

**DEHN FILLING, VOLUME, AND  
THE JONES POLYNOMIAL**

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**Abstract**

Given a hyperbolic 3-manifold with torus boundary, we bound the change in volume under a Dehn filling where all slopes have length at least  $2\pi$ . This result is applied to give explicit diagrammatic bounds on the volumes of many knots and links, as well as their Dehn fillings and branched covers. Finally, we use this result to bound the volumes of knots in terms of the coefficients of their Jones polynomials.

**1. Introduction**

It is well-known that the volumes of hyperbolic 3-manifolds form a closed, well-ordered subset of  $\mathbb{R}$  [39]. However, 3-manifolds are often described combinatorially, and it remains hard to translate the combinatorial data into explicit information on volume. In this paper, we prove results that bound the volumes of a large class of manifolds with purely combinatorial descriptions.

There are other recent theorems relating volumes to combinatorial data. Brock and Souto have proved that the volume of a hyperbolic 3-manifold is coarsely determined by the complexity of a Heegaard splitting [13]. Costantino and Thurston have related volume to the complexity of a shadow complex [18]. Despite the general power of these theorems, the constants that bound volume from below remain mysterious.

This paper provides explicit and readily applicable estimates on the volume of hyperbolic manifolds obtained using Dehn filling. We apply these estimates to a large class of knot and link complements, obtaining bounds on their volume based purely on the combinatorics of a diagram of the link. We then use these results to relate the volume of a large class of knots to the coefficients of the Jones polynomial.

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The volume conjecture [27, 32] asserts that the volume of hyperbolic knots is determined by certain asymptotics of the Jones polynomial and its relatives. At the same time, a wealth of experimental evidence suggests a direct correlation between the coefficients of the Jones polynomial and the volume of hyperbolic knots. For example, Champanerkar, Kofman, and Patterson have computed the Jones polynomials of all the hyperbolic knots whose complements can be decomposed into seven or fewer ideal tetrahedra [15]. Although some of these Jones polynomials have large spans, their non-zero coefficients have small values, suggesting a relationship between small volume and small coefficients. Dasbach and Lin have proved that such a connection does in fact exist for alternating links [22]; our results extend this relationship to many non-alternating links.

**1.1. Volume change.** Given a 3-manifold  $M$  with  $k$  torus boundary components, we use the following standard terminology. For the  $i$ -th torus  $T_i$ , let  $s_i$  be a *slope* on  $T_i$ , that is, an isotopy class of simple closed curves. Let  $M(s_1, \dots, s_k)$  denote the manifold obtained by Dehn filling  $M$  along the slopes  $s_1, \dots, s_k$ .

When  $M$  is hyperbolic, each torus boundary component of  $M$  corresponds to a cusp. Taking a maximal disjoint horoball neighborhood about the cusps, each torus  $T_i$  inherits a Euclidean structure, well-defined up to similarity. The slope  $s_i$  can then be given a geodesic representative. We define the *slope length* of  $s_i$  to be the length of this geodesic representative. Note that when  $k > 1$ , this definition of slope length depends on the choice of maximal horoball neighborhood.

**Theorem 1.1.** *Let  $M$  be a complete, finite-volume hyperbolic manifold with cusps. Suppose  $C_1, \dots, C_k$  are disjoint horoball neighborhoods of some subset of the cusps. Let  $s_1, \dots, s_k$  be slopes on  $\partial C_1, \dots, \partial C_k$ , each with length greater than  $2\pi$ . Denote the minimal slope length by  $\ell_{\min}$ . If  $M(s_1, \dots, s_k)$  satisfies the geometrization conjecture, then it is a hyperbolic manifold, and*

$$\text{vol}(M(s_1, \dots, s_k)) \geq \left(1 - \left(\frac{2\pi}{\ell_{\min}}\right)^2\right)^{3/2} \text{vol}(M).$$

Note that when at least one cusp of  $M$  is left unfilled, the manifold  $M(s_1, \dots, s_k)$  is Haken, and thus satisfies geometrization by Thurston's theorem [40]. In the general case, the hyperbolicity of  $M(s_1, \dots, s_k)$  would follow from Perelman's work [34, 35].

