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TOTALLY GEODESIC SEIFERT SURFACES IN HYPERBOLIC KNOT AND LINK COMPLEMENTS II

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Abstract

We generalize the results of Adams–Schoenfeld [**2**], finding large classes of totally geodesic Seifert surfaces in hyperbolic knot and link complements, each covering a rigid 2-orbifold embedded in some hyperbolic 3-orbifold. In addition, we provide a uniqueness theorem and demonstrate that many knots cannot possess totally geodesic Seifert surfaces by giving bounds on the width invariant in the presence of such a surface. Finally, we utilize these examples to demonstrate that the Six Theorem is sharp for knot complements in the 3-sphere.

1. Introduction

Define a knot or link in S^3 to be hyperbolic if its complement is a hyperbolic 3-manifold of finite volume. This implies that there is a covering map p from $\mathbf{H}^3$ to $S^3 - K$ such that the covering translations are isometries of $\mathbf{H}^3$. We say that an embedded or immersed surface S in $S^3 - K$ is totally geodesic if it is isotopic to a surface that is covered by a set of geodesic planes in $\mathbf{H}^3$. Throughout this paper, we will be using the upper half-space model of $\mathbf{H}^3$, where the pre-image of a particular cusp neighborhood is a union of horoballs. In particular, we employ pictures generated by Jeff Weeks' program SnapPea [**8**] displaying the pattern of horoballs in the pre-images of the cusps by looking down at the $\{xy\}$-plane from above.

In Adams–Schoenfeld [**2**], the first examples of totally geodesic Seifert surfaces in knot complements were produced. The main idea of these examples is that certain knot complements cover hyperbolic 3-orbifolds. If a surface S in the knot complement projects to a rigid 2-orbifold under the covering map, then S must indeed be totally geodesic.

Section 2 generalizes this class of examples, utilizing spherical 3-orbifolds as listed in Dunbar [**4**], and rigid 2-orbifolds as appear in Thurston [**7**]. A spherical 3-orbifold O has universal cover S^3, so if

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we drill out appropriate curves from O to obtain a hyperbolic 3-orbifold O', then O' will be covered by a hyperbolic knot or link complement in S^3. Any rigid 2-orbifold embedded in O' will be covered by a totally geodesic surface in the knot or link complement. We will be particularly interested in surfaces which are bounded by the knot or link.

Definition 1.1. A **Seifert surface** for a knot or link is an orientable surface whose boundary is the knot or link. A nonorientable surface which is bounded by the knot or link is called a **nonorientable Seifert surface**.

We often consider Seifert surfaces in the knot or link exterior, $S^3 - N(L)$. In this case, the boundary of S is a union of l-**curves**, one in each cusp boundary, where an l-curve is defined to be a closed curve in the boundary of a cusp neighborhood which intersects the meridian exactly once. It is a fact that the boundary of a Seifert surface in a knot complement is a longitude, defined as the l-curve which has linking number 0 with the missing core curve of the cusp.

Additionally, we often need to distinguish between different types of surfaces using the following categorization:

Definition 1.2. Let S be an embedded surface in the complement of a link L. Then S is **free** if $S^3 - N(L) - N(S)$ is a handlebody. We say S is **totally knotted** if $S^3 - N(L) - N(S)$ has incompressible boundary. We say S is **semifree** if there exists a compressing disk for $\partial(S^3 - N(L) - N(S))$. Note that free implies semifree.

In Subsection 2.3 we provide a series of examples of orientable and nonorientable totally geodesic Seifert surfaces (both free and totally knotted) in knot and link complements using the methods described in Section 2. It remains an open question as to whether a knot can have a nonorientable totally geodesic Seifert surface.

Additionally, in Adams–Schoenfeld [**2**] the search for further examples was narrowed through a proof that there are no totally geodesic Seifert surfaces in two-bridge knot complements. With the same goal of limiting the existence of totally geodesic Seifert surfaces in mind, we have the following theorem of Section 3:

Theorem 3.3. *Given a semifree totally geodesic Seifert surface S embedded in the complement of a knot or link L, there exists no other totally geodesic Seifert surface embedded in $S^3 - L$ with the same boundary slope on each component of L.*

Indeed, knowing that all Seifert surfaces in knot complements have the same boundary slope leads us to the following corollary:

Corollary 3.4. *Given a semifree totally geodesic orientable Seifert surface S embedded in the complement of a knot K, there exists no other totally geodesic orientable Seifert surface embedded in $S^3 - K$.*

With a similar goal in mind, Section 4 defines the **width** invariant for surfaces, which is itself motivated by a definition of width for an l-curve.

Definition 1.3. Given a nontrivial, minimal length closed curve γ on a maximal cusp boundary C, we call the length of the shortest path in C which starts and ends on γ, but which is not isotopic into γ, the **width** with respect to γ, sometimes denoted w_γ. We sometimes discuss the width of a knot K, denoted $w(K)$, by which we mean the width with respect to the longitude of K.

Definition 1.4. Let S be a Seifert surface or a nonorientable Seifert surface in the complement of a hyperbolic knot or link L. Then, by definition, ∂S is a union of l-curves, with exactly one l-curve on each cusp. We can choose the cusps to be disjoint from one another such that the widths of all the l-curves are equal. We can then expand the cusps while keeping the widths of the l-curves equal, until there is a cusp tangency. We call the resulting width the **balanced width** of the surface S. Note that by definition, the width of a knot must be balanced.

It turns out that the balanced width of a totally geodesic surface has a very predictable behavior, leading to the following series of theorems.

Theorem 4.1. *Consider a hyperbolic knot or link L. If there exists a semifree totally geodesic Seifert surface S, orientable or nonorientable, with balanced width w in $S^3 - L$, then $w < 2$.*

This bound is actually the best possible, as demonstrated by the free totally geodesic surfaces in the (p, p, p)-pretzel knots (Example 2.2), which have width approaching 2 from below as p approaches infinity. On the other hand, the semifree restriction is indeed necessary, as shown by the totally knotted totally geodesic surface in Example 2.3 which has width greater than 2.

Theorem 4.2. *Consider a hyperbolic knot or link L. If there exists an embedded totally geodesic Seifert surface S, orientable or nonorientable, with balanced width w in $S^3 - L$, then $w \geq 1$.*

Indeed, knowing that $w(S) \geq 1$ for any totally geodesic Seifert surface allows us to eliminate a very large class of knots from having totally geodesic Seifert surfaces via the following theorem.

