J. DIFFERENTIAL GEOMETRY
79 (2008) 167-183

ON THE STEINNESS OF A CLASS OF KÄHLER MANIFOLDS

Albert Chau & Luen-Fai Tam

Abstract

Let (M^n, g) be a complete non-compact Kähler manifold with non-negative and bounded holomorphic bisectional curvature. We prove that M is holomorphically covered by a pseudoconvex domain in $\mathbb{C}^n$ which is homeomorphic to $\mathbb{R}^{2n}$, provided (M^n, g) has uniform linear average quadratic curvature decay.

1. Introduction

Let (M^n, g_0) be a complete non-compact Kähler manifold with complex dimension n and with bounded nonnegative holomorphic bisectional curvature. Let R be the scalar curvature, and define

$$k(x, r) := \frac{1}{V_x(r)} \int_{B_x(r)} R dV.$$

In [**8**], it was proved by the authors that if M has maximum volume growth, then M is biholomorphic to $\mathbb{C}^n$. There, the authors used a result of Ni in [**22**] (see also [**10**, **13**]), which states that the condition of maximum volume growth on M implies that

$$k(x, r) \le \frac{C}{1 + r^2} \tag{1.1}$$

for some C for all x and r. In [**9**], the authors proved that condition (1.1) implies that M is holomorphically covered by $\mathbb{C}^n$, without assuming the maximum volume growth condition. The proof is obtained by studying the Kähler-Ricci flow

$$\frac{dg_{i\bar{j}}}{dt} = -R_{i\bar{j}} \tag{1.2}$$

with initial data g_0. It is well-known by [**30**] that if the scalar curvature decays linearly in the average sense:

$$k(x, r) \le C/(1 + r) \tag{1.3}$$

Research of the first author was partially supported by NSERC grant no. # 327637-06. Research of the second author was partially supported by Earmarked Grant of Hong Kong #CUHK403005.

Received 01/08/2007.

for some constant C for all x and r, then (1.2) has a long time solution with uniformly bounded curvature. By the results in [**13**, **26**], the linear decay condition (1.3) is true in most cases, at least for a constant C which may depend on x.

In this paper, we will prove the following:

Theorem 1.1. *Let (M^n, g_0) be a complete non-compact Kähler manifold with bounded non-negative holomorphic bisectional curvature. Suppose the scalar curvature of g_0 satisfies the linear decay condition* (1.3)*. Then M is holomorphically covered by a pseudoconvex domain in $\mathbb{C}^n$ which is homeomorphic to $\mathbb{R}^{2n}$. Moreover, if M has positive bisectional curvature and is simply connected at infinity, then M is biholomorphic to a pseudoconvex domain in $\mathbb{C}^n$ which is homeomorphic to $\mathbb{R}^{2n}$, and in particular, M is Stein.*

Remark 1.1. By a result of Yau [**32**], the pseudoconvex domain in Theorem 1.1 has infinite Euclidean volume. The authors would like to thank Shing Tung Yau for providing this information.

If we assume that $k(r) = \frac{C}{1+r^{1+\epsilon}}$, for $\epsilon > 0$, the result that M is biholomorphic to a pseudoconvex domain was proved by Shi [**30**] under the additional assumption that (M, g) has positive sectional curvature. Note that if M has positive sectional curvature, then it is well-known that M is diffeomorphic to $\mathbb{R}^{2n}$ by [**16**], and is Stein by [**15**]. Under the same decay condition and assuming maximum volume growth, similar results were obtained by Chen-Zhu [**11**]. All these works are before [**8**, **9**].

As in the above mentioned works, our proof of Theorem 1.1 is based on the Kähler-Ricci flow (1.2). In fact, Theorem 1.1 will be proved as a consequence of the following more general:

Theorem 1.2. *Let M^n be a complex noncompact manifold. Suppose there exist a sequence of complete Kähler metrics g_i, for $i \geq 1$, on M such that*

- **(a1)** $cg_i \leq g_{i+1} \leq g_i$ *for some* $1 > c > 0$ *for all* i,
- **(a2)** $|Rm(g_i)| + |\nabla Rm(g_i)| \leq c'$ *for some* c' *on* $B_i(p, r_0)$ *for some* $p \in M$, $r_0 > 0$ *and all* i *where* $B_i(p, r_0)$ *is the geodesic ball with respect to* g_i,
- **(a3)** g_i *is contracting in the following sense: For any* ϵ, *for any* i, *there exists* $i' > i$ *with*
$$g_{i'} \leq \epsilon g_i$$
in $B_i(p, r_0)$.

Then M is covered by a pseudoconvex domain in $\mathbb{C}^n$ which is homeomorphic to $\mathbb{R}^{2n}$.

To prove Theorem 1.1, the solution $g(t)$ to the Kähler-Ricci flow on M will be used to produce a sequence of Kähler metrics g_i satisfying the hypothesis of the Theorem 1.2. The main steps in proving Theorem 1.2 can be sketched as follows. The idea is to consider the sequence of holomorphic normal "coordinate charts" around some $p \in M$ corresponding to the sequence g_i. We then use this sequence of charts, together with a gluing technique as in [**30**], to build a map from an open set in $\mathbb{C}^n$ onto M. In general, however, these charts will only be locally biholomorphic, and to build such a map one generally needs to control the sets around p on which these charts are injective[1]. We will not assume any control on these sets. Instead, we will control the sets where corresponding coordinate transition functions are injective using a method developed by the authors in [**9**]. Once these transition functions are established, we then follow techniques similar to those in [**30**] and [**9**] to build a covering map from an open set Ω in $\mathbb{C}^n$ onto M. By its construction, Ω will be shown to be pseudoconvex and homeomorphic to $\mathbb{R}^{2n}$.

2. Holomorphic coordinate "covering" charts

Let M, g_i, p, r_0 be as in Theorem 1.2 and let $D(r)$ be the Euclidean open ball of radius r with center at the origin in $\mathbb{C}^n$.

Lemma 2.1. *There exists $r > 0$, and a family of holomorphic maps*

$$\Phi_i : D(r) \to M \tag{2.1}$$

for all $i \geq 1$ with the following properties:

(i) *Φ_i is a local biholomorphism from $D(r) \subset \mathbb{C}^n$ onto its image,*
(ii) *$\Phi_i(0) = p$,*
(iii) *$\Phi_i^*(g_i)(0) = g_e$,*
(iv) *$\frac{1}{C} g_e \leq \Phi_i^*(g_i) \leq C g_e$ in $D(r)$,*

where g_e is the standard metric on $\mathbb{C}^n$ and C is a constant independent of t and p.

Proof. Using condition **(a2)** and considering the pullback metric under the exponential map within the conjugate locus, one can apply Proposition 2.1 in [**8**] to obtain the results. q.e.d.

Corollary 2.1. *$B_i(C^{-1}\rho) \subset \Phi_i(D(\rho)) \subset B_i(C\rho)$ for some $C > 0$ for all $0 < \rho < r$ and $i \geq 1$.*

The following two lemmas are from [**9**, Lemmas 3.2 and 3.3].

Lemma 2.2. *Let r be as in Lemma 2.1. For any $0 < \rho \leq r$, there exists $0 < \rho_1 < r_0$, independent of i, satisfying the following:*

(i) *For any $q \in B_i(p, \rho_1)$, there is $z \in D(\frac{\rho}{8})$ such that $\Phi_i(z) = q$.*

[1] In [**30**], positive sectional curvature was used to produce a sequence of strictly convex domains around p exhausting M, which were then used to control the injectivity of the charts. In [**11**], maximal volume growth was used to control the injectivity radius under the Kähler-Ricci flow.