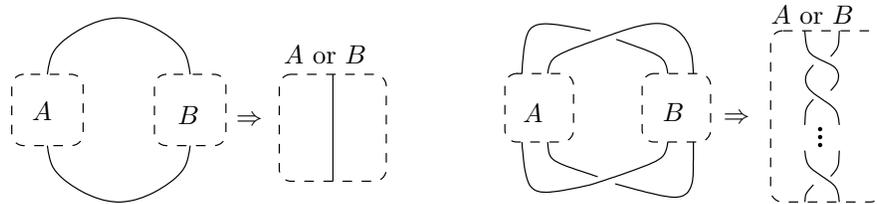
Theorem 1.1 should be compared with other known results. Neumann and Zagier have found asymptotic changes in volume under Dehn filling as slope lengths become long [33]. They show that the change in volume is asymptotically of order  $O(1/\ell_{\min}^2)$ . Although Theorem 1.1 was not meant to analyze the asymptotic behavior of volume, it also gives an

$O(1/\ell_{\min}^2)$  estimate. However, our constants are not sharp. See Section 2.3 for a more detailed discussion of the sharpness and asymptotic behavior of our estimate.

Hodgson and Kerckhoff have also found bounds on volume change under Dehn filling, provided that the filling is obtained via cone deformation [25]. They show that if the normalized slope length is at least 7.515, the cone deformation exists and their volume estimates apply. However, the normalized slope length is typically much smaller than the actual slope length. Thus Theorem 1.1 applies in many more cases than their results.

**1.2. Twist number and volumes.** We will apply Theorem 1.1 to link complements in  $S^3$ . Consider a diagram of a knot or link  $K$  as a 4-valent graph in the plane, with over-under crossing information associated to each vertex. A bigon region is a region of the graph bounded by only two edges. A *twist region* of a diagram consists of maximal collections of bigon regions arranged end to end. A single crossing adjacent to no bigons is also a twist region. Let  $D(K)$  denote the diagram of  $K$ . We denote the number of twist regions in a diagram by  $\text{tw}(D)$ .

Our statements concern the number of twist regions of a diagram. We rule out extraneous twist regions by requiring our diagram to be reduced in the sense of the following two definitions, illustrated in Figure 1.



**Figure 1.** Left: A prime diagram. Right: A twist reduced diagram.

First, we require the diagram to be *prime*. That is, any simple closed curve which meets two edges of the diagram transversely must bound a region of the diagram with no crossings.

Second, we require the diagram to be *twist-reduced*. That is, if any simple closed curve meets the diagram transversely in four edges, with two points of intersection adjacent to one crossing and the other two adjacent to another crossing, then that simple closed curve must bound a (possibly empty) collection of bigons arranged end to end between the crossings.

In the remainder of this paper, we will implicitly assume that all link diagrams are connected, and that a diagram is alternating within each twist region.

**Theorem 1.2.** *Let  $K \subset S^3$  be a link with a prime, twist-reduced diagram  $D(K)$ . Assume that  $D(K)$  has  $\text{tw}(D) \geq 2$  twist regions, and that each region contains at least 7 crossings. Then  $K$  is a hyperbolic link satisfying*

$$0.70735 (\text{tw}(D) - 1) < \text{vol}(S^3 \setminus K) < 10 v_3 (\text{tw}(D) - 1),$$

where  $v_3 \approx 1.0149$  is the volume of a regular ideal tetrahedron.

The upper bound on volume is due to Agol and D. Thurston [29, Appendix], improving an earlier estimate by Lackenby [29]. For alternating diagrams, Agol, Storm, and W. Thurston [7] have proved a sharper lower bound of  $1.83(\text{tw}(D) - 2)$ , again improving an earlier estimate by Lackenby [29]. Theorem 1.2 is also an improvement of a recent theorem of Purcell [37]. A linear lower bound was also obtained in that paper, but the results applied only to links with significantly more crossings per twist region.

Theorem 1.1 also leads to lower bounds on the volumes of Dehn fillings of link complements in  $S^3$  and branched coverings of  $S^3$  over links. For example, combining Theorem 1.1 with the orbifold theorem [10, 16] and a result of Adams on the waist size of knots [3] yields the following result.

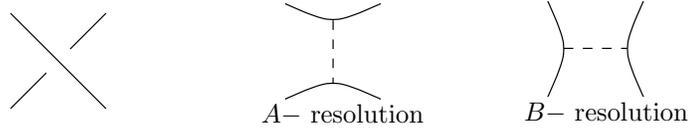
**Theorem 1.3.** *For a hyperbolic knot  $K$  in  $S^3$  and an integer  $p > 0$ , let  $M_p$  denote the  $p$ -fold cyclic cover of  $S^3$  branched over  $K$ . If  $p \geq 7$ , then  $M_p$  is hyperbolic, and*

$$\left(1 - \frac{4\pi^2}{p^2}\right)^{3/2} \text{vol}(S^3 \setminus K) \leq \frac{\text{vol}(M_p)}{p} < \text{vol}(S^3 \setminus K).$$

For further applications and discussion, including a sharper version of Theorem 1.3, we refer the reader to Section 3.