Theorem 4.5. *Consider a hyperbolic knot K with an oriented projection P. Form a sequence of knots $\{K_i\}$ by twisting similarly oriented strands incident to the same region in the projection plane about each other so as to add an even number of crossings, as in Figure* 13*. If, for some $N > 0$, $n > N$ implies K_n is hyperbolic, then*

$$\lim_{i \to \infty} w(K_i) = 0.$$

Corollary 4.6. *With K_n as above, for some positive integer N and all $n > N$, the complement of K_n does not possess a totally geodesic Seifert surface.*

Note that in the case that the initial projection is a reduced prime alternating projection that does not correspond to a 2-braid knot, and we twist to create alternating knots, all of the knots will be hyperbolic by results of Menasco. And if the twist makes the resulting projection nonalternating, it is still true that for enough twists, the resulting knots will all be hyperbolic.

The final theorem regarding width requires a technical definition that will be useful throughout the paper.

Definition 1.5. Let S be a semifree surface with boundary in the complement of a link L, and let D be a compressing disk for $\partial(S^3 - N(L) - N(S))$. Since S is itself incompressible and boundary incompressible, ∂D alternates between n arcs in S and n arcs in the cusp boundaries for some $n > 1$. Then if n cannot be reduced through isotopy while preserving the property that D is a compressing disk, we say that D is an **essential n-gon** in the complement of S.

Theorem 4.3. *Let L be a hyperbolic knot or link and S a totally geodesic Seifert surface, orientable or nonorientable, embedded in $S^3 - L$ with balanced width w. Then $w = 1$ if and only if there is an essential 3-gon in the complement of S.*

Finally, in Section 5, we look at an application of the examples. The Six Theorem, proven independently by Ian Agol [**3**] and Mark Lackenby [**5**], shows that for a finite volume hyperbolic 3-manifold N with single embedded horocusp C, performing Dehn surgery on a curve α such that the length of α is strictly greater than six, always yields a hyperbolike manifold. (See Section 5 for more details.) Moreover, Agol demonstrated that this bound is sharp by giving an explicit example of a hyperbolic 3-manifold with two cusps and a curve in its cusp boundary of length exactly six. Dehn fillings with high coefficients on the remaining cusp yields manifolds with one cusp and a curve on the boundary of length arbitrarily closer to six such that Dehn surgery on the curve yielded a non-hyperbolike manifold. His example was not a knot complement in the 3-sphere. In this section, we prove:

Theorem 5.3. *The Six Theorem is sharp for knot complements in the 3-sphere, with (p, p, p) pretzel knots as examples, for every odd $p \geq 3$.*

Acknowledgements. Thanks to Ian Agol for his generous help in improving the exposition of this paper.

2. Generating Totally Geodesic Seifert Surfaces

2.1. Background on 2-Orbifolds and 3-Orbifolds. An n-orbifold is a Hausdorff space X^n, along with neighborhoods locally modelled on R^n/Γ where Γ is a finite group action. We define the singular set of an orbifold to be the set of points in X^n that are locally modelled on R^n/Γ where Γ is not the identity. Specifically, the singular set of a 2-orbifold may contain the following:

- Cone points of order n - modelled on R^2/Z_n, where Z_n acts by rotations,
- Corner reflectors of order n - modelled on R^2/D_n, where D_n is the dihedral group of order n, and
- Mirrors - modelled on R^2/Z_2, Z_2 acts by reflection.

Likewise, the singular set for an orientable 3-orbifold consists of a trivalent graph. The edges of order n are modelled on R^3/Z_n, where Z_n acts by rotations. We label each such edge by n, except when the edge is modelled on R^3/Z_2. The vertices are modelled on the quotient of R^3 by either the dihedral group of order $2n$, the tetrahedral group, the octahedral group, or the icosahedral group. Thus the edges emanating from a vertex must be one of the following combinations: (2, 2, n), (2, 3, 3), (2, 3, 4), or (2, 3, 5). For more details on orbifolds, see [**7**, Chapter 13] or [**4**].

In this paper we will denote 2-orbifolds as $X^2(;)$, where X^2 is the underlying Hausdorff space, the numbers before the ";" are cone points and numbers after the ";" are the corner reflectors. In our notation all points in the boundary of X^2 which are not corner reflectors are mirror points.

We are specifically interested in spherical 3-orbifolds and rigid 2-orbifolds. A spherical 3-orbifold is an orbifold with an orbifold covering map from S^3. The spherical 3-orbifolds are partitioned into a finite number of classes and the complete list of these classes can be found in [**4**]. Figure 1 contains examples of spherical 3-orbifolds.

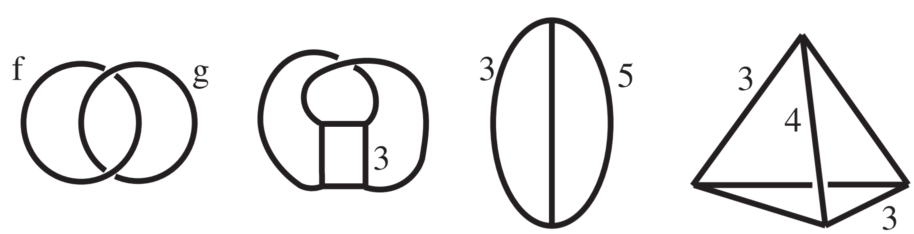

Figure 1. A few examples of spherical 3-orbifolds. On the left, $f, g \in Z^+$.

A 2-orbifold is rigid if it is hyperbolic and its Teichmüller space has dimension 0. In other words, the orbifold has a unique hyperbolic structure. In a hyperbolic 2-orbifold, the dimension of the Teichmüller space can be obtained from the function $-3\chi(X^2)+2k+l$ [**7**], where $\chi(X^2)$ is the Euler characteristic of the underlying space, k is the number of cone points, and l is the number of corner reflectors. The orbifolds in Table 1 are the only 2-orbifolds for which the Teichmüller space has dimension 0.

Table 1. Table of Rigid 2-Orbifolds.

Hyperbolic Rigid 2-Orbifolds	Exceptions (these are not hyperbolic)
$S^2(n,m,p)$	$S^2(2,2,n), S^2(2,3,3),$ $S^2(2,3,4), S^2(2,3,5),$ $S^2(2,3,6), S^2(2,4,4), S^2(3,3,3)$
$D^2(n;m)$	$D^2(2;n), D^2(3;2), D^2(3;3), D^2(4;2)$
$D^2(;n,m,p)$	$D^2(;2,2,n), D^2(;2,3,3),$ $D^2(;2,3,4), D^2(;2,3,5),$ $D^2(;2,3,6), D^2(;2,4,4), D^2(;3,3,3)$

Note that a cone point labeled with a positive integer n corresponds to an elliptic isometry of order n. A cone point labeled with "∞" corresponds to a parabolic isometry and can be thought of as a puncture in the interior of the 2-orbifold. Similarly, a corner reflector point labeled "∞" is thought of as a puncture on the boundary of the 2-orbifold. We define the infinity set of a hyperbolic 2-orbifold to be the set of infinity cone points and corner reflectors. For instance, the 2-orbifold $S^2(2,3,\infty)$ is equivalent to an open disk (a sphere with a puncture) with cone points of order 2 and 3, and its infinity set is the boundary of this disk.

