{\Gamma_t\cap C_0} \big(H_{\Gamma_t}(x)\big)^n \frac{\Big(1+\frac{1}{(n+1)}\,H_{\Gamma_t}(x)\,s\Big)^{n+1}}{\Big(1+\frac{1}{n}H_{\Gamma_t}(x)\,s\Big)^{n}}\,d\mathcal{H}^n(x)\\
&\leqslant \int\limits_{\Gamma_t} \big|\mathbf{H}_{\Gamma_t}\big|^n\,d\mathcal{H}^n \;+o(1).
\end{aligned}
$$

Note that these calculations are justified since the mean curvature of Γ_t is in $L^{n+1}(\Gamma_t)$. q.e.d.

The estimate (2) in the case that the ambient manifold is a 3-dimensional Hadamard manifold $(N^3, \bar{g})$, is a modification of an argument due to L. Simon, see [**22**]. The main tool is, as in the monotonicity formula, to use a well chosen vectorfield in the the first variation identity

$$
\int\limits_M \mathrm{div}_M(Y)\,d\mu = -\int\limits_M \bar{g}(Y,\mathbf{H})\,d\mu \tag{67}
$$

which holds for any $C^{0,1}$ vector field Y, defined in a neighborhood of M. As before, we write for $p\in M$:

$$
\mathrm{div}_M(Y)(p) = \sum_{i=1}^{2} \bar{g}(\bar{\nabla}_{e_i}Y, e_i),
$$

where $\bar{\nabla}$ is the covariant derivative on N and e_1, e_2 form an orthonormal basis of T_pM. Since we still can make sense of (67) in the varifold setting, the estimate needs much less regularity than in the case we treated before.

Lemma 6.7. *Let N^3 be a complete, simply connected 3-manifold, with nonpositive sectional curvatures. Let $M\subset N^3$ be a bounded integer 2-rectifiable varifold, carrying a weak mean curvature $\mathbf{H}\in L^2(\mu)$. Then*

$$
\int\limits_M |\mathbf{H}|^2\,d\mu \geqslant 16\pi. \tag{68}
$$

Proof. For the computation in the case of a flat Euclidean ambient space one uses the position vectorfield X, centered at a point x_0. The calculation then depends on the fact that

$$
\mathrm{div}_M(X)(x) = 2,
$$

for all $x\in M$, $x\neq x_0$, such that the tangent space of M exists at x. Now on a complete, simply-connected manifold N, we replace this vectorfield by

$$X := r\bar{\nabla}r,$$

where $r(p) := \mathrm{dist}_N(p, p_0)$ for a fixed $p_0 \in N$. Here $\bar{\nabla}$ denotes the gradient operator on N. Let us assume that the sectional curvatures of N are bounded above by $-\kappa$ for some $\kappa \geqslant 0$. The distance function to a point on such a manifold has two important properties, see, for example, [**19**]:

$$\begin{aligned}\bar{\nabla}r &\neq 0\\ \mathrm{Hess}(r) = \bar{\nabla}^2 r &\geqslant \Psi(r)\big(\mathrm{id} - \bar{\nabla}r \otimes \bar{\nabla}r\big),\end{aligned}$$

for $p \neq p_0$, where $\Psi(r) := \sqrt{\kappa}\cosh(\sqrt{\kappa}r)/\sinh(\sqrt{\kappa}r)$ and the second inequality holds w.r.t. an orthonormal basis of T_pN. For a point $p \in M$, such that the tangent space of M exists at p, choose a normal vector ν to T_pM and compute

$$\begin{aligned}\mathrm{div}_M(X) &= \mathrm{div}_M(r\bar{\nabla}r)\\ &= r\,\mathrm{div}_M(\bar{\nabla}r) + \bar{g}(\nabla_M r, \bar{\nabla}r)\\ &= r\,\mathrm{tr}_{T_pM}\big(\mathrm{Hess}(r)\big) + 1 - \bar{g}(\bar{\nabla}r, \nu)^2\\ &\geqslant r\,\Psi(r)\,\mathrm{tr}_{T_pM}\big(\mathrm{id} - \bar{\nabla}r \otimes \bar{\nabla}r\big) + 1 - \bar{g}(\bar{\nabla}r, \nu)^2\\ &= \Big(\mathrm{tr}_{T_pN}\big(\mathrm{id} - \bar{\nabla}r \otimes \bar{\nabla}r\big) + \bar{g}(\bar{\nabla}r, \nu)^2 - 1\Big) + 1\\ &\quad - \bar{g}(\bar{\nabla}r, \nu)^2 + \big(r\,\Psi(r) - 1\big)\mathrm{tr}_{T_pM}\big(\mathrm{id} - \bar{\nabla}r \otimes \bar{\nabla}r\big)\\ &= 2 + \big(r\,\Psi(r) - 1\big)\Big(1 + \bar{g}(\bar{\nabla}r, \nu)^2\Big) \geqslant 2,\end{aligned} \tag{69}$$