(ii) *For any* $q \in B_i(p, \rho_1)$, $z \in D(\frac{\rho}{8})$ *with* $\Phi_i(z) = q$, *and any smooth curve* γ *in* M *with* $\gamma(0) = q$ *such that* $L_i(\gamma) < \rho_1$, *there is a unique lift* $\widetilde{\gamma}$ *of* γ *by* Φ_i *so that* $\widetilde{\gamma}(0) = z$ *and* $\widetilde{\gamma} \subset D(\frac{\rho}{2})$.

Lemma 2.3. *Fix* $i \geq 1$. *Let* r *be as in Lemma* 2.1, *let* $0 < \rho \leq r$ *be given and let* ρ_1 *be as in Lemma* 2.2. *Given any* $\epsilon > 0$, *there exists* $\delta > 0$, *which may depend on* i, *satisfying the following properties:*

Let $\gamma(\tau)$, $\beta(\tau)$, $\tau \in [0,1]$ *be smooth curves from* $q \in B_i(p, \rho_1)$ *with length less than* ρ_1 *with respect to* g_i *and let* $z_0 \in D(\frac{1}{8}\rho)$ *with* $\Phi_i(z_0) = q$. *Let* $\widetilde{\gamma}$, $\widetilde{\beta}$ *be the liftings from* z_0 *of* γ *and* β *as described in Lemma* 2.2. *Suppose* $d_i(\gamma(\tau), \beta(\tau)) < \delta$ *for all* $\tau \in [0,1]$; *then* $d_e(\widetilde{\gamma}(1), \widetilde{\beta}(1)) < \epsilon$. *Here* d_i *is the distance in* g_i *and* d_e *is the Euclidean distance.*

Corollary 2.2. *Let* r *be as in Lemma* 2.1, *let* $0 < \rho \leq r$ *be given and let* ρ_1 *be as in Lemma* 2.2. *Let* $\gamma : [0,1] \times [0,1] \to M$ *be smooth homotopy such that*

(a) $\gamma(0, \tau) = q_1$ *and* $\gamma(1, \tau) = q_2$ *for all* τ.
(b) $q_1 \in B_i(p, \rho_1)$ *and* $\Phi_i(z_0) = q_1$ *for some* $z_0 \in D(\frac{1}{8}\rho r)$.
(c) *For all* $0 \leq \tau \leq 1$, *the length of* $\gamma(\cdot, \tau)$ *is less than* ρ_1.

For all τ, *let* $\widetilde{\gamma}_\tau$ *be the lift of* $\gamma(\cdot, \tau)$ *as in Lemma* 2.2 *from* z_0. *Then* $\widetilde{\gamma}_\tau(1) = \widetilde{\gamma}_0(1)$ *for all* τ.

Proof. By Lemma 2.2, $\widetilde{\gamma}_\tau(1) \in D(\frac{1}{2}\rho)$ for all τ. Let $\epsilon > 0$ be such that Φ_i is injective on $D(w, \epsilon)$ for all $w \in D(\frac{1}{2}\rho)$. Let $\delta > 0$ be as in Lemma 2.3. Let m be large enough, such that $d_i(\gamma(s, j/m), \gamma(s, (j+1)/m)) < \delta$ for all s for $0 \leq j \leq m-1$. By Lemma 2.3, we have $|\widetilde{\gamma}_{j/m}(1) - \widetilde{\gamma}_{(j+1)/m}(1)| < \epsilon$ for $0 \leq j \leq m-1$. Since $\Phi_i \circ \widetilde{\gamma}_\tau(1) = q_2$, and Φ_i is injective in $D(\widetilde{\gamma}_\tau(1), \epsilon)$, we have

$$\widetilde{\gamma}_{j/m}(1) = \widetilde{\gamma}_{(j+1)/m}(1).$$

From this the corollary follows. q.e.d.

3. Holomorphic transition functions

Let M, g_i, p, r_0 be as in Theorem 1.2.

Lemma 3.1. *Let* r *be as in Lemma* 2.1. *There exists* $0 < \rho < r$ *such that for every* $i \geq 1$, *there is a map* F_{i+1} *from* $D(\rho)$ *to* $D(r)$ *such that* $\Phi_i = \Phi_{i+1} \circ F_{i+1}$ *on* $D(\rho)$. *Moreover,* $\Phi_i(D(\rho)) \subset B_i(p, r_0)$ *where* r_0 *is the constant in* **(a3)**.

Proof. In Lemma 2.2, let $\rho = r$ and let ρ_1 be as in the conclusion of the Lemma. Note that ρ_1 is independent of i. Now let $0 < K < 1$ be a constant to be determined. For any $z \in D(K\rho_1)$, let $\gamma^*(\tau)$, $0 \leq \tau \leq 1$, be the line segment from 0 to z, and let $\gamma = \Phi_i \circ \gamma^*$. By **(a1)** and Lemma 2.1, we see that there is a constant $C_1 > 0$ independent of i

such that

$$L_{i+1}(\gamma) \leq L_i(\gamma) < C_1 K \rho_1. \tag{3.1}$$

Now choose K so that $C_1K < 1$, and redefine ρ as $\rho = K\rho_1$. Since $\gamma(0) = p$, by Lemma 2.2, there is a unique lift $\widetilde{\gamma}$ of γ by Φ_{i+1} so that $\widetilde{\gamma}(0) = 0$ and $\widetilde{\gamma} \subset D(\frac{1}{2}r)$. We define $F_{i+1}(z) = \widetilde{\gamma}(1)$. F_{i+1} is then a well-defined map from $D(\rho)$ to $D(r)$ by the uniqueness of the lifting. Also, by construction we have $\Phi_i = \Phi_{i+1} \circ F_{i+1}$ on $D(\rho)$. By choosing a smaller ρ, we also have $\Phi_i(D(\rho)) \subset B_i(p, r_0)$. This completes the proof of the lemma. q.e.d.

Lemma 3.2. *Let ρ be as in Lemma* 3.1. *Then for any $i \geq 1$, the map F_{i+1} satisfies the following:*

(a) *$F_{i+1}(0) = 0$.*

(b) *F_{i+1} is a local biholomorphism.*

(c)
$$b_1|v| \leq |F'_{i+1}(0)v| \leq b_2|v|$$
for some $0 < b_1 \leq b_2 \leq 1$ independent of i, and for all vectors $v \in \mathbb{C}^n$, where F' is the Jacobian of F.

(d) *There exist ρ_1 and ρ_2 independent of i, each in $(0, \rho)$, such that*
$$F_{i+1}(D(\rho_1)) \subset D(\rho_2),$$
and F_{i+1}^{-1} exists on $D(\rho_2)$.

Proof. (a) follows from the definition of F_{i+1}.

(b) can be proved as in the proof of Lemma 3.4 part (b) in [**9**], using Lemmas 2.3 and 3.1.

(c) follows from **(a1)** and Lemma 2.1.

(d) follows from the proof of part (d) of Lemma 3.4 in [**9**]. q.e.d.