2.2. Embedding of 2-Orbifolds inside 3-Orbifolds.

Theorem 2.1. *Let J be a collection of disjoint arcs and simple closed curves in a spherical 3-orbifold N, such that their complement is a hyperbolic 3-orbifold Q containing a rigid 2-orbifold O with non-empty infinity set. If the prei-mage of J in the covering of N by S^3 is a knot or link and if the pre-image of O is a Seifert surface S for that knot or link then S is isotopic to a totally geodesic Seifert surface.*

This appears as Corollary 2.2 in [**2**]. Note that when we speak of a 2-orbifold embedded in a 3-orbifold, we are assuming that the 2-orbifold inherits its singular set from the 3-orbifold.

To apply this theorem, we need to consider rigid 2-orbifolds with nonempty infinity set. Since we consider a sphere with one "∞" cone point to be the same as an open disk, we think of a puncture from an "∞" cone point in a 2-orbifold as removing a closed disk from the interior of the 2-orbifold. We will often represent "∞" cone points as "∞" closed loops. Likewise, an "∞" corner reflector can be thought of as removing a closed disk, centered at the corner reflector, from the 2-orbifold. We will often represent an "∞" corner reflector as an "∞" arc in the 2-orbifold.

Because we want the pre-image of the infinity set in S^3 to be a link, any "∞" arc in a 2-orbifold must end on a 2-axis of the 3-orbifold. When a 2-orbifold embedded in a 3-orbifold has two "∞" corner reflectors with a path connecting them that consists of only mirror points, then their corresponding "∞" arcs must end on a common 2-axis in the 3-orbifold. An orbifold with three infinity corner reflectors will have three "∞" arcs where each pair of arcs end on a common 2-axis in the 3-orbifold. Since $D(n;\infty)$ has only one corner reflector, its "∞" arc must start and end on the same 2-axis in the 3-orbifold. Cone points that are not in the infinity set, say of degree n, are realized in the 2-orbifold by an intersection with an axis of order n in the 3-orbifold. Corner reflectors of order n are realized in the 2-orbifold as the intersection point of two 2-axes in the 3-orbifold. Examples of these can be seen in Figure 2. The order of the corner reflector corresponds to the angle π/n between the 2-axes.

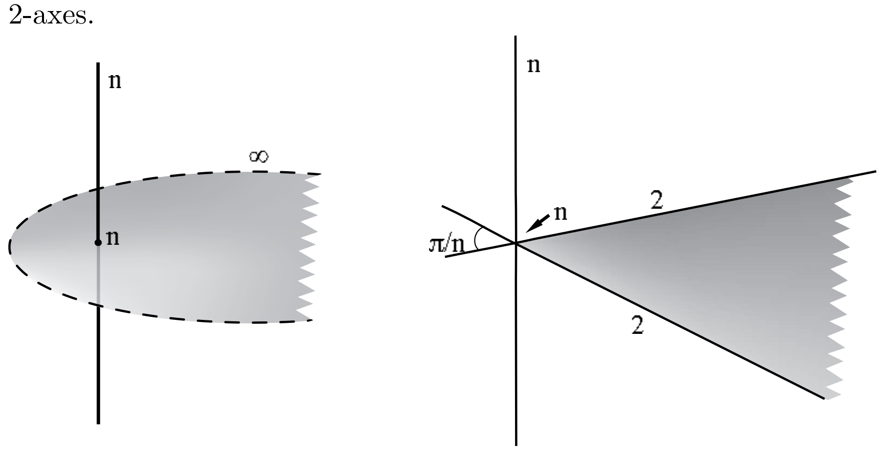

Figure 2. Left: cone point of order n. Right: Corner reflector of order n.

Finally, we must ensure that the resulting link is hyperbolic. The creation of essential tori, annuli and spheres must be avoided.

Immersed totally geodesic surfaces can also be generated from a similar process using immersed rigid 2-orbifolds. Examples of immersed surfaces appear in the following examples section.

2.3. Examples. Now that we have the background, we can look at a few interesting examples of totally geodesic surfaces generated with this method.

Example 2.2. Pretzel knots.

An $(n, n, \ldots, n)$ pretzel knot is a knot with some number of arms, each of which contains n crossings. The $(n, n, \ldots, n)$ pretzel knot is shown in Figure 3 with a generating orbifold. The grey surface in the figure is a totally geodesic Seifert surface in the knot complement. Note that there is more than one way to embed a rigid 2-orbifold in a hyperbolic 3-orbifold such that it is covered by the $(3, 3, 3)$ pretzel knot: in Figure 3 the rigid 2-orbifold is a $S^2(\infty, 3, 3)$, while in Figure 4 it is a $D^2(;\infty, 2, 3)$. In Figure 4, the knot is drawn with symmetry axes corresponding to the axes of the generating 3-orbifold.

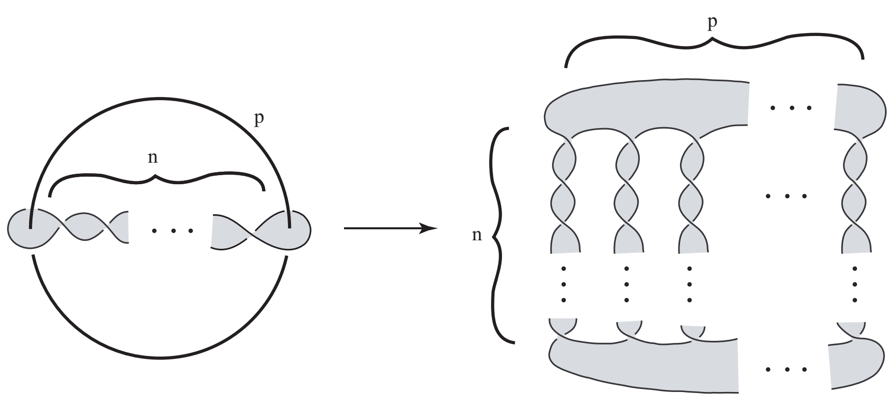

Figure 3. The $(n, n, \ldots, n)$ pretzel knot, and a generating orbifold.

Similar surfaces can be made in any $(p, p, \ldots, p)$ pretzel knot complement.

Example 2.3. A totally knotted surface.

Another way to make more complicated knots is to knot up the $S^2(\infty, 3, 3)$ orbifold, as in Figure 5. The result is a totally knotted totally geodesic surface. This surface, as in the previous example, is orientable. It is of note that this knot has width greater than 2; thus 2 as an upper bound on width for semifree surfaces from Theorem 4.1 does not hold in the totally knotted case.

Example 2.4. The Whitehead link.

An example of a nonorientable totally geodesic checkerboard surface in a link complement is found in the Whitehead link, shown in Figure 7.