since $r\Psi(r) \geqslant 1$. The computation now proceeds as in the Euclidean case: Pick any $p_0 \in M$ such that the density $\Theta(p_0)$ exists and $\Theta(p_0) \geqslant 1$. For $0 < \sigma < \rho$ we can substitute in (67) the vectorfield $Y(p) \equiv (|X|_\sigma^{-2} - \rho^{-2})_+ X$ where $|X|_\sigma = \max(|X|, \sigma)$. A direct computation, using (69), then yields

$$\begin{aligned}&2\sigma^{-2}\mu(B_\sigma) + 2\int_{B_\rho\setminus B_\sigma}\left|\frac{1}{4}\mathbf{H} + \frac{X^\perp}{|X|^2}\right|^2 d\mu\\ &\leqslant 2\rho^{-2}\mu(B_\rho) + \frac{1}{8}\int_{B_\rho\setminus B_\sigma}|\mathbf{H}|^2\,d\mu - \sigma^{-2}\int_{B_\sigma}\bar{g}(X, \mathbf{H})\,d\mu + \rho^{-2}\int_{B_\rho}\bar{g}(X, \mathbf{H})\,d\mu,\end{aligned}$$

where we assume that all balls B_σ, B_ρ are centered at p_0. Since

$$\lim_{\sigma\to 0}\sigma^{-2}\mu(B_\sigma) \geqslant \pi,$$

we can take the limits $\sigma \to 0$ and $\rho \to \infty$ to prove the estimate (68). q.e.d.

In the case that N^3 has sectional curvatures bounded above by $-\kappa$ for some $\kappa > 0$ there is a stronger estimate by the Gauss-Bonnet formula. This estimate was used by B. Kleiner in [**17**] to show that on such a manifold N^3 the isoperimetric inequality of the model space with constant sectional curvatures κ holds. The estimate goes as follows. Let $M \subset N^3$ be a closed hypersurface, diffeomorphic to a sphere and denote with R_{int} the intrinsic scalar curvature of M. Then by the Gauss equations

$$4\pi = \int_M R_{\mathrm{int}}\, d\mathcal{H}^2 \leqslant \int_M G - \kappa\, d\mathcal{H}^2 \leqslant \int_M \frac{1}{4}H^2 - \kappa\, d\mathcal{H}^2. \tag{70}$$

We now want to modify the proof of (62) somewhat to show that the above estimate holds for a.e. t along a weak H^k-flow.

Lemma 6.8. *Let N^3 be complete, simply connected manifold with sectional curvatures bounded above by $-\kappa$ for some $\kappa > 0$. Let $\Omega \subset N^3$ be a bounded, open set with smooth boundary and $H|_{\partial\Omega} > 0$, which minimizes area from the outside in N^3. Let u be a weak H^k-flow generated by Ω and $k > 2$. Then for all $t \in \tilde{B}$ we have*

$$\int_{\Gamma_t} \frac{1}{4}|\mathbf{H}|^2 - \kappa\, d\mathcal{H}^2 \geqslant 4\pi.$$

Proof. The proof is modification of Proposition 6.6. We first need the following claim.

Claim 1: The sets E_t minimize area from the outside in N^3 for all $t \in (0,T)$.