Corollary 3.1. *Let ρ_1 be as in Lemma* 3.2. *Then for any $i \geq 1$, $F_{i+1}(D(\rho_1))$ is Runge in $\mathbb{C}^n$.*

Proof. Let $i \geq 1$ be given. Then given any holomorphic function f on $F_{i+1}(D(\rho_1)) \subset \mathbb{C}^n$, we must show that f can be approximated by entire functions on $\mathbb{C}^n$ uniformly on compact subsets of $F_{i+1}(D(\rho_1))$. Consider the holomorphic function $f \circ F_{i+1}$ defined on $D(\rho_1)$. Since $D(\rho_1)$ is just a ball in $\mathbb{C}^n$, $f \circ F_{i+1}$ can be approximated uniformly on compact subsets of $D(\rho_1)$ by entire functions h on $\mathbb{C}^n$. By part (d) of Lemma 3.2, $h \circ F_{i+1}^{-1}$ are defined on $D(\rho_2)$ and holomorphic. We see that these approximate f uniformly on compact subsets of $F_{i+1}(D(\rho_1)) \subset D(\rho_2)$. Finally, as $D(\rho_2)$ is just a ball in $\mathbb{C}^n$, we see that the functions $h \circ F_{i+1}^{-1}$ can themselves be approximated uniformly on compact subsets of $D(\rho_2)$ by entire functions. Thus, by part (d) of Lemma 3.2, f can be uniformly approximated on compact subsets of $F_{i+1}(D(\rho_1))$ by entire functions. This completes the proof of the corollary. q.e.d.

Corollary 3.2. *Let ρ_1 be as in Lemma 3.2. Then for any $i \geq 1$, F_{i+1} can be approximated uniformly on compact subsets of $D(\rho_1)$ by elements of $\mathrm{Aut}(\mathbb{C}^n)$.*

Proof. This follows from Corollary 3.1 and Theorem 2.1 in [**1**]. q.e.d.

4. Construction of a map onto M

Let M, g_i, p, r_0 be as in Theorem 1.2. We begin with the following lemma, which basically says that the maps F_i are contracting.

Lemma 4.1. *Let ρ_1 be as in Lemma 3.2. Then there exists positive constants ρ_3 and C such that $C > 1$, $C\rho_3 < \rho_1$, and for every i and $k \geq 1$:*

$$F_{i+k} \circ \cdots \circ F_{i+1}(D(\rho_3)) \subset D(C\rho_3). \tag{4.1}$$

Proof. Let $C_1 > 1$ be the constant in property (iv) of Φ_i in Lemma 2.1 and ρ_1 as in Lemma 3.2. Let $C = C_1^2$ and $\rho_3 = \rho_1/(2C)$. We first want to prove that for any i,

$$F_{i+1}(D(\rho_3)) \subset D(C\rho_3).$$

Let

$$A = \{\eta \in (0, \rho_3] |\ F_{i+1}(D(\eta)) \subset D(C\rho_3)\}.$$

Since $F_{i+1}(0) = 0$ and F_{i+1} is a local biholomorphism, it is easy to see that A is nonempty in $(0, \rho_3]$. Let $\eta \in A$ and $z \in D(\eta)$. Since $C\rho_3 < \rho_1$, the curves $\Phi_i(tz)$ and $\Phi_{i+1} \circ F_{i+1}(tz)$, for $0 \leq t \leq 1$, are defined and are equal by Lemmas 3.1 and 3.2. By Lemma 2.1, we have

$$\begin{aligned}
\left\|\frac{d}{dt}F_{i+1}(tz)\right\|_{g_e} &\leq C_1 \left\|\frac{d}{dt}F_{i+1}(tz)\right\|_{\Phi^*_{i+1}(g(i+1))} \\
&= C_1 \left\|\frac{d}{dt}\Phi_{i+1} \circ F_{i+1}(tz)\right\|_{g(i+1)} \\
&\leq C_1 \left\|\frac{d}{dt}\Phi_i(tz)\right\|_{g(i)} \\
&= C_1 \left\|\frac{d}{dt}(tz)\right\|_{\Phi^*_i(g(i))} \\
&\leq C_1^2|z| \\
&< C_1^2\eta \\
&\leq C\rho_3,
\end{aligned} \tag{4.2}$$

where we have used **(a1)**. From this, it is easy to see that A is both open and closed in $(0, \rho_3]$, and thus $F_{i+1}(D(\rho_3)) \subset D(C\rho_3)$.

Now suppose $k > 1$ such that

$$F_{i+l} \circ \cdots \circ F_{i+1}(D(\rho_3)) \subset D(C\rho_3) \tag{4.3}$$

for all $1 \leq l < k$. As before, let

$$B = \{\eta \in (0, \rho_3] |\ F_{i+k} \circ \cdot \cdot \circ F_{i+1}(D(\eta)) \subset D(C\rho_3)\}.$$

Again, B is nonempty in $(0, \rho_3]$. Suppose $\eta \in B$ and $\eta \in D(r)$. Then $\Phi_i(tz)$ and $\Phi_{i+k} \circ F_{i+k} \circ \cdot \cdot \circ F_{i+1}(tz)$, $0 \leq t \leq 1$, are well-defined and equal. As before, we can prove that B is open and closed in $(0, \rho_3]$ and

$$F_{i+k} \circ \cdot \cdot \circ F_{i+1}(D(\rho_3)) \subset D(C\rho_3). \tag{4.4}$$

This completes the proof of the lemma. q.e.d.

Remark 4.1. For later use, we will assume that $\rho_3 < \frac{1}{8}r$, where r is as in Lemma 2.1.

Lemma 4.2. *Let ρ_3 as in Lemma 4.1 There exists a positive increasing sequence n_i for $i \geq 1$ such that $n_1 = 1$ and*

$$F_{n_{i+1}} \circ \cdot \cdot F_{n_i+2} \circ F_{n_i+1}(D(\rho_3)) \subset D\left(\frac{\rho_3}{2}\right) \tag{4.5}$$

for every i.

Proof. Let C be the constant in Lemma 4.1. Let $n_1 = 1$. By Lemma 4.1, for all k, $F_{n_1+k} \circ \cdots \circ F_{n_1+1}$ is defined in $D(\rho_3)$ for all $k \geq 1$. As in the proof of Lemma 4.1, for all $z \in D(\rho_3)$, and $0 \leq t \leq 1$:

$$\begin{aligned}
&\left\| \frac{d}{dt} F_{n_1+k} \circ \cdots \circ F_{n_1+1}(tz) \right\|_{g_e} \\
&\leq C_1 \left\| \frac{d}{dt} F_{n_1+k} \circ \cdots \circ F_{n_1+1}(tz) \right\|_{\Phi^*_{k+n_1}(g_{n_1+1})} \\
&= C_1 \left\| \frac{d}{dt} \Phi_{k+n_1} \circ F_{n_1+k} \circ \cdots \circ F_{n_1+1}(tz) \right\|_{g_{k+n_1}} \\
&= C_1 \left\| \frac{d}{dt} \Phi_{n_1}(tz) \right\|_{g_{k+n_1}},
\end{aligned} \tag{4.6}$$

where C_1 is as in the proof Lemma 4.1. Since $\Phi_{n_1}(D(\rho_3)) \subset B_{n_1}(p, r_0)$ by the choice of ρ in Lemma 3.1, by **(a3)** and (iv) in Lemma 2.1, we can find $n_2 > n_1$ such that

$$F_{n_2} \circ \cdots \circ F_{n_1+1}(D(\rho_3)) \subset D\left(\frac{1}{2}\rho_3\right).$$

Similarly, one can choose $n_3, n_4, \ldots$ inductively which satisfy the conclusion of the lemma. q.e.d.