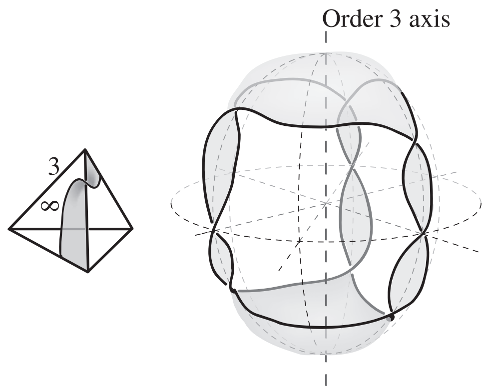

Figure 4. Another view of the $(3, 3, 3)$ pretzel knot with another generating orbifold.

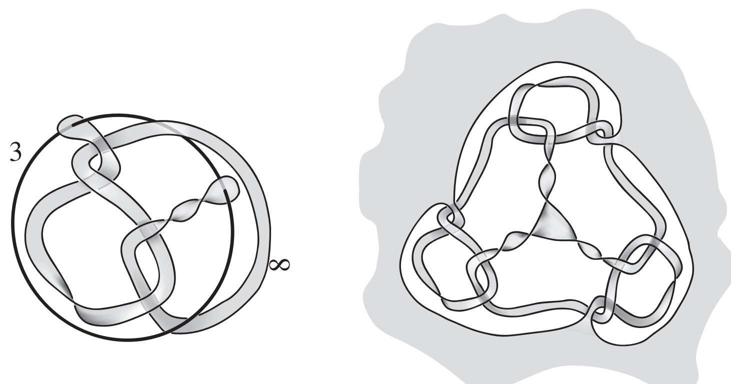

Figure 5. A totally knotted surface.

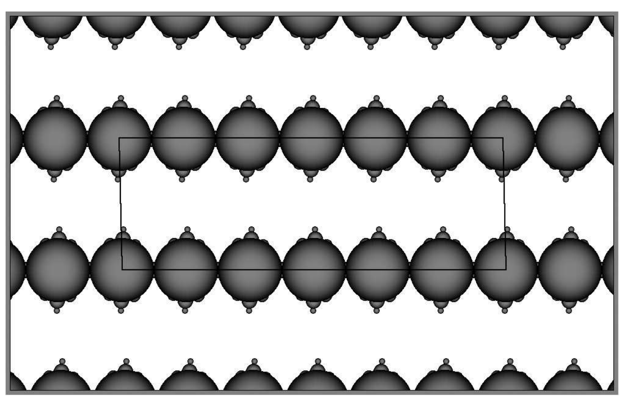

Figure 6. The horoball diagram for the totally knotted surface.

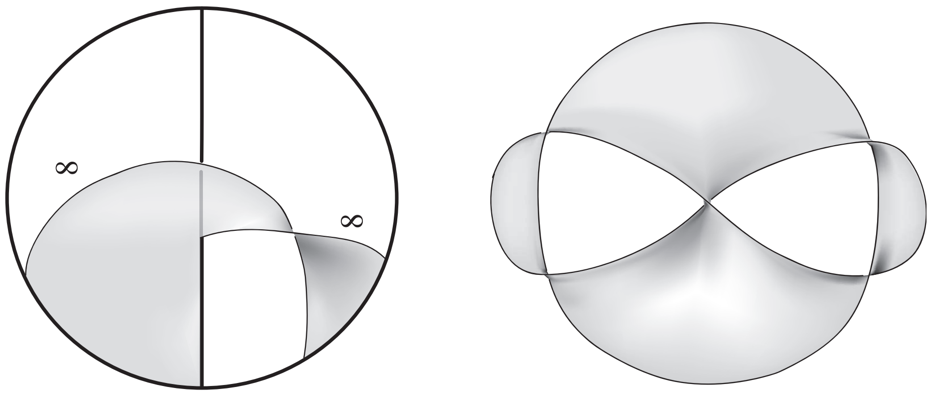

Figure 7. The Whitehead link.

It is not known whether nonorientable totally geodesic Seifert surfaces exist in knot complements.

By twisting up the two infinity arcs, or by twisting one arc around the 2-axis it ends on, we can create a family of links containing totally geodesic surfaces. These are (2p, 2q + 1, 2p) pretzel links. These orbifolds will always have link complement covers, and they always generate nonorientable totally geodesic surfaces in those link complements.

Example 2.5. Multiple totally geodesic surfaces.

The link in Figure 8 is of particular interest because it has two totally geodesic checkerboard surfaces. The link can be realized as the cover of a 3-orbifold containing two rigid orbifolds at once–a $D^2(3;\infty)$ and a $D^2(4;\infty)$. Two other links are known to have two totally geodesic checkerboard surfaces; they are generated by a similar configuration: by orbifolds sitting in a spherical 3-orbifold with axes labelled (2, 3, 3) (this gives the Borromean rings) and (2, 3, 5).

There is also an immersed totally geodesic surface in this link complement. The surface that comes from $D^2(3;\infty)$ also covers the $S^2(\infty,3,3)$ shown in Figure 9. The surface in grey is a self-intersecting $S^2(2,\infty,\infty)$. It is covered by a set of four thrice-punctured disks, all intersecting each other, each with with a different link component as boundary.

Pre-images of all three of these totally geodesic surfaces can be seen in the horoball diagram in Figure 10. Vertical planes covering the immersed surfaces have boundary lines that cover the meridian and longitude, while vertical planes covering the embedded surface have boundaries given by the lines drawn in white.

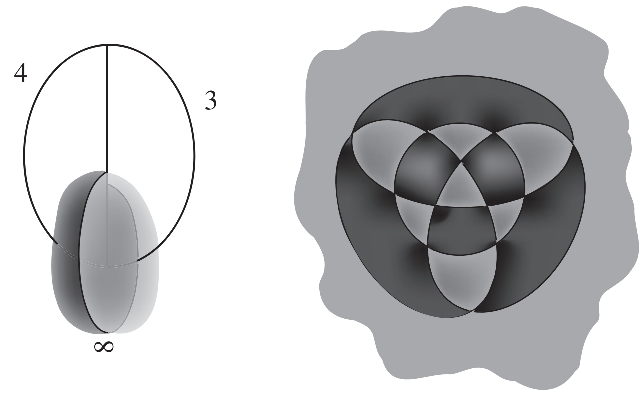

Figure 8. A link with two embedded totally geodesic checkerboard surfaces.

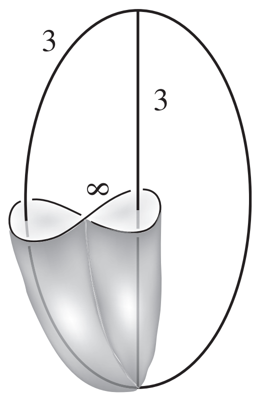

Figure 9. Another orbifold that is covered by this link.