Let $F \subset N^3$ be a comparison set for E_t, $t \in (0,T)$, i.e,. $E_t \subset F$. Now pick a time τ with $t < \tau < T$. Then $E_\tau \Subset F$; thus we can pick a sequence of sets F_i with smooth boundary, such that $E_\tau \subset F_i$ for all i and $|\partial F_i| \to |\partial^* F|$. By inequality (36) we have that $|\partial^*(F_i \cap \Omega)| \leqslant |\partial F_i|$. Since $\partial\Omega$ and ∂F_i are smooth we can approximate $F_i \cap \Omega$ with sets $F^i_j \Subset \Omega$ such that $E_\tau \subset F^i_j$ for all j and $|\partial^* F^i_j| \to |\partial^*(F_i \cap \Omega)|$. Now the sets F^i_j are valid comparison sets for E_τ in Ω, and thus taking the limits $j \to \infty$ and $i \to \infty$ we see that

$$|\partial^* E_\tau| \leqslant |\partial^* F|.$$

Now take a sequence $\tau_i \nearrow t$, then by non-fattening $E_{\tau_i} \to E_t$ and by outward minimizing $|\partial^* E_{\tau_i}| \to |\partial^* E_t|$, which proves the claim.

We will now show the corresponding result to Lemma 6.2, with a modification since the ambient space N^3 is not flat.

Claim 2: Under the same conditions and with the same notation as in Corollary 6.5 we have :

$$\int\limits_{L_s(\Gamma_t)} f(y)\, d\mathcal{H}^2(y) \leqslant \int\limits_{\Gamma_t} \tilde{f}(x)\Big(\cosh(\gamma s) + \frac{1}{3\gamma}\sinh(\gamma s)|\mathbf{H}(x)|\Big)^3\, d\mathcal{H}^2(x),$$

where $\gamma = \sqrt{(C/3)}$, $C := -\inf\{\,\bar{\mathrm{R}}\mathrm{ic}(X,X) \mid X \in T_pN^3,\ |X| = 1,\ \mathrm{dist}(p,\Gamma_t) \leqslant s\}$.

The proof of this claim is nearly identical to the proof of Lemma 6.2. The only difference is that instead of (57) we have

$$\frac{\partial}{\partial s}H = -|A|^2 - \bar{\mathrm{R}}\mathrm{ic}(\nu,\nu) \leqslant -\frac{1}{3}H^2 + C,$$

which can be integrated to give

$$H(s) \leqslant \frac{\sqrt{3C}\sinh\big(\sqrt{(C/3)}\cdot s\big) + \cosh\big(\sqrt{(C/3)}\cdot s\big)H(0)}{\cosh\big(\sqrt{(C/3)}\cdot s\big) + (\sqrt{3C})^{-1}\sinh\big(\sqrt{(C/3)}\cdot s\big)H(0)}. \tag{71}$$

Inserting this into (59) we obtain

$$d\mu(s) \leqslant \Big(\cosh\big(\sqrt{(C/3)}\cdot s\big) + (\sqrt{3C})^{-1}\sinh\big(\sqrt{(C/3)}\cdot s\big)H(0)\Big)^3\, d\mu(0).$$

Note that Lemma 6.4 remains true, so the rest of the proof follows again in the same way.

We now have to rework Proposition 6.6. Claim 1 in the proof there remains true, but we have to replace the second claim by the following.

Claim 3: Under the same conditions and with the same notation as in Claim 2 of the proof of Proposition 6.6, we have

$$\frac{\cosh\big(\sqrt{(C/2)}s\big)H_{\Gamma_t}(p)}{\cosh\big(\sqrt{(C/2)}s\big) + (\sqrt{2C})^{-1}\sinh\big(\sqrt{(C/2)}s\big)H_{\Gamma_t}(p)} + Cs \geqslant H_{C_s}(\pi_s^{-1}(p)).$$

As in the proof there let Σ be a supporting hypersurface of C_s at q such that (64) holds. We may assume that Σ is a convex hypersurface, with principal curvatures locally bounded by $1/s - \delta/4$. Since in N^3 there is no notion of translating Σ to touch Γ_t in p from the outside, we directly evolve Σ equidistantly towards Γ_t, i.e., until Σ_s touches Γ_t in p. Since in an ambient Hadamard manifold N, focal points develop later than in Euclidean space, the conditions above guarantee that Σ_s remains smooth. Thus

$$H_{\Gamma_t}(p) \geqslant H_{\Sigma_s}(p).$$

By (71), where now the 3's are replaced by 2's, since the surfaces are 2-dimensional,