We now want to construct an appropriate sequence $\tilde{F}_j \in \text{Aut}(\mathbb{C}^n)$ that will approximate the sequence F_j for $j \geq 2$.

Let n_i be as in Lemma 4.2. By Lemmas 4.1, 4.2 and Corollary 3.2, we can find $\tilde{F}_2, \ldots, \tilde{F}_{n_2}$ in $\text{Aut}(\mathbb{C}^n)$ such that

$$\tilde{F}_{k+1} \circ \cdots \circ \tilde{F}_2(D(\rho_3)) \subset D(\rho_1) \tag{4.7}$$

for $2 \le k \le n_2$ and

$$\tilde F_{n_2} \circ \cdots \circ \tilde F_2(D(\rho_3)) \subset D(\rho_3). \tag{4.8}$$

Since Φ_{n_i} is a local biholomorphism, we have

$$\|D(\Phi_{n_2} \circ \tilde F_{n_2} \circ \cdots \circ \tilde F_2)(z)(v)\|_{g_1} \ge b_2 > 0 \tag{4.9}$$

for all $z \in D(\rho_3)$ and unit vectors $v \in \mathbb{C}^n$.

Let $S_2 = (\tilde F_{n_2} \circ \cdots \circ \tilde F_2)^{-1}(D(\rho_3))$. Use Lemmas 4.1, 4.2 and Corollary 3.2 again, we can find $\tilde F_{n_2+1}, \ldots, \tilde F_{n_3}$ in $\mathrm{Aut}(\mathbb{C}^n)$ such that

$$\tilde F_{n_3} \circ \cdots \circ \tilde F_{n_2+1}(D(\rho_3)) \subset D(\rho_3). \tag{4.10}$$

Since

$$\Phi_{n_2} = \Phi_{n_3} \circ F_{n_3} \circ \cdots \circ F_{n_2+1}$$

on $D(\rho_3)$, we may choose $\tilde F_{n_2+1}, \ldots, \tilde F_{n_3}$ such that they also satisfy:

(4.11)

$$d_{g_1}(\Phi_{n_3} \circ \tilde F_{n_3} \circ \cdots \circ \tilde F_{n_2+1} \circ \tilde F_{n_2} \circ \cdots \circ \tilde F_2(z), \Phi_{n_2} \circ \tilde F_{n_2} \circ \cdots \circ \tilde F_2(z)) \le \frac{1}{2^2}$$

for $z \in S_2$ and

$$\begin{aligned} &\|D(\Phi_{n_3} \circ \tilde F_{n_3} \cdots \circ \tilde F_{n_2+1} \circ \tilde F_{n_2} \circ \cdots \circ \tilde F_2)(z)(v)\|_{g_1} \\ &\qquad - \|D(\Phi_{n_2} \circ \tilde F_{n_2} \circ \cdots \circ \tilde F_2)(z)(v)\|_{g_1} \le \frac{b_2}{2^2} \end{aligned} \tag{4.12}$$

for all $z \in S_2$, and for all unit vectors in $\mathbb{C}^n$.

Now let $0 < b_3 < b_2$ be such that

$$\|D\Phi_{n_3} \circ \tilde F_{n_3} \circ \cdots \circ \tilde F_2(z)(v)\|_{g_1} \ge b_3 > 0 \tag{4.13}$$

for all $z \in S_2$ and unit vectors $v \in \mathbb{C}^n$. Let $S_3 = (\tilde F_{n_3} \circ \cdots \circ \tilde F_2)^{-1}(D(\rho_3))$. An inductive argument based on the above construction gives:

Lemma 4.3. *There exist $\tilde F_2, \tilde F_3, \ldots, \tilde F_j, \ldots$ in $\mathrm{Aut}(\mathbb{C}^n)$, such that the following conditions are satisfied for all $i \ge 2$:*

$$\tilde F_{n_{i+1}} \circ \cdots \circ \tilde F_{n_i+1}(D(\rho_3)) \subset D(\rho_3)). \tag{4.14}$$

$$\begin{aligned} d_{g_1}\Big(&\Phi_{n_{i+1}} \circ \tilde F_{n_{i+1}} \cdot \cdot \circ \tilde F_{n_i+1} \circ \tilde F_{n_i} \circ \cdots \circ \tilde F_{n_1+1}(z), \\ &\qquad \Phi_{n_i} \circ \tilde F_{n_i} \circ \cdots \circ \tilde F_{n_1+1}(z)\Big) \le \frac{1}{2^{i+1}} \end{aligned} \tag{4.15}$$

for all $z \in S_i$.

$$\begin{aligned} &\|D(\Phi_{n_{i+1}} \circ \tilde F_{n_{i+1}} \cdots \circ \tilde F_{n_i+1} \circ \tilde F_{n_i} \circ \cdots \circ \tilde F_{n_1+1})(z)(v)\|_{g_1} \\ &\qquad - \|D(\Phi_{n_i} \circ \tilde F_{n_i} \circ \cdots \circ \tilde F_{n_1+1})(z)(v)\|_{g_1} \le \frac{b_i}{2^{i+1}} \end{aligned} \tag{4.16}$$

for all $z \in S_i$ and Euclidean unit vectors v, where the sequence b_i is positive, decreases, and satisfies

$$\|D\Phi_{n_i} \circ \tilde F_{n_i} \circ \cdot \cdot \circ \tilde F_2(z)(v)\|_{g_1} \ge b_i \tag{4.17}$$

for all $z \in S_i$ and Euclidean unit vectors v and

$$S_i = \left(\tilde{F}_{n_i} \circ \cdots \circ \tilde{F}_2\right)^{-1} (D(\rho_3)).$$

Corollary 4.1. *Let S_i be as above. Then S_i is an increasing sequence of open sets in $\mathbb{C}^n$.*

Proof. From (4.14) we have

(4.18)
$$\tilde{F}_{n_{i+1}} \circ \cdots \circ \tilde{F}_{n_i+1} \circ \tilde{F}_{n_i} \circ \cdots \circ \tilde{F}_2(S_i) = \tilde{F}_{n_{i+1}} \circ \cdots \circ \tilde{F}_{n_i+1}(D(\rho_3)) \subset D(\rho_3),$$

and thus

$$S_i \subset \tilde{F}_2^{-1} \circ \cdots \circ \tilde{F}_{n_{i+1}}^{-1}(D(\rho_3)) = S_{i+1} \tag{4.19}$$

q.e.d.

Definition 4.1. Let the sequences S_i and n_i be as above. Let

$$\Omega = \bigcup_{i=2}^{\infty} S_i$$

Corollary 4.2. *Ω is pseudoconvex and is homeomorphic to $\mathbb{R}^{2n}$.*

Proof. Since each S_i is pseduoconvex in $\mathbb{C}^n$, Ω is pseudoconvex. Since each S_i is homeomorphic to the unit ball in $\mathbb{R}^{2n}$, Ω is also homeomorphic to $\mathbb{R}^{2n}$ by [**2**]. This completes the proof of the corollary. q.e.d.

We now begin to use the maps $\tilde{F}_i$ to construct a map from Ω onto M. We need the following lemma.