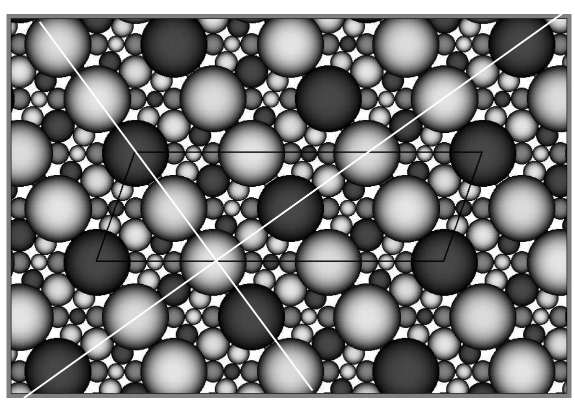

Figure 10. Horoball diagram.

3. Using Topological Means to Disprove the Existence of Totally Geodesic Surfaces

There are some properties, both in knot and link complements and in the surfaces themselves, that allow us to eliminate surfaces from contention as totally geodesic candidates, and eventually to prove a fact about the uniqueness of totally goedesic surfaces in a given knot or link complement. In [**2**], the authors use the topological properties of totally geodesic surfaces to show that no such surface can have a bigon in its complement.

Theorem 3.1. *Any surface S with an essential bigon in its complement cannot be totally geodesic.*

Proof. Cut the manifold open along the totally geodesic surface and double it. The essential bigon doubles to an essential annulus, contradicting the hyperbolicity of the doubled manifold. q.e.d.

In the case of a checkerboard surface for an alternating knot or link, these bigons can occur in any reduced alternating projection.

Theorem 3.2. *An n-gon region R in the projection plane of a reduced alternating diagram in the complement of a totally geodesic checkerboard surface S must correspond to an essential n-gon.*

Proof. Each of the crossing arcs corresponding to this projection is an essential arc in the surface S. Since S is totally geodesic, they must isotope to geodesics, and two of the geodesics are isotopic in the link complement if and only if they are isotopic on S. Two of these geodesics are isotopic on S if and only if the corresponding crossing arcs are related through a sequence of bigonal regions making up the checkerboard surface S. Each geodesic arc lifts to a collection of geodesics in hyperbolic 3-space, each of which connects two distinct horoballs in $\mathbf{H}^3$. Since the pre-image of R is a collection of disks in $\mathbf{H}^3$, a given component $\widetilde{R}$ is a disk that is bounded by an alternating sequence of n horoballs and geodesics connecting the horoballs. If S consists of no bigonal regions when realized as a checkerboard surface, then the geodesics corresponding to the crossing arcs are all distinct. If there is at least one bigon making up S, then two crossing arcs that share a bigon will be isotopic to the same geodesic. However, the boundary of R will then run up the geodesic from one end to the other, around a meridian on the cusp boundary and then down the same geodesic before continuing on. Hence, the boundary of $\widetilde{R}$ will run along a lift of the geodesic, then along an arc on a horosphere covering a meridian, and then along a distinct lift of the geodesic. Hence all n of the geodesics in the boundary of $\widetilde{R}$ are distinct and R cannot be isotoped to decrease n. Hence, it is an essential n-gon. q.e.d.

Observe that in a given reduced alternating projection P of a link L, the checkerboard surfaces can only be totally geodesic if they contain no bigons in their complement in P. Since checkerboard surfaces are complementary in P, any projection P for which both checkerboard surfaces contain bigons cannot generate a totally geodesic checkerboard surface, although a different reduced alternating projection may generate such. However, since all reduced alternating projections are related through flypes, it is straightforward to check whether or not every reduced alternating projection has this property. Note that Example 2.5 yields an example of an alternating link with no bigons in a reduced alternating projection, and both of the corresponding checkerboard surfaces are in fact totally geodesic.

3.1. Uniqueness of Totally Geodesic Seifert Surfaces for Knots and Links. In many cases, if there exists a totally geodesic Seifert surface, it is unique.

Theorem 3.3. *Given a semifree totally geodesic Seifert surface S embedded in the complement of a knot or link L, there exists no other totally geodesic Seifert surface embedded in that complement with the same boundary slope on each component of L.*

Proof. Since S is semifree, there exists a compressing disk D. Let $\widetilde{D}$ be a particular component of $p^{-1}(D)$. The boundary of $\widetilde{D}$ is a curve that lies alternately on a cyclic sequence of geodesic planes in $p^{-1}(S)$ and the series of horoballs, not necessarily distinct, that occur at their points of tangency. Let that cyclic sequence of geodesic planes be denoted $\widetilde{S}_1$, $\ldots$, $\widetilde{S}_n$, where $n \geq 3$. Denote the horoball that occurs at the point of tangency between $\widetilde{S}_i$ and $\widetilde{S}_{i+1}$ as H_i, and let P_i be that tangency point. We consider H_1 to be the horoball at infinity. (See Figure 11.)

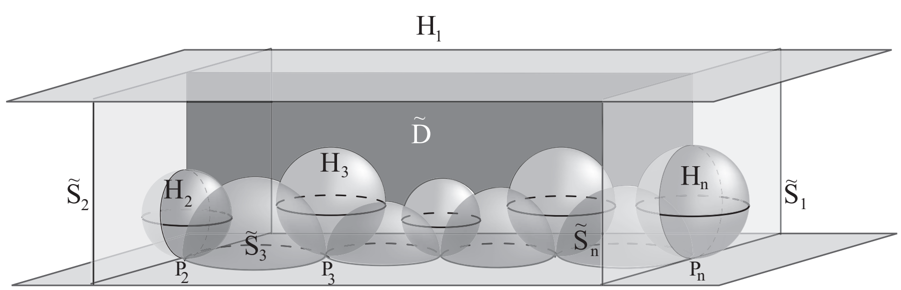

Figure 11. The disk $\widetilde{D}$ borders on a series of geodesic planes and horoballs.

Now, assume there exists another totally geodesic Seifert surface T embedded in $S^3 - L$ which has the same boundary slope as S for each component of L, but is distinct from S. A meridian of L thus intersects the boundaries of T and S exactly once each. As we continue to trace the meridian multiple times, we alternate between intersections with the boundary of T and intersections with the boundary of S. Thus, for any two geodesic planes $\widetilde{S}_i$ and $\widetilde{S}_{i+1}$ tangent at P_i, there must be a geodesic plane $\widetilde{T}_i$ covering T that contains P_i, is distinct from each of $\widetilde{S}_i$ and $\widetilde{S}_{i+1}$, and that separates them. Note that these planes may coincide for distinct values of i. Since the resulting collection of hemispheres is finite and disjoint, there exists one that contains no others in the ball that it bounds in H^3. However, because it cannot coincide with either of the two geodesic planes it is separating, it must contain some P_j in the interior of the disk that is its projection. However, this contradicts the fact that there must be a separating geodesic plane in $p^{-1}(T)$ with P_j on its boundary. q.e.d.