$$
\begin{aligned}
H_{C_s} - \delta &\leqslant H_\Sigma(q)\\
&\leqslant \frac{\sqrt{2C}\sinh\big(\sqrt{(C/2)}s\big) + \cosh\big(\sqrt{(C/2)}s\big)H_{\Sigma_s}(p)}{\cosh\big(\sqrt{(C/2)}s\big) + (\sqrt{2C})^{-1}\sinh\big(\sqrt{(C/2)}s\big)H_{\Sigma_s}(p)}\\
&\leqslant \frac{\cosh\big(\sqrt{(C/2)}s\big)H_{\Sigma_s}(p)}{\cosh\big(\sqrt{(C/2)}s\big) + (\sqrt{2C})^{-1}\sinh\big(\sqrt{(C/2)}s\big)H_{\Sigma_s}(p)} + Cs\\
&\leqslant \frac{\cosh\big(\sqrt{(C/2)}s\big)H_{\Gamma_t}(p)}{\cosh\big(\sqrt{(C/2)}s\big) + (\sqrt{2C})^{-1}\sinh\big(\sqrt{(C/2)}s\big)H_{\Gamma_t}(p)} + Cs,
\end{aligned}
$$

which proves Claim 3 as $\delta \to 0$.

To finally prove the lemma we combine (70) and Claim 1 to estimate

$$
\begin{aligned}
4\pi \leqslant \int\limits_{C_s} G - \kappa\, d\mathcal{H}^2 &= \int\limits_{C_s\cap L_s(\Gamma_t)} G\, d\mathcal{H}^2 + \int\limits_{C_s\setminus L_s(\Gamma_t)} G\, d\mathcal{H}^2 - \int\limits_{C_s} \kappa\, d\mathcal{H}^2\\
&\leqslant \int\limits_{C_s\cap L_s(\Gamma_t)} \frac{1}{4}H^2\, d\mathcal{H}^2 + \int\limits_{C_s\setminus L_s(\Gamma_t)} G\, d\mathcal{H}^2 - \int\limits_{\Gamma_t} \kappa\, d\mathcal{H}^2.
\end{aligned}
$$

By an argument of B. Kleiner (see the last part of the proof of Proposition 8 in [**17**]), the second term in the second line goes to zero as $s \to 0$. So we can apply Claim 2 and 3 to estimate further

$$
\begin{aligned}
4\pi &\leqslant \frac{1}{4}\int\limits_{\Gamma_t\cap C_0} \big(H_{C_s}(\pi_s^{-1}(x))\big)^2\Big(\cosh(\gamma s) + \frac{1}{3\gamma}\sinh(\gamma s)|\mathbf{H}(x)|\Big)^3\, d\mathcal{H}^2\\
&\quad - \int\limits_{\Gamma_t} \kappa\, d\mathcal{H}^2 + o(1)\\
&\leqslant \frac{1}{4}\int\limits_{\Gamma_t\cap C_0} (\cosh(\gamma s))^2|\mathbf{H}|^2\, d\mathcal{H}^2 - \int\limits_{\Gamma_t} \kappa\, d\mathcal{H}^2 + o(1)\\
&\leqslant \frac{1}{4}\int\limits_{\Gamma_t} |\mathbf{H}|^2\, d\mathcal{H}^2 - \int\limits_{\Gamma_t} \kappa\, d\mathcal{H}^2 + o(1).
\end{aligned}
$$

Taking the limit $s \to 0$ we obtain the estimate. q.e.d.

7. The monotonicity of the isoperimetric difference

Let $\varphi \in C^1_c(\mathbb{R})$, $\varphi \geqslant 0$ such that $\int_{\mathbb{R}} \varphi\, dx \geqslant 1$. Choose a function $\phi := \varphi(x_{n+2}) \in C^1(\Omega \times \mathbb{R})$ to define the approximative area and volume by

$$
A^i_t := \int_{N^i_t} \varphi\, d\mathcal{H}^{n+1}\ , \quad V^i_t := \int_{E^i_t} \varphi\, d\mathcal{H}^{n+2} \tag{72}
$$

and the approximate isoperimetric difference

$$I_t^i := \left(A_t^i\right)^{\frac{n+1}{n}} - c_{n+1} V_t^i$$

where c_{n+1} is defined as in the introduction. Let $t_1, t_2 \in (0, T)$. Assuming that i is big enough, the boundary (in $N \times \mathbb{R}$) of the graphs N_t^i does not intersect $\operatorname{supp}(\varphi)$ for $t \in (t_1, t_2)$. By the Coarea formula and the evolution equations we have