Lemma 4.4. *Let M, g_i, p be as in Theorem 1.2. Then for all $\epsilon > 0$, $\bigcup_i B_i(\epsilon) = M$, where $B_i(\epsilon) = B_i(p, \epsilon)$.*

Proof. Let $0 < 3\epsilon < r_0$, where r_0 is as in **(a3)**. Obviously, $B_1(\epsilon) \subset \bigcup_i B_i(\epsilon)$. We claim that if $B_1(k\epsilon) \subset \bigcup_i B_i(\epsilon)$, $k \geq 1$, then $B_1((k+1)\epsilon) \subset \bigcup_i B_i(\epsilon)$.

Suppose $B_1(k\epsilon) \subset \bigcup_i B_i(\epsilon)$; then $\overline{B_1(k\epsilon - \frac{1}{2}\epsilon)} \subset B_i(\epsilon)$ provided i is large enough. Hence $B_1((k+1)\epsilon) \subset B_i(\epsilon + \frac{3}{2}\epsilon) \subset B_i(3\epsilon)$ for i large enough by **(a1)**. By **(a3)**, we can find i such that $B_1((k+1)\epsilon) \subset B_i(\epsilon)$. This completes the proof of the lemma. q.e.d.

Lemma 4.5. *Let $\Gamma_i := \Phi_{n_i} \circ \tilde{F}_{n_i} \circ \cdot \cdot \circ \tilde{F}_2$. Then the following map $\Psi : \Omega \to M$ is well defined:*

$$\Psi(z) = \lim_{i \to \infty} \Gamma_i(z). \tag{4.20}$$

Proof. This follows from (4.15) in Lemma 4.3, Corollary 4.1 and the definition of the maps Γ_i. q.e.d.

Lemma 4.6. *Ψ is a local biholomorphism and onto.*

Proof. By property (iv) of the maps Φ_i, and the fact that $\bigcup_i B_i(p,\epsilon) = M$ for all ϵ, given any $R > 0$ we can find n_i such that

$$B_1(p,R) \subset B_{n_i}(p,\epsilon) \subset \Phi_{n_i}(D(C_1\epsilon)), \tag{4.21}$$

for some constant C_1^2 is the constant in Lemma 2.1(iv). Here we have used Corollary 2.1 provided $C_1\epsilon < \rho_3$. Choose such an ϵ. Then

$$\Gamma_i(S_i) = \Phi_{n_i} \circ \tilde{F}_{n_i} \circ \cdots \circ \tilde{F}_2(S_i) = \Phi_{n_i}(D(\rho_3) \supset B_1(p,R).$$

Thus, by (4.15) and the fact that the S_i's are increasing, it follows that

$$B_1(p,R-1) \subset \Gamma_j(S_i) \tag{4.22}$$

for all $j \geq i$. From the definition of the map Ψ, we see that

$$B_1(p,R-1) \subset \Psi(\Omega). \tag{4.23}$$

Hence $\Psi(\Omega) = M$.

We now show that Ψ is a local biholomorphism. Observe that Ω is open and Ψ is a holomorphic map. Now to show Ψ is a local biholomorphism on Ω, it will be sufficient to show it is a local biholomorphism on the sets S_i for each i. Fix some i. Then by (4.16) and the fact that the b_i's are decreasing,

$$\|D(\Gamma_j)(z)(v)\|_{g_1} \geq b_i - \frac{b_i}{2} \tag{4.24}$$

for all $j \geq i$, $z \in S_i$ and all unit vectors v at z. Thus, by the definition of Ψ, (4.24) implies Ψ is a local biholomorphism on S_i. Noting that i is arbitrary, this completes the proof of the lemma. q.e.d.

5. Proof of Theorem 1.2

Let M and g_i satisfy **(a1)–(a3)** and let Ψ be the map constructed in the previous section. If we take $\pi : \widehat{M} \to M$ to be a universal holomorphic covering of M and let $\widehat{g}_i = \pi^*(g_i)$, then $(\widehat{M}, \widehat{g}_i)$ will still satisfy **(a1)–(a3)**. Thus to prove Theorem 1.2, it will be sufficient to prove that Ψ is injective assuming that M is simply connected. Before we prove this, let us first prove the following:

Lemma 5.1. *Let $\alpha(s)$, $0 \leq s \leq 1$ be a smooth curve in M. Then there exists $\epsilon > 0$ such that if $\beta(s)$ is another smooth curve M with same end points as $\alpha(s)$ such that $d_1(\alpha(s), \beta(s)) < \epsilon$ for all s, then there is a smooth homotopy $\gamma(s,\tau)$ with end points fixed such that $\gamma(s,0) = \alpha(s)$ and $\gamma(s,1) = \beta(s)$. Moreover, there is a constant L depending only on (M, g_1), $\max_{0\leq 1 \leq 1}\{|\alpha'(s)|_{g_1} + |\beta'(s)|_{g_1}\}$, such that the length of $\gamma(\cdot,\tau)$ with respect to g_1 is bounded above by L.*

Proof. In the following, all lengths on M will be computed with respect to the metric g_1. Let $\alpha(s)$ be given. Then there is $R > 0$ such that $\alpha \subset B_1(p, R/2)$. First let $\epsilon > 0$ be a lower bound for the injectivity radius of $B_1(p,R)$. Suppose $\beta(s)$ is another smooth curve on M with the same end points as $\alpha(s)$ such that $d_1(\alpha(s),\beta(s))<\epsilon$ for all s. Then there is a smooth homotopy $\gamma(s,\tau)$ such that $\gamma(s,\tau)$, for $0\le\tau\le 1$ is the minimal geodesic from $\alpha(s)$ to $\beta(s)$. Then for each s, $J=\gamma_s$ is a Jacobi field along the geodesic $\gamma(s,\tau)$ for $0\le\tau\le 1$, with boundary value $J(0)=\alpha'(s)$ and $J'(1)=\beta'(s)$. With respect to an orthonormal frame $\{e_i\}$ parallel along $\gamma(s,\tau)$, $0\le\tau\le 1$, the component y_i of J satisfies

$$\begin{bmatrix} y_1'' \\ \vdots \\ y_{2n}'' \end{bmatrix} = A \begin{bmatrix} y_1 \\ \vdots \\ y_{2n} \end{bmatrix},$$

where $A_{ij}=\langle R(\gamma_\tau,e_i)\gamma_\tau,e_j\rangle$. Here $'$ means derivatives with respect to τ. Note that $|\gamma_\tau|\le\epsilon$ and the curvature is bounded from below, and we have

$$\left(\sum_i y_i^2\right)'' \ge -C_1\epsilon^2\sum_i y_i^2$$

for some constant $C_1>0$ depending only on the lower bound of the curvature and n. Hence, if $\epsilon>0$ is small enough depending only on the curvature, we can compare $\sum_i y_i^2$ with the solution f of $f''=-C_1\epsilon^2 f$ with the same boundary value as $\sum_i y_i^2$. Hence

$$|\gamma_s|^2=|J|^2=\sum_i y_i^2\le C_2$$

for some C_2 depending only on g_0 and $\max_{0\le 1\le 1}\{|\alpha'(s)|_{g_1}+|\beta'(s)|_{g_1}\}$.
q.e.d.