In the case that L is a knot, and S is orientable, we have the following corollary:

Corollary 3.4. *Given a semifree totally geodesic orientable Seifert surface S embedded in the complement of a knot K, there exists no other totally geodesic orientable Seifert surface embedded in $S^3 - K$.*

Proof. Since S is an orientable Seifert surface in a knot complement, it has boundary slope parallel to the longitude. Any other such surface must also have the same boundary slope. By the previous theorem, there can be no such surface distinct from S. q.e.d.

4. The Width Invariant for Totally Geodesic Surfaces

In this section, we consider how width can impact the possible existence of totally geodesic Seifert surfaces.

4.1. Bounds on Width.

Theorem 4.1. *Consider a hyperbolic knot or link L. If there exists a semifree totally geodesic Seifert surface S, orientable or nonorientable, with balanced width w in $S^3 - L$, then $w < 2$. This upper bound is best possible.*

Proof. Since S is semifree, there is a compressing disk D in $S^3 - (N(K) \cup N(S))$. The boundary of D consists of arcs which alternate between lying in the boundary of the cusp set $\{C_i\}$ and lying in the surface. Indeed, the set $\partial D \cap (\cup\{C_i\})$ is a collection of arcs each of which travels on some element of $\{C_i\}$ nontrivially from S back to S. Thus, the length of each arc in this set is greater than or equal to the balanced width w of the cusp set $\{C_i\}$. Choose D so that the number of arcs in $\partial D \cap (\cup\{C_i\})$ is minimized.

The disk D is covered by a collection of closed disks in $\mathbf{H}^3$. Let $\widetilde{D}$ be one such copy. Then $\partial\widetilde{D}$ alternates between travelling along the boundaries of horoballs covering the cusp set and geodesic planes covering the surface. Choose one such horoball to be centered at $\{\infty\}$, similar to Figure 11.

Since there are only a finite number of horoballs in this chain, there must be some horoball A with Euclidean height less than or equal to the Euclidean height of every other horoball in the chain, excluding the horoball at infinity. Then $\partial\widetilde{D}$ enters A from a geodesic plane with a boundary point that is the center of A and leaves A on a distinct geodesic plane with a boundary point that is the center of A. Let $\partial\widetilde{D} \cap A = \gamma$. Since $\partial\widetilde{D}$ must have come from a horoball that is no bigger than A and it must go to a horoball that is no bigger than A, γ starts and ends at or above the equator of A. By the triangle inequality and the fact that the distance from the top of a horoball to the equator is always exactly 1, we see that $|\gamma| \leq 2$. But $w \leq |\gamma|$ and so we see that $w \leq 2$.

Now consider the case where $w = 2$. Then $|\gamma| = 2$, and since no other horoball can be strictly smaller than A we see that the horoballs on either side of A in the sequence are actually both the same height as A and tangent to A. In fact, we are forced to have a sequence of equal height, tangent horoballs. But now consider a horoball B adjacent in the sequence to the horoball at $\{\infty\}$. Then $\partial\widetilde{D} \cap B$ is a curve which starts at the equator of B and ends at the top of B, forcing w to be less than or equal to 1, contradicting the assumption that $w = 2$.

The (n, n, n) pretzel knots of Example 2.2 yield a sequence of hyperbolic knots with width approaching 2 from below and with free (and hence semifree) totally geodesic Seifert surfaces, demonstrating that the upper bound of 2 is best possible. q.e.d.

Theorem 4.2. *Consider a hyperbolic knot or link L. If there exists an embedded totally geodesic Seifert surface S, orientable or nonorientable, with balanced width w in $S^3 - L$, then $w \geq 1$.*

Proof. Assume $w < 1$. The totally geodesic surface S has boundary a union of l-curves. We maximize the cusps while forcing the widths with respect to these l-curves to be equal. Hence, not every cusp will necessarily have a point of tangency. Let C be a cusp with a point of tangency with itself or another cusp and let $\widetilde{C}$ be a horoball covering C centered at $\{\infty\}$, and normalized to have boundary a horizontal plane of Euclidean height 1. There is a horoball A tangent to $\widetilde{C}$.

The surface S is covered by a set of geodesic planes containing two vertical planes that are a distance w apart. The width curve with respect to the resulting l-curves on the horosphere at infinity has a well defined direction. If we travel along a great circle in this direction from the top of A a distance at most w, we will have reached a hemisphere $\widetilde{S}$ contained in the pre-image of S. Since $w < 1$ and the hyperbolic distance from the top of A to the equator is 1, $\widetilde{S}$ must intersect A above the equator, and hence $\widetilde{S}$ has radius that is greater than $\frac{1}{2}$. The surface is embedded, thus $\widetilde{S}$ is contained between two vertical planes contained in the pre-image of S. The distance between the two vertical planes is equal to w, both in the Euclidean and hyperbolic length since the maximal cusp is normalized to height 1. But since the radius of $\widetilde{S}$ is greater than $\frac{1}{2}$, this implies that $w > 1$, which contradicts the assumption. q.e.d.

Theorem 4.3. *Let L be a hyperbolic knot or link and S a totally geodesic Seifert surface, orientable or nonorientable, embedded in $S^3 - L$ with balanced width w. Then $w = 1$ if and only if there is an essential 3-gon in the complement of S, and this can occur only if S is nonorientable.*

Proof. First, assume there is an essential 3-gon D in the complement. Note that this implies S is nonorientable, since otherwise, let S_+ and S_- be the two copies of S on the boundary of the regular neighborhood of S. Arcs in $\partial D \cap \partial N(S)$ must alternate between lying in S_+ and S_-. Hence there must be an even number of them, contradicting the fact D is a 3-gon.

A component in the pre-image of the essential 3-gon D is a disk $\widetilde{D}$ in $\mathbf{H}^3$ bounded by two vertical planes V_1 and V_2 and a hemisphere $\widetilde{S}$ covering the totally geodesic surface and three horospheres covering the cusp boundary, one of which is centered at ∞ and is denoted $\widetilde{C}$ and the other two of which are denoted A and B. (See Figure 12.)

The hemisphere $\widetilde{S}$ meets each vertical plane only at the center of A and B since the totally geodesic surface is embedded. The 3-gon D has arcs in its boundary that lie in the surface. These boundary curves cannot be isotoped to the surface since the number of boundary curves of D would then not be minimal and so D would not be essential. Hence for the lifts, c_i, of the boundary curves, $|c_i| \geq w$ for all i. Let the origin of the upper-half space model of $\mathbf{H}^3$ be taken as the center of A on the boundary of the xy plane such that A and $\widetilde{S}$ are centered on the y axis. Simple calculations show that for the height, z, of the point of intersection, (x, y, z), of A with $\widetilde{S}$ and the yz plane, $z = \frac{r}{a}y$, where r is the radius of $\widetilde{S}$ and a is the radius of A. Since $\widetilde{C}$ is at Euclidean height 1, $2r \geq w$. Therefore $\frac{1}{2} \leq r \leq 1$, and $0 < a \leq \frac{1}{2}$. Hence $z \geq a$. Since w is also realized as a segment of a great circle running from the top of A to (x, y, z), it follows from the lower bound on z that $w \leq 1$, since the segment of a great circle on A to the equator of A is 1 in hyperbolic distance and (x, y, z) is above or at the equator of A. Theorem 4.2 then implies that $w = 1$.