$$V_{t_2}^i - V_{t_1}^i = -\int_{t_1}^{t_2} \int \varphi H_i^k \, d\mu_t^i \, dt$$

and

$$\begin{aligned} \left(A_{t_2}^i\right)^{\frac{n+1}{n}} - \left(A_{t_1}^i\right)^{\frac{n+1}{n}} &= -\frac{n+1}{n} \int_{t_1}^{t_2} \left(A_t^i\right)^{\frac{1}{n}} \int \varphi H_i^{k+1} \, d\mu_t^i \, dt \\ &\quad - \frac{n+1}{n} \int_{t_1}^{t_2} \left(A_t^i\right)^{\frac{1}{n}} \int \langle \nabla\varphi, \nu \rangle H_i^k \, d\mu_t^i \, dt \ . \end{aligned}$$

By Hölder's inequality we can estimate

$$\begin{aligned} \frac{I_{t_2}^i - I_{t_1}^i}{c_{n+1}} \leqslant \int_{t_1}^{t_2} &\Bigg(\left(\int \varphi H_i^{k+1} \, d\mu_t^i \right)^{\frac{k}{k+1}} \left(A_t^i\right)^{\frac{1}{k+1}} \\ &- \frac{n+1}{n c_{n+1}} \left(A_t^i\right)^{\frac{1}{n}} \int \varphi H_i^{k+1} \, d\mu_t^i \Bigg) dt \\ &- \frac{n+1}{n c_{n+1}} \int_{t_1}^{t_2} \left(A_t^i\right)^{\frac{1}{n}} \int \langle \nabla\varphi, \nu \rangle H_i^k \, d\mu_t^i \, dt. \end{aligned} \tag{73}$$

Lemma 7.1. *Let $k \geqslant n-1$. Assume that for a.e. $t \in [0, T]$ the estimate*

$$\int_{\Gamma_t} |\mathbf{H}|^n \, d\mu \geqslant \left(\frac{n}{n+1} \, c_{n+1} \right)^n \tag{74}$$

holds. Then for any $t_1, t_2 \in (0, T)$, $t_1 < t_2$,

$$\limsup_{i \to \infty} \int_{t_1}^{t_2} L_t^i \, dt \leqslant 0,$$

where

$$L_t^i := \left(\int \varphi H_i^{k+1} \, d\mu_t^i \right)^{\frac{k}{k+1}} \left(A_t^i\right)^{\frac{1}{k+1}} - \frac{n+1}{n c_{n+1}} \left(A_t^i\right)^{\frac{1}{n}} \int \varphi H_i^{k+1} \, d\mu_t^i.$$

Proof. Since $t_2 < T$ and $\phi \not\equiv 0$ there is $\delta > 0$ such that $A_t^i \geqslant \delta$ for all $t \in [t_1, t_2]$ and all i. Since $k/(k+1) < 1$, there is a $C_1 \geqslant 0$ such that

$$L_t^i \leqslant C_1$$

for all $t \in [t_1, t_2]$ and all i. To prove the lemma, it thus suffices by Fatou's lemma to show that $\limsup_{i \to \infty} L_t^i \leqslant 0$ for all $t \in \tilde{B} \cap [t_1, t_2]$

such that (74) holds. Fix such a t. Arguing as above, we see that there is a $C_2 > 0$, independent of i, such that

$$L_t^i \leqslant 0 \quad \text{if} \quad \int \varphi H_i^{k+1}\, d\mu_t^i \geqslant C_2.$$

Thus we can assume that

$$\int \varphi H_i^{k+1}\, d\mu_t^i \leqslant C_2 \quad \text{for all } i.$$

We write L_t^i in the form $L_t^i = a_i \cdot b_i$, where

$$a_i := \Big(\big(A_t^i\big)^{\frac{1}{k}} \int \varphi H_i^{k+1}\, d\mu_t^i \Big)^{\frac{k}{k+1}}$$

and

$$b_i := 1 - \frac{n+1}{nc_{n+1}} \big(A_t^i\big)^{\frac{1}{n} - \frac{1}{k+1}} \Big(\int \varphi H_i^{k+1}\, d\mu_t^i \Big)^{\frac{1}{k+1}}$$

with the bounds $0 \leqslant a_i \leqslant C_2'$, $|b_i| \leqslant C_3$. By the lower semicontinuity of the L^{k+1}-norm of H^i and (74) we see that