**1.3. Twist number and Jones polynomials.** Let  $D$  be a link diagram, and  $x$  a crossing of  $D$ . Associated to  $D$  and  $x$  are two link diagrams, each with one fewer crossing than  $D$ , called the  $A$ -resolution and  $B$ -resolution of the crossing. See Figure 2. Starting with any  $D$ , let  $s_A(D)$  (resp.  $s_B(D)$ ) denote the crossing-free diagram obtained by applying the  $A$ -resolution (resp.  $B$ -resolution) to all the crossings of  $D$ .

**Definition 1.4.** Given a link diagram  $D$  we obtain graphs  $\mathbb{G}_A, \mathbb{G}_B$  as follows. The vertices of  $\mathbb{G}_A$  are in one-to-one correspondence with the components of  $s_A(D)$ . Every crossing of  $D$  gives rise to two arcs of the  $A$ -resolution. These will each be associated with a component of  $s_A(D)$ , and thus correspond to a vertex of  $\mathbb{G}_A$ . Add an edge to  $\mathbb{G}_A$  connecting these two vertices for each crossing of  $D$ , as in Figure 2. We will refer to  $\mathbb{G}_A$  as the  $A$ -graph associated to  $D$ . In a similar manner, construct the  $B$ -graph  $\mathbb{G}_B$  by considering components of  $s_B(D)$ .



**Figure 2.** A crossing and its  $A$ -,  $B$ -resolutions. The dashed lines show the edges of the graphs  $\mathbb{G}_A, \mathbb{G}_B$  corresponding to the crossing.

A link diagram  $D$  is called  $A$ -adequate (resp.  $B$ -adequate) if the graph  $\mathbb{G}_A$  (resp.  $\mathbb{G}_B$ ) contains no loops (i.e., edges with both of their endpoints on the same vertex). The diagram  $D$  is called adequate if it is both  $A$ -adequate and  $B$ -adequate. A link is called adequate if it admits an adequate diagram.

The class of adequate links includes all alternating links and all  $n$ -string parallels of alternating links, as well as most pretzel knots and links and most arborescent links. For more information, see for example the paper of Lickorish and Thistlethwaite [30].

For any link  $K \subset S^3$ , let

$$J_K(t) = \alpha t^n + \beta t^{n-1} + \dots + \beta' t^{s+1} + \alpha' t^s$$

denote the Jones polynomial of  $K$ , so that  $n$  (resp.  $s$ ) is the highest (resp. lowest) power in  $t$ . We will always denote the second and next-to-last coefficients of  $J_K(t)$  by  $\beta$  and  $\beta'$ , respectively.

**Theorem 1.5.** *Let  $K$  be a link in  $S^3$  with an adequate diagram  $D(K)$ , such that every twist region of  $D(K)$  contains at least 3 crossings. Then*

$$\frac{1}{3} \text{tw}(D) + 1 \leq |\beta| + |\beta'| \leq 2 \text{tw}(D).$$

By putting together Theorem 1.2 and Theorem 1.5, we obtain the following result relating the volume and the Jones polynomial of a hyperbolic link.

**Corollary 1.6.** *Let  $K \subset S^3$  be a link with a prime, twist-reduced, adequate diagram  $D(K)$ . Assume that  $D(K)$  has  $\text{tw}(D) \geq 2$  twist regions, and that each region contains at least 7 crossings. Then  $K$  is a hyperbolic link, satisfying*

$$0.35367 (|\beta| + |\beta'| - 2) < \text{vol}(S^3 \setminus K) < 30 v_3 (|\beta| + |\beta'| - 1).$$

Here,  $\beta$  and  $\beta'$  are the second and next-to-last coefficients of the Jones polynomial of  $K$ , and  $v_3 \approx 1.0149$  is the volume of a regular ideal tetrahedron.

Dasbach and Lin [22] showed that the twist number of a twist-reduced alternating diagram is exactly  $|\beta| + |\beta'|$ . Combined with work

of Lackenby [29], this led to two-sided bounds on the volume of alternating links in terms of these coefficients of the Jones polynomial. Theorem 1.5 and Corollary 1.6 extend these results into the realm of non-alternating links.

**1.4. Organization of the paper.** In Section 2, we prove Theorem 1.1 and provide some experimental data. The proof of 1.1 requires a careful analysis of the properties of solutions to certain differential equations; due to their technical nature, these details are postponed until Section 5. In Section 3, we apply Theorem 1.1 to knots and links, their Dehn fillings, and their branched covers. In particular, we prove Theorem 1.2 and several other applications. In Section 4, we relate the twist number of a diagram to the Jones polynomial, proving Theorem 1.5.