We now complete the proof of Theorem 1.2 by proving the following:

Lemma 5.2. *If M is simply connected, then Ψ is injective.*

Proof. Suppose the lemma is false. Then there are distinct points $z_1,z_2\in\Omega$ such that $\Psi(z_1)=\Psi(z_2)=q$. Let $\tilde\gamma(s)$ be a smooth curve in Ω for $s\in[0,1]$, joining z_1 to z_2 parametrized proportional to arc length with respect to the Euclidean metric, and let $\gamma(s)=\Psi\circ\tilde\gamma(s)$. Then $\gamma(0)=\gamma(1)=q$. Let $\gamma(s,\tau)$ be a smooth homotopy of γ for $(s,\tau)\in[0,1]\times[0,1]$ such that $\gamma(s,0)=\gamma(s)$, $\gamma(s,1)=q$ for all $s\in[0,1]$, and $\gamma(0,\tau)=\gamma(1,\tau)=q$ for all $\tau\in[0,1]$. Let $L_1=\max\{l(\gamma(\cdot,t)|\,\tau\in[0,1]\}$, where $l(\gamma(\cdot,\tau))$ is the length of $\gamma(\cdot,\tau)$ with respect to g_1.

Let $R>0$ be fixed, such that $\gamma(s,\tau)\in B_1(p,R)$ for all $0\le s,\tau\le 1$.

By (4.22) and the fact that $S_i\subset S_{i+1}$ for all i, there exists i_0 such that

$$B_1(p,R)\subset\Gamma_j(S_i) \tag{5.1}$$

for all $j\ge i\ge i_0$, and that $\widetilde{\gamma}\subset S_{i_0}$.

Since Ψ is a local biholomorphism, it is easy to see that for any $a > 0$ there is a $b > 0$ such that for all i large enough, $\Gamma_i(D(z_k, a)) \supset B_1(\Gamma_i(z_k), b)$, $k = 1, 2$. Since $\Gamma_i(z_k) \to \Psi(z_k) = q$, by choosing an even larger i_0, for all $i \geq i_0$ there exist $\zeta_{1,i} \neq \zeta_{2,i} \in S_{i_0}$ such that $\Gamma_i(\zeta_{1,i}) = \Gamma_i(\zeta_{2,i}) = q$ and that $\zeta_{1,i} \to z_1$ and $\zeta_{2,i} \to z_2$. Now for i large enough, we can join $\zeta_{1,i}$ to $\zeta_{2,i}$ by first joining $\zeta_{1,i}$ to z_1, then z_1 to z_2 along $\widetilde{\gamma}$, then z_2 to $\zeta_{2,i}$. Let us denote this curve by $\widetilde{\gamma}_i(s)$, $s \in [0, 1]$ parametrized proportional to arc length. We may assume $\widetilde{\gamma}_i(s)$ is smooth, $\widetilde{\gamma}_i(s) \subset K \subset S_{i_0}$ for some compact set K, and $|\widetilde{\gamma}_i'| \leq C_1$ for some constant independent of i for all $i \geq i_0$. Moreover, we have $|\widetilde{\gamma}(s) - \widetilde{\gamma}_i(s)| \to 0$ uniformly over s as $i \to \infty$. Since Ψ is a local biholomorphism, there is a constant C_2 independent of i such that if $\gamma_i = \Psi \circ \widetilde{\gamma}_i$, then

$$|\gamma_i'(s)|_{g_1} \leq C_2. \tag{5.2}$$

For the curve $\gamma(s)$, let ϵ be as in Lemma 5.1. Since Γ_i converge to Ψ uniformly on compact sets together with first derivatives, if i_0 is chosen large enough, then the following are true:

(i) $d_1(\gamma(s), \gamma_i(s)) < \frac{\epsilon}{2}$,
(ii) $d_1(\Gamma_i \circ \widetilde{\gamma}_j(s), \gamma_j(s)) = d_1(\Gamma_i \circ \widetilde{\gamma}_j(s), \Psi \circ \widetilde{\gamma}_j(s)) < \frac{\epsilon}{2}$,
(iii) $|(\Gamma_i \circ \widetilde{\gamma}_j)'(s)|_{g_1} \leq |(\Psi \circ \widetilde{\gamma}_j)'(s)|_{g_1} + C_2 = |\gamma_i'(s)|_{g_1} + C_2 \leq 2C_2$

for $i, j \geq i_0$.

By (i) and (ii), we have:

$$d_1(\gamma(s), \Gamma_i \circ \widetilde{\gamma}_i(s)) < \epsilon$$

for all $i \geq i_0$. Thus by Lemma 5.1 and (5.2), for each $i \geq i_0$ we can find a homotopy which deforms $\gamma(s)$ to $\Gamma_i \circ \widetilde{\gamma}_i(s)$, with end points fixed, so that each curve in the homotopy has length (with respect to g_1) bounded by some constant L independent of i.

Now in Lemma 2.2, let $\rho = r$ and let ρ_1 be as in the conclusion of the lemma. Then we can choose $i \geq i_0$ large enough but fixed, such that $B_1(p, L + L_1 + R + 1) \subset \Phi_{n_i}(D(\rho_3))$, and any curve β in the above homotopies is in $B_1(p, L + L_1 + R + 1)$ and satisfies $L_i(\beta) \leq 1/(L + L_1 + R + 1)\rho_3$. Here we have used **(a3)**.

Let $w_k = \tilde{F}_{n_i} \circ \cdots \circ \tilde{F}_2(\zeta_{k,i})$, $k = 1, 2$. Then $w_1 \neq w_2$. Note that we have

$$\tilde{F}_{n_i} \circ \cdots \circ \tilde{F}_2(S_{i_0}) \subset \tilde{F}_{n_i} \circ \cdots \circ \tilde{F}_2(S_i) \subset D(\rho_3),$$

and $\rho_3 < \frac{1}{8}r$ (see Remark 4.1).

By Corollary 2.2, since the lift of $\Gamma_i \circ \widetilde{\gamma}_i(s)$ in Lemma 2.2 from w_1 by Φ_{n_i} is $\tilde{F}_{n_i} \circ \cdots \circ \tilde{F}_2 \circ \widetilde{\gamma}_i(s)$, the lift $\widetilde{\sigma}(\cdot)$ of $\gamma(\cdot, 1)$ satisfies $\widetilde{\sigma}(1) = \tilde{F}_{n_i} \circ \cdots \circ \tilde{F}_2 \circ \widetilde{\gamma}_i(1) = w_2$. But this would give $\widetilde{\sigma}(0) = w_1 \neq w_2 = \widetilde{\sigma}(1)$, which is impossible because $\Phi_{n_i} \circ \widetilde{\sigma}(s) = \gamma(s, 1)$ is a constant map and Φ_{n_i} is a local biholomorphism. q.e.d.

6. Proof of Theorem 1.1

In this section we prove Theorem 1.1. We begin proving a general theorem on complete solutions to the Kähler-Ricci flow

$$\frac{\partial g_{i\bar{j}}}{\partial t} = -R_{i\bar{j}}. \tag{6.1}$$

Theorem 6.1. *Let $g(t)$ be a complete solution to* (6.1) *with nonnegative holomorphic bisectional curvature such that $g(0)$ has bounded curvature. Fix some $p \in M$ and let $\lambda_i(t)$ be the eigenvalues of $Rc(p,t)$ arranged in increasing order. Then*

$$t\lambda_k(t)$$

is nondecreasing in t for all $1 \le k \le n$.

Proof. To prove the theorem, we may assume again that M is simply connected and by the result of [**5**], we may further assume that the Ricci curvature is positive for all $x \in M$ and for all $t > 0$.