$$\limsup_{i\to\infty} b_i \leqslant 1 - \frac{n+1}{nc_{n+1}} \big(\mu_t(\varphi)\big)^{\frac{1}{n} - \frac{1}{k+1}} \Big(\int \varphi |\mathbf{H}|^{k+1}\, d\mu_t \Big)^{\frac{1}{k+1}} \leqslant 0,$$

since $\mu_t = \mathcal{H}^{n+1} \llcorner (\Gamma_t \times \mathbb{R})$ and $\varphi(x) = \phi(x_{n+2})$ with $\int_{\mathbb{R}} \phi\, dx \geqslant 1$. Since the a_i are nonnegative and uniformly bounded this implies also $\limsup_{i\to\infty} L_t^i \leqslant 0$. q.e.d.

Proposition 7.2. *Let $k \geqslant n-1$ and u be a weak H^k-flow generated by Ω. Assume that* (74) *holds for a.e. $t \in [0,T]$. Then the isoperimetric difference*

$$I_t := \big(\mathcal{H}^n(\partial^* E_t)\big)^{\frac{n+1}{n}} - c_{n+1}\mathcal{H}^{n+1}(E_t)$$

is monotonically decreasing along u for $t \in [0,T)$.

Proof. Let $l > 1$. Pick a $\phi \in C_c^1(\mathbb{R})$, $0 \leqslant \phi \leqslant 1/(2l)$ with $\phi = 1/(2l)$ on $[-l,l]$, $\phi = 0$ on $\mathbb{R} \setminus [-l-1, l+1]$, and $|D\phi| \leqslant 1/l$. By the mass bound we have

$$\Big| \int_{t_1}^{t_2} (A_t^i)^{\frac{1}{n}} \int \langle \nabla\phi, \nu\rangle H_i^k\, d\mu_t^i\, dt \Big| \leqslant \frac{C_3}{l}.$$

For $t \in \tilde{B}$ we have

$$I_t^i \to \Big(\int_{\Gamma_t \times \mathbb{R}} \varphi\, d\mathcal{H}^{n+1} \Big)^{\frac{n+1}{n}} - c_{n+1} \int_{E_t'} \varphi\, d\mathcal{H}^{n+2} \;:= I_t'.$$

By the above estimate, equation (73), and Lemma 7.1, we can deduce that in the limit

$$I_{t_2}' \leqslant I_{t_1}' + \frac{C_3}{l}$$

for any $t_1, t_2 \in \tilde{B},\ t_1 < t_2$. Since $I_t^l \to I_t$ as $l \to \infty$ this implies that

$$I_{t_2} \leqslant I_{t_1}.$$

Since u is non-fattening, and all sets E_t minimize area from the outside in Ω, we can approximate times in $[0, T] \setminus \tilde{B}$ with times in $\tilde{B}$ to see that this monotonicity holds for all $t \in [0, T)$. q.e.d.

Proof of Theorem 1.1. By Lemma 6.7 and Proposition 6.6 together with Proposition 7.2 the isoperimetric difference is monotonically decreasing in all of the above cases. Since u is non-fattening we have $\lim_{t \to T} I_t \geqslant 0$. q.e.d.

We can now use the above theorem to a give a proof of the isoperimetric inequality.

Proof of Corollary 1.2. We can first assume that U is connected. Let Ω be the outer minimizing hull of U; see the proof of Theorem 4.4 for more details on minimizing hulls. Since $U \subset \Omega$ and $|\partial^* \Omega| \leqslant |\partial U|$ we can replace U by Ω. We first treat the case $n \leqslant 6$. Here we can apply a result of Sternberg, Williams and Ziemer [**23**], which shows that $\partial\Omega$ is a $C^{1,1}$-hypersurface. Thus $\partial\Omega$ carries a weak mean curvature in L^∞, which is non-negative, since Ω minimizes area from the outside. We can thus apply a result of G. Huisken and T. Ilmanen, Lemma 2.5 in [**12**], which states that starting from such a hypersurface there exists a smooth solution to mean curvature flow $(M_t)_{0<t<\varepsilon}$, where all the M_t have strictly positive mean curvature for $t > 0$. Furthermore, the initial datum $M_0 = \partial\Omega$ is attained in $C^{1,\alpha}$ and since the mean curvature is positive, the hypersurfaces M_t foliate a neighborhood of $\partial\Omega$ in Ω. The lemma in [**12**] is stated only if the ambient space is flat, but by a closer examination one sees that with some minor modifications the same proof also works in N^3 as above. Then for $0 < t < \varepsilon$ let $\Omega_t := \Omega \setminus \big(\cup_{0<\tau\leqslant t} M_t\big)$, thus $\partial\Omega_t = M_t$. Pick any $t \in (0, \varepsilon)$. Since $\partial\Omega_t$ has strictly positive mean curvature, there exists a weak H^k-flow, generated by Ω_t. Taking $k > n$ if $n \geqslant 4$, or $k \geqslant 1$ in the case $n = 3$, we can apply Theorem 1.1 to see that Ω_t satisfies (3). We then take the limit $t \searrow 0$.