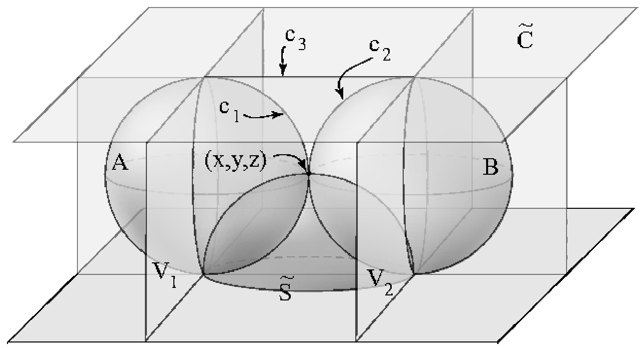

Figure 12. The case when an essential 3-gon is present.

Now, assume there is an embedded totally geodesic surface S and $w = 1$. Let C be a cusp with a point of tangency with itself or another cusp and let $\widetilde{C}$ be a horoball covering C. Center $\widetilde{C}$ at $\{\infty\}$ and normalize it to height 1. There is a horoball A tangent to $\widetilde{C}$. The surface S is covered by a set of geodesic planes containing two vertical planes that are a distance w apart. The width curve with respect to the resulting l-curves on the horoball at infinity has a well defined direction.

If the cusp does not touch itself in S, we may travel along a great circle on A in the well defined direction a distance at most w and we will have come to a hemisphere. Since we assume a vertical plane does not intersect the top of A, there must be a hemisphere intersecting A above its equator, as the hyperbolic distance from the top of A to the equator is 1. This implies that the radius of the hemisphere is greater than $\frac{1}{2}$. But since S is embedded, the hemisphere is also contained between two vertical planes a Euclidean and hyperbolic distance of $w = 1$ apart, which is impossible.

Thus the cusp must touch itself in the surface S. Hence, there is a vertical plane, V_1, containing a boundary curve of $\widetilde{C}$, centered on A. If we travel along a great circle on A in a well defined direction with respect to the resulting l-curve on the horoball at $\{\infty\}$ from the top of A a hyperbolic distance of 1 we will have reached a geodesic plane, since the width, w, is equal to 1. Thus there is a hemisphere, $\widetilde{S}$, intersecting A at Euclidean height $\frac{1}{2}$. If we travel in the same direction as the great circle along a straight line in $\widetilde{C}$ a distance of 1, we will again have reached a geodesic plane, and hence there is a second vertical plane, V_2, a distance of 1, both in Euclidean and hyperbolic distance, from V_1. The surface is embedded, therefore $\widetilde{S}$ does not intersect V_1 or V_2 except perhaps at a point. It follows that the radius of $\widetilde{S}$ is $\frac{1}{2}$, and V_1, V_2, and $\widetilde{S}$ then bound a disk that is an essential 3-gon in $\mathbf{H}^3$ with V_1 and V_2 intersecting $\widetilde{S}$ at distinct points on the boundary of $\mathbf{H}^3$ and meeting each other at $\{\infty\}$. This disk projects to an essential 3-gon in the manifold. q.e.d.

4.2. An Application of Width.

Definition 4.4. A **minimal l-curve** for the maximal cusp C of a hyperbolic knot K is its l-curve of shortest length.

Under the above definition, a minimal l-curve always exists, although there could potentially be two l-curves of shortest length. We will use minimal l-curves to show that the width for certain knots is small. (Recall that when we speak of the width of a knot, without regard to a particular l-curve, we mean width with respect to the longitude.)

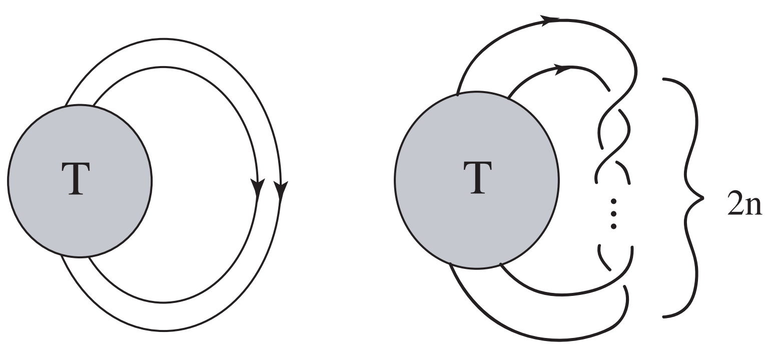

Figure 13. The second knot is obtained from the first by adding an even number of crossings to two similarly oriented strands.

Consider a knot K in an oriented projection P. Given two similarly oriented strands of the knot that are both on the boundary of the same region within the projection, we can form a sequence of knots $\{K_p\}$ by twisting the strands about each other so as to add an even number of crossings, as in Figure 13.

Theorem 4.5. *If, for some $N > 0$, $n > N$ implies K_n is hyperbolic, then*

$$\lim_{p\to\infty} w(K_p) = 0.$$

Proof. We first describe a method for obtaining the knots K_p described above. Begin with the knot K and create a link L by drilling out a curve γ which bounds a disk D that is punctured by the two strands of K we wish to twist, so that the strands have the same orientation as they pass through D. Then, upon performing $(1, p)$ Dehn filling on γ, we obtain the knot K_p. Working directly with the link L will help us to show that the width for these knots becomes small.

From the link L, before the Dehn filling, we can form a series of links, the complements of which are all homeomorphic. Let L_0 denote L and let L_p denote the result of adding $2p$ twists to L_0 in the above manner. The complements are indeed homeomorphic, because they are formed by cutting the complement of L open along the disk D, twisting one copy of D p times, and gluing together again in the original manner.

Let α denote K's longitude and β denote K's minimal l-curve. Let α_p and β_p denote respectively the images of α and β under the homeomorphisms h_p from L_0 to L_p. Mostow's Rigidity Theorem guarantees that the hyperbolic structures of the link complements are the same. For this reason, the image of L_0's minimal l-curve under h_p will also be the minimal l-curve for L_p. However, this does not hold true for the longitude. Let η_p denote the longitude of L_p.