Now let $k \ge 1$, and let $h(t)$ be any positive function with $h'(t) > 0$ for all t. We claim that for any t_0 there is $\epsilon > 0$ such that $th(t)\lambda_k(t) < t_0 h(t_0)\lambda_k(t_0)$ for all $t \in (t_0 - \epsilon, t_0)$. By taking $h(t) = 1 + \delta t$ with $\delta > 0$ and then let $\delta \to 0$, we see that the theorem will follow from this claim which we now prove.

For any t, let $0 < \lambda_1(t) \le \cdots \le \lambda_n(t)$ be the eigenvalues of $R_{i\bar{j}}(p,t)$. For any $\sigma > 0$, let $E_\sigma(t)$ be the direct sum of the corresponding eigenspaces with eigenvalues $\lambda < \sigma$. Now let t_0 be fixed and let $m \ge k$ be the largest integer such that $\lambda_j(t_0) = \lambda_k(t_0)$ for $m \ge j \ge k$. Let $\sigma > 0$ be such that $\lambda_k(t_0) < \sigma < \lambda_{m+1}(t_0)$ if $m < n$ and $\sigma > \lambda_k(t_0)$ if $m = n$. Then there exists $\epsilon > 0$ such that for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$, $\lambda_m(t) < \sigma < \lambda_{m+1}(t)$ if $m < n$ and $\sigma > \lambda_n(t)$ if $m = n$. In any case, the orthogonal projection $P_\sigma(t)$ onto $E_\sigma(t)$ is smooth in $(t_0 - \epsilon, t_0 + \epsilon)$, see [**8**, p.501] for example. For any $t_1 \in (t_0 - \epsilon, t_0 + \epsilon)$, let v_1 be an eigenvector of $R_{i\bar{j}}(t_1)$ corresponding to $\lambda_m(t_1)$ with length 1. Let

$$v(t) = \frac{P_\sigma(t)v_1}{|P_\sigma(t)v_1|_t}.$$

Note that for t close to t_1, $P_\sigma(t)v_1 \ne 0$. In local holomorphic coordinates z^i, let $a(t) = R_{i\bar{j}}v^i\bar{v}^j$ where $v(t) = v^i(t)\frac{\partial}{\partial z^i}$. Note that $P_\sigma(t_1)(v_1) = v_1$ and thus $a(t_1) = \lambda_m(t_1)$. Also note that we have $a(t) \le \lambda_m(t)$ for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$ where $a(t)$ is defined. Now note that

$$0 = \frac{d}{dt}\langle v(t), v(t)\rangle_t = -R_{i\bar{j}}(p,t)v^i v^{\bar{j}} + 2Re\left(g_{i\bar{j}}\frac{dv^i}{dt}v^{\bar{j}}\right). \tag{6.2}$$

Hence, at t_1 we have

$$Re\left(g_{i\bar{j}}\frac{dv^i}{dt}v^{\bar{j}}\right) = \frac{a(t_1)}{2} = \frac{\lambda_m(t_1)}{2}. \tag{6.3}$$

By the Li-Yau-Hamilton type inequality in [**4**], we have

$$\frac{\partial R_{i\bar{j}}}{\partial t} + g^{k\bar{l}} R_{i\bar{l}} R_{k\bar{j}} + \frac{R_{i\bar{j}}}{t} \geq 0 \tag{6.4}$$

for all t. Thus, at t_1 we have

(6.5)

$$\begin{aligned}
0 &\leq \frac{\partial R_{i\bar{j}}}{\partial t} v^i v^{\bar{j}} + g^{k\bar{l}} R_{i\bar{l}} R_{k\bar{j}} v^i v^{\bar{j}} + \frac{R_{i\bar{j}}}{t_1} v^i v^{\bar{j}} \\
&= \frac{d}{dt}(R_{i\bar{j}} v^i v^{\bar{j}}) - 2Re\left(R_{i\bar{j}}\left(\frac{d}{dt}v^i\right)v^{\bar{j}}\right) + g^{k\bar{l}} R_{i\bar{l}} R_{k\bar{j}} v^i v^{\bar{j}} + \frac{R_{i\bar{j}}}{t} v^i v^{\bar{j}} \\
&= \frac{d}{dt}(R_{i\bar{j}} v^i v^{\bar{j}}) - \lambda_m^2(t_1) + \lambda_m^2(t_1) + \frac{R_{i\bar{j}}}{t} v^i v^{\bar{j}} \\
&= \frac{d}{dt}(R_{i\bar{j}} v^i v^{\bar{j}}) + \frac{R_{i\bar{j}}}{t_1} v^i v^{\bar{j}} \\
&= \frac{d}{dt} a + \frac{a}{t_1},
\end{aligned}$$

where the third equality follows from writing the expressions in a holomorphic coordinate z^i such that $\frac{\partial}{\partial z^i}$ form a basis of eigenvectors of $Rc(p,T)$ with $v_1 = \frac{\partial}{\partial z^1}$ at p, and (6.3). Since $a(t_1) > 0$, $h(t) > 0$ and $h'(t) > 0$, we conclude that

$$\frac{d}{dt}\left(th(t)a(t)\right) > 0$$

at t_1 and hence $th(t)a(t)$ is increasing in t for $t \in (t_1 - \epsilon_1, t_1 + \epsilon_1)$ for some $\epsilon_1 > 0$. Hence,

$$\begin{aligned}
t_1 h(t_1)\lambda_m(t_1) &= t_1 h(t_1) a(t_1) \\
&< th(t)a(t) \\
&\leq th(t)\lambda_m(t)
\end{aligned} \tag{6.6}$$

for all $t_1 < t < t_1 + \epsilon_1$. As t_1 was chosen arbitrarily in $(t_0 - \epsilon, t_0 + \epsilon)$, we conclude that $th(t)\lambda_m(t)$ is increasing in $(t_0 - \epsilon, t_0 + \epsilon)$. In particular,

$$\begin{aligned}
t_0 h(t_0)\lambda_k(t_0) &= t_0 h(t_0)\lambda_m(t_0) \\
&> th(t)\lambda_m(t) \\
&\geq th(t)\lambda_k(t)
\end{aligned} \tag{6.7}$$

for all $t \in (t_0 - \epsilon, t_0)$. This proves the claim and the theorem. q.e.d.

Proof of Theorem 1.1. We begin by observing that if M has positive holomorphic bisectional curvature and is simply connected near infinity, then it is actually simply connected. Indeed, if M were not simply connected there would exist a nontrivial minimizer of a free homotopy class. This, however, is impossible by the fact that the bisectional curvature is positive, and an argument as in the proof of Sygne theorem. So the second assertion of the theorem follows from the first.

To prove the first assertion, by the remarks at the beginning of § 5, we may assume that M is simply connected in Theorem 1.1 and by [**5**] we may also assume that the Ricci curvature is positive in spacetime.

Now by the long time existence results in [**30**], we know that under the hypothesis of Theorem 1.1, (6.1) has a long time solution $g(t)$ with uniformly bounded non-negative holomorphic bisectional curvature together with the covariant derivatives of the curvature tensor. Since $Rc > 0$, Theorem 6.1 implies that given any compact set Ω we can find $C > 0$ such that $Rc(t) \geq \frac{C}{t} g(t)$ on Ω for all t. From this, (6.1) and recalling that the curvature of $g(t)$ is uniformly bounded on $[0, \infty) \times M$, it is not hard to see that the sequence of metrics $g_i = g(i)$ on M satisfies the hypothesis of Theorem 1.2, and thus Theorem 1.1 follows. q.e.d.