For $n = 7$ the outer minimizing hull can have isolated singularities in the part where it does not touch the obstacle, i.e., in the part where its boundary constitutes a minimal surface. To treat this case let

$$U_\tau := \{x \in U \mid \operatorname{dist}(x, \partial U) > \tau\},$$

which has a smooth boundary for small enough $\tau > 0$. Let E be the outer minimizing hull of U and E_τ be the outer minimizing hulls of U_τ. We have $E_s \subset E_\tau \subset E$ for $0 < \tau < s$, which implies by Lemma 5.6 that $E_\tau \to E$ in L^1 as $\tau \searrow 0$ as well as $\partial^* E_\tau \to \partial^* E$ in the sense of radon measures. We now apply a strong maximum principle of Ziemer and Zumbrun [**25**] (which actually is an application of the strong maximum principle of Moschen-Simon) to show that $E_\tau \Subset E$ for $\tau > 0$. This maximum principle states that if two sets F, G are outer minimizing and minimizing, respectively, with respect to an open set V and $G \cap V \subset F \cap V$, then either $\partial F = \partial G$ in V or $\partial F \cap \partial G = \emptyset$ relative to V, provided $\partial F \cap V$ and $\partial G \cap V$ are connected. Note that E_τ minimizes area in $N \setminus U_\tau$ and that ∂E is connected in $N \setminus U_\tau$ and does not touch U_τ. Thus applying the maximum principle to every connected component of $\partial E_\tau \cap (N \setminus U_\tau)$ we see that $E_\tau \Subset E$ for all $\tau > 0$.

These properties enable us to argue as Hardt-Simon in [**8**], Theorem 5.6, to deduce that ∂E_τ is $C^{1,1}$ for all τ small enough. Note that ∂E is $C^{1,1}$ in a neighborhood of ∂U, and thus also ∂E_τ for small enough τ. Thus we can replace U by U_τ and argue as for $n \leqslant 6$, finally taking the limit $\tau \searrow 0$. q.e.d.

In the case that N^3 has sectional curvatures bounded above by $-\kappa$, $\kappa > 0$, we aim to apply estimate (70) to show how one can use a weak H^k-flow to prove that the isoperimetric profile of N always lies above the isoperimetric profile of the model space with constant curvature $-\kappa$.

Let $(M_t)_{0 \leqslant t < T}$ be a smooth H^k-flow in N^3 of hypersurfaces with positive mean curvature. Let us assume all the M_t are diffeomorphic to a sphere. We then can apply (70) to estimate

$$
\begin{aligned}
-\frac{d}{dt}V &= \int\limits_{M_t} H^k \, d\mathcal{H}^2 \\
&\leqslant \Big(\int\limits_{M_t} H^{k+1} \, d\mathcal{H}^2\Big)^{\frac{k}{k+1}} A^{\frac{1}{k+1}} (16\pi + 4\kappa A)^{-\frac{1}{2}} \Big(\int\limits_{M_t} H^{k+1} \, d\mathcal{H}^2\Big)^{\frac{1}{k+1}} A^{\frac{1}{2} - \frac{1}{k+1}} \\
&= (16\pi + 4\kappa A)^{-\frac{1}{2}} A^{\frac{1}{2}} \int\limits_{M_t} H^{k+1} \, d\mathcal{H}^2 = -\frac{d}{dt} f_\kappa(A),
\end{aligned}
$$

where f_κ is defined by (4). Thus $f_\kappa(A) - V$ is monotonically decreasing under the flow. Consider the case that N^3_κ is the model space of constant curvature $-\kappa$ and let M_t be the H^k-flow of geodesic spheres contracting to a point. Then (70) holds with equality for all M_t and also the above calculation is an equality. Using that in the model space geodesic balls optimize the isoperimetric ratio, we have

$$\mathcal{H}^3(U) \leqslant f_\kappa(\mathcal{H}^2(\partial U)),$$

for all open and bounded $U \subset N^3_\kappa$, with equality on geodesic balls. Arguing as in the beginning of this section, and using Lemma 6.8, we arrive at the following proposition.