Assume the intersection number of α with β is x. Computations show that the linking number of α_p with the core curve of K's image is $\pm 4p$, where the sign depends on the direction in which the twisting is done. (This computation does not work if the strands are oppositely oriented, in which case the linking number is zero.) The above core curve has by definition linking number zero with η_p. Thus, the intersection number of β_p with η_p is $x \pm 4p$, because homeomorphisms preserve intersection number. So, as p approaches infinity, the linking number of the curve β_p (which remains constant on the cusp) with η_p approaches infinity. Therefore, $|\eta_p| \to \infty$ as $p \to \infty$. Because the cusp area A must remain constant under the homeomorphisms, and because A is equal to the product of the longitude length and the width, we must have $w_p \to 0$, where this width is with respect to η_p.

Performing $(1, 0)$ Dehn filling on the image of γ under the map h_p, which corresponds to performing $(1, p)$ surgery on the original γ, gives a *knot* complement which, in general, could have a quite different hyperbolic structure from $\mathbf{S}^3 - L_p$. However, if we take p large enough, then these structures get arbitrarily close by Thurston's Hyperbolic Dehn Surgery Theorem [**7**]. In particular, we can choose p so that the width after the filling also gets arbitrarily close to zero, as required. q.e.d.

The above construction of knots K_p relies on the fact that all of them past a certain point are hyperbolic. One instance where this occurs is when the knot K is a reduced prime alternating knot that is not a two braid knot. Then, twisting similarly oriented strands in any reduced alternating projection so that the resulting knots are alternating will yield a sequence of knots, all of which are hyperbolic, by results in Menasco [**6**]. Hence, the knots will have width approaching zero. Moreover, in the same case, if we twist in the direction that yields nonalternating knots, the resulting knots will still be hyperbolic for large enough twists, since they will be limiting toward an augmented alternating link, which was shown to be hyperbolic in Adams [**1**].

Note that if the orientations of the two strands do not match, the resulting sequence of knots need not have width approaching 0, as occurs for the sequence of twist knots.

Our main interest in Theorem 4.5 lies in the following corollary:

Corollary 4.6. *Consider, as above, a sequence of knots* $\{K_p\}$ *obtained by twisting similarly oriented strands about each other in a projection of a knot* K*, as in Figure* 13*. If, eventually, all knots past a certain point in the sequence are hyperbolic, then we can find* $N > 0$ *so that* $n > N$ *implies the knot* K_n *cannot possess any totally geodesic orientable Seifert surfaces.*

Proof. This follows immediately from Theorem 4.2 and Theorem 4.5, since we can make width arbitrarily small. q.e.d.

5. Application: The Six Theorem is Sharp for Knot Complements

A manifold is said to be *hyperbolike* if it is irreducible with infinite word-hyperbolic fundamental group. Under this definition, hyperbolic and hyper*bolike* manifolds are very similar: for instance, neither can possess an essential torus. In fact, a proof of Thurston's geometrization conjecture would imply that hyperbolic and hyperbolike manifolds are exactly the same.

The Six Theorem, proven independently by Ian Agol [**3**] and Mark Lackenby [**5**], showed that, for a finite volume hyperbolic 3-manifold N with single embedded horocusp C, performing Dehn surgery on a curve α in the cusp boundary such that the length of α is strictly greater than six always yields a hyperbolike manifold. Moreover, Agol demonstrated that this bound is sharp by giving an explicit example of a hyperbolic 3-manifold of two cusps such that high surgery on one of the cusps yields a curve of length approaching six on the other cusp such that Dehn surgery on that curve yields a non- hyperbolike manifold. In this section, we demonstrate that, furthermore, the bound is sharp for hyperbolic knot complements, in the sense that there is a hyperbolic knot complement with a curve of length exactly six in the cusp boundary, such that surgery along that curve yields a non-hyperbolike manifold, a case that was not covered by Agol's example. Our first task is to find a candidate curve on which we can perform the surgery.

Lemma 5.1. *For all (p, p, p) pretzel knots, with odd $p \geq 3$, the longitude length is greater than or equal to six.*

Proof. Figure 4 from an earlier section shows symmetries of a $(3, 3, 3)$ pretzel knot. The (p, p, p) case with odd $p \geq 3$ is completely analogous. The vertical axis represents a rotational symmetry of order three, and the circular axis running horizontally along the equator is a rotational symmetry of order two. (There are other symmetries, but these are the two that will concern us.) These correspond to isometries of $\mathbf{H}^3$. Because these symmetries preserve the totally geodesic surface which has boundary along a longitude, they must send longitude to longitude. Since neither of the symmetry axes touches the knot, they must both correspond to parabolic isometries. Combining this information, we have that the parallelogram corresponding to the fundamental domain of the cusp should contain symmetries realized as longitudinal translations of order two and order three.

As usual, consider the horoball at infinity to be normalized so that its height is one, and consider any full-sized horoball tangent to it. Because the symmetries preserve the horoball diagram, they force a minimum of six full-sized horoballs with centers along a line that is in the preimage of the longitude. As mentioned in the preceding paragraph, these symmetries all occur within one fundamental domain of the cusp, and so the longitude length must have room for all six full-sized balls. Because they all have diameter one, this forces the longitude length to be greater than or equal to six, as desired. q.e.d.

We can see this phenomenon explicitly in Figure 14, provided by SnapPea (see [**8**]).

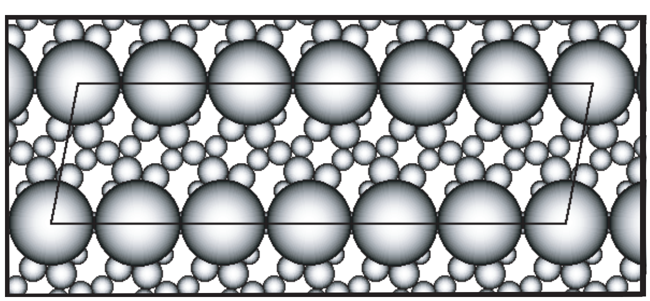

Figure 14. The horoballs do indeed satisfy an order six translational symmetry along a longitude.

One more lemma is needed before the main result, which will follow immediately.

Lemma 5.2. *Performing Dehn surgery on the longitude of the* (p, p, p) *pretzel knot, for odd* $p \geq 3$*, yields a non-hyperbolike manifold.*

Proof. Consider the totally geodesic surface in the (p, p, p) pretzel knot complement. It is a once-punctured torus. Theorem 7.1 in Agol [**3**] guarantees that the punctured torus, which is Fuchsian, remains essential under Dehn filling along the puncture. This filling results in an essential torus, which shows that the resulting manifold cannot be hyperbolike. q.e.d.

By the Six Theorem, Lemma 5.2 shows that the longitude must have length at most six, which combines with the result of Lemma 5.1 to give that the longitude length for all of these knots is precisely six. We obtain the following:

Theorem 5.3. *The Six Theorem is sharp for knot complements, with* (p, p, p) *pretzel knots all as examples, for every odd* $p \geq 3$.

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[8] Jeff Weeks' SnapPea program is available at www.geometrygames.org/SnapPea/.

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