References

[1] E. Anderson & L. Lempert, *On the group of holomorphic automorphisms of* $\mathbb{C}^n$, Invent. Math. **110**(**2**) (1992) 371–388, MR 1185588, Zbl 0770.32015.

[2] M. Brown, *The monotone union of open n-cells is an open n-cell*, Proc. Amer. Math. Soc. **12** (1961) 812–814, MR 0126835, Zbl 0103.39305.

[3] H.-D. Cao, *On Harnack's inequality for the Kähler-Ricci flow*, Invent. Math. **109** (1992) 247–263, MR 1172691, Zbl 0779.53043.

[4] ———, *Limits of solutions to the Kähler-Ricci flow*, J. Differential Geom. **45** (1997) 257–272, MR 1449972, Zbl 0889.58067.

[5] ———, *On Dimension reduction in the Kähler-Ricci flow*, Comm. Anal. Geom. **12** (2004) 305–320, MR 2074880, Zbl 1075.53058.

[6] A. Chau & L.-F. Tam, *Gradient Kähler-Ricci soliton and a uniformization conjecture*, arXiv eprint 2002, arXiv:math.DG/0310198.

[7] ———, *A note on the uniformization of gradient Kähler-Ricci solitons*, Math. Res. Lett. **12** (2005) 19–21, MR 2122726, Zbl 1073.53082.

[8] ———, *On the complex structure of Kähler manifolds with non-negative curvature*, J. Differential Geom. **73** (2006) 491–530, MR 2228320.

[9] ———, *Non-negatively curved Kähler manifolds with average quadratic curvature decay*, Comm. Anal. Geom. **15**(**1**) (2007) 121–146, MR 2301250.

[10] B.L. Chen, S.H. Tang, & X.P. Zhu, *A Uniformization Theorem Of Complete Noncompact Kähler Surfaces With Positive Bisectional Curvature*, J. Differential Geom. **67** (2004) 519–570, MR 2153028.

[11] B.L. Chen & X.P. Zhu, *On complete noncompact Kähler manifolds with positive bisectional curvature*, Math. Ann. **327** (2003) 1–23, MR 2005119, Zbl 1034.32015.

[12] ———, *Positively Curved Complete Noncompact Kähler Manifolds*, arXiv: math.DG/0211373.

[13] ———, *Volume Growth and Curvature Decay of Positively Curved Kähler manifolds*, Q. J. Pure Appl. Math. **1** (2005) 68–108, MR 2154333.

[14] R.E. Greene & H. Wu, *Analysis on noncompact Kähler manifolds*, Proc. Sympos. Pure Math., **30**(**2**) (1977) 69–100, MR 0460699, Zbl 0383.32005.

[15] ______, *C^∞ convex function and the manifolds of positive curvature*, Acta. Math. **137** (1976) 209–245, MR 0458336.

[16] D. Gromoll & W. Meyer, *On complete open manifolds of positive curvature*, Ann. of Math. **90** (1969) 75–90, MR 0247590, Zbl 0191.19904.

[17] R.S. Hamilton, *Formation of Singularities in the Ricci Flow*, Surveys in differential geometry **II** (1995) 7–136, MR 1375255, Zbl 0867.53030.

[18] N. Mok, *An embedding theorem of complete Kä manifolds of positive bisectional curvature onto affine algebraic varieties*, Bull. Soc. Math. France. **112** (1984) 179–258, MR 0788968, Zbl 0536.53062.

[19] ______, *An embedding theorem of complex Kähler manifolds of positive Ricci curvature onto quasi-projective varieties*, Math. Ann. **286**(**1-3**) (1990) 373–408, MR 1032939, Zbl 0711.53057.

[20] N. Mok, Y.-T. Siu, & S.-T. Yau, *The Poincaré-Lelong equation on complete Kähler manifolds* , Comp. Math. **44** (1981) 183–218, MR 0662462, Zbl 0531.32007.

[21] L. Ni, *Vanishing theorems on complete Kähler manifolds and their applications*, J. Differential Geom. **50** (1998) 89–122, MR 1678481, Zbl 0963.32010.

[22] ______, *Ancient solutions to Kähler-Ricci flow*, Math. Res. Lett. **12** (2005) 633–653, MR 2189227.

[23] ______, *A matrix Li-Yau-Hamilton estimate for Kähler-Ricci flow*, J. Differential Geom. **75**(**2**) (2007) 303–358, MR 2286824.

[24] L. Ni, Y.-G. Shi, & L.-F. Tam, *Poisson equation, Poincaré-Lelong equation and curvature decay on complete Kähler manifolds*, J. Differential Geom. **57** (2001) 339–388, MR 1879230, Zbl 1046.53025.

[25] L. Ni & L.-F. Tam, *Kähler-Ricci flow and the Poincaré-Lelong equation*, Comm. Anal. Geom. **12** (2004) 111–141, MR 2074873, Zbl 1067.53054.

[26] ______, *Plurisubharmonic functions and the structure of complete Kähler manifolds with nonnegative curvature*, J. Differential Geom. **64** (2003) 457–524, MR 2032112.

[27] W.-X. Shi, *Ricci deformation of the metric on complete noncompact Riemannian manifolds*, J. Differential Geom. **30** (1989) 223–301, MR 1010165, Zbl 0686.53037.

[28] ______, *Ricci deformation of the metric on complete noncompact Kähler manifolds*, Ph.D. thesis, Harvard University, 1990.

[29] ______, *Complete noncompact Kähler manifolds with positive holomorphic bisectional curvature*, Bull. Amer. Math. Soc. (N. S.) **23** (1990) 437–400, MR 1044171, Zbl 0719.53043.

[30] ______, *Ricci Flow and the uniformization on complete non compact Kähler manifolds*, J. Differential Geom. **45** (1997) 94–220, MR 1443333, Zbl 0954.53043.

[31] Y.-T. Siu, *Pseudoconvexity and the problem of Levi*, Bull. Amer. Math. Soc. **84** (1978) 481–512, MR 0477104, Zbl 0423.32008.

[32] S.-T. Yau, *Some Function-Theoretic Properties of Complete Riemannian Manifold and Their Applications to Geometry*, Indiana University Mathematics Journal **25**(**7**) (1976) 659–670, MR 0417452, Zbl 0335.53041.

[33] ______, *A review of complex differential geometry*, Proc. Sympos. Pure Math. **52**(**2**) (1991) 619–625.

[34] H. Wu, *An elementary methods in the study of nonnegative curvature*, Acta. Math. **142** (1979) 57–78, MR 1128577, Zbl 0739.32001.

[35] X.P. Zhu, *The Ricci Flow on Complete Noncompact Kähler Manifolds*, Ser. Geom. Topol., **37**, 525–538, Int. Press, Somerville, MA, 2003, MR 2143257.

Department of Mathematics
The University of British Columbia
Room 121, 1984 Mathematics Road
Vancouver, B.C., Canada V6T 1Z2

E-mail address: chau@math.ubc.ca

The Institute of Mathematical Sciences
and Department of Mathematics
The Chinese University of Hong Kong
Shatin, Hong Kong, China

E-mail address: lftam@math.cuhk.edu.hk