Proposition 7.3. *Let $k > 2$ and u be a weak H^k-flow generated by Ω, where $\Omega \subset N^3$ is open and bounded, and N^3 has sectional curvatures* *bounded above by $-\kappa$. Furthermore, assume that $\partial\Omega$ is smooth with strictly positive mean curvature and that Ω minimizes area from the outside in N^3. Then*

$$I_t^\kappa := f_\kappa(\mathcal{H}^2(\partial^* E_t)) - \mathcal{H}^3(E_t)$$

is a nonnegative, monotonically decreasing function along u for $t \in [0,T)$.

Proof. We define A_t^i and V_t^i as in (72). Then take

$$I_t^{\kappa,i} := f_\kappa(A_t^i) - V_t^i.$$

Arguing as before, we can estimate

$$\begin{aligned} I_{t_2}^{\kappa,i} - I_{t_1}^{\kappa,i} \leqslant \int_{t_1}^{t_2} \Bigg(\Big(\int \varphi H_i^{k+1}\, d\mu_t^i \Big)^{\frac{k}{k+1}} \big(A_t^i\big)^{\frac{1}{k+1}} \\ - \frac{\big(A_t^i\big)^{\frac{1}{2}}}{\big(16\pi + 4\kappa A_t^i\big)^{\frac{1}{2}}} \int \varphi H_i^{k+1}\, d\mu_t^i \Bigg) dt \\ - \int_{t_1}^{t_2} \frac{\big(A_t^i\big)^{\frac{1}{2}}}{\big(16\pi + 4\kappa A_t^i\big)^{\frac{1}{2}}} \int \langle \nabla\varphi, \nu\rangle H_i^k\, d\mu_t^i\, dt. \end{aligned}$$

Analogous to the proof of Lemma 7.1 we can use Lemma 6.8 to show that for $t_1, t_2 \in (0,T),\ t_1 < t_2$, we have

$$\begin{aligned} \limsup_{i\to\infty} \int_{t_1}^{t_2} \Bigg(\Big(\int \varphi H_i^{k+1}\ \ d\mu_t^i \Big)^{\frac{k}{k+1}} \big(A_t^i\big)^{\frac{1}{k+1}} \\ - \frac{\big(A_t^i\big)^{\frac{1}{2}}}{\big(16\pi + 4\kappa A_t^i\big)^{\frac{1}{2}}} \int \varphi H_i^{k+1}\, d\mu_t^i \Bigg) dt \leqslant 0. \end{aligned}$$

The rest follows as in Proposition 7.2 and Theorem 1.1. q.e.d.

This enables us to also give a new proof of the stronger result in the case that the sectional curvatures of N^3 are bounded above by $-\kappa < 0$.

Proof of Theorem 1.3. We can use Proposition 7.3 to give an analogous proof as in Corollary 1.2. We use the same terminology as in the proof there. The only missing bit to apply Proposition 7.3 is to show that the sets Ω_t are minimizing area from the outside in N^3. Fix a $t > 0$. Take F to be a comparison set for Ω_t, i.e., $\Omega_t \subset F$. Again by (36), we have that $|\partial^*(F \cap \Omega)| \leqslant |\partial^* F|$ since Ω minimizes area from the outside in N^3. Note that the surfaces $M_\tau = \partial\Omega_\tau,\ \tau \in (0,t)$ smoothly foliate $\Omega \setminus \Omega_t$ and $M_t \to \partial\Omega$ in C^1 as $t \to 0$. Since all M_t have nonnegative mean curvature, we can use the outer unit normal vectorfield ν to these hypersurfaces as a calibration on $\Omega \setminus \Omega_t$ which satisfies $\mathrm{div}_{N^3}(\nu) \geqslant 0$.

Using this calibration we see that $|\partial\Omega_t| \leqslant |\partial^*(F\cap\Omega)|$, i.e., Ω_t minimizes area from the outside in N^3. q.e.d.

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