**1.5. Acknowledgements.** We thank Marc Lackenby for pointing us in the right direction with differential equation arguments in the proof of Theorem 2.1. We thank Nathan Dunfield for helping us set up the numerical experiments to check the sharpness of our volume estimate. Finally, we are grateful to Lawrence Roberts, Peter Storm, and Xiaodong Wang for their helpful suggestions.

## 2. Volume change under filling

In this section, we prove Theorem 1.1, by employing the following strategy. For every cusp of  $M$  that we need to fill, we will explicitly construct a negatively curved solid torus, following the proof of Gromov and Thurston's  $2\pi$ -theorem [9]. When we sew in these solid tori, we obtain a negatively curved Riemannian metric on  $M(s_1, \dots, s_k)$ . Then, we will use a theorem of Boland, Connell, and Souto [11] to compare the volume of this metric with the true hyperbolic volume of the filled manifold.

This strategy is similar to that of Agol in [6]. However, while Agol starts with closed hyperbolic manifolds and constructs negatively curved metrics on cusped ones, we begin with cusped hyperbolic manifolds and construct negatively curved metrics on their Dehn fillings.

**2.1. Negatively curved metrics on a solid torus.** Our main tool in the proof of Theorem 1.1 is the following result, inspired by Cooper and Lackenby [17, Proposition 3.1]. To simplify exposition, we define a function

$$h(x) := 1 - \left(\frac{2\pi}{x}\right)^2.$$

**Theorem 2.1.** *Let  $V$  be a solid torus. Assume that  $\partial V$  carries a Euclidean metric, in which the Euclidean geodesic representing a meridian has length  $\ell_1 > 2\pi$ . Then, for any constant  $\zeta \in (0, 1)$ , there exists a smooth Riemannian metric  $\tau$  on  $V$ , with the following properties:*

- (a) On a collar neighborhood of  $\partial V$ ,  $\tau$  is a hyperbolic metric, whose restriction to  $\partial V$  is the prescribed flat metric.
- (b) The sectional curvatures of  $\tau$  are bounded above by  $-\zeta h(\ell_1)$ .
- (c) The volume of  $V$  in this metric is at least  $\frac{1}{2}\zeta \text{area}(\partial V)$ .

*Proof.* Following Bleiler and Hodgson’s proof of the  $2\pi$  Theorem [9], we will explicitly construct a metric on  $\tilde{V}$ , the universal cover of  $V$ . First, give  $\tilde{V}$  cylindrical coordinates  $(r, \mu, \lambda)$ , where  $r \leq 0$  is the radial distance measured *outward* from  $\partial\tilde{V}$ ,  $0 \leq \mu \leq 1$  is measured around each meridional circle, and  $-\infty < \lambda < \infty$  is measured in the longitudinal direction, perpendicular to  $\mu$ . We normalize the coordinates so that the generator of the deck transformation group on  $\tilde{V}$  changes the  $\lambda$  coordinate by 1.

The Riemannian metric on  $\tilde{V}$  is given by

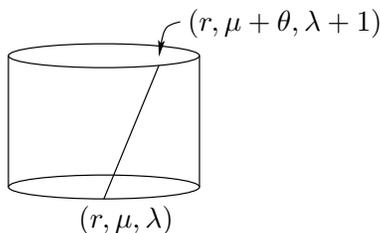
$$(1) \quad ds^2 = dr^2 + (f(r))^2 d\mu^2 + (g(r))^2 d\lambda^2,$$

where  $f$  and  $g$  are smooth functions that we will construct in the course of the proof. In order to obtain the prescribed Euclidean metric on  $\partial\tilde{V}$ , we must set  $f(0) = \ell_1$  and  $g(0) = \ell_2$ , where  $\ell_2 := \text{area}(\partial V)/\ell_1$ .

With this metric, the deck transformation group on  $\tilde{V}$  is generated by the isometry

$$(r, \mu, \lambda) \mapsto (r, \mu + \theta, \lambda + 1),$$

where the *shearing factor*  $\theta \in [0, 1)$  is chosen so that the fundamental domain of  $\partial V$  becomes a parallelogram of the correct shape. See Figure 3. The metric on  $\tilde{V}$  descends to give a smooth metric on  $V$ , and the coordinates  $(r, \mu, \lambda)$  give local cylindrical coordinates on  $V$ .



**Figure 3.** The fundamental domain for the action of the deck transformation group on  $\tilde{V}$ .

In order to give conclusions (a)–(c) of the theorem, the functions  $f$  and  $g$  must satisfy several conditions:

- $f$  and  $g$  must give a hyperbolic metric near  $\partial V$ , such that the induced metric on  $\partial V$  gives a Euclidean torus with the right shape. In other words, we must have  $f(r) = \ell_1 e^r$  and  $g(r) = \ell_2 e^r$  near  $r = 0$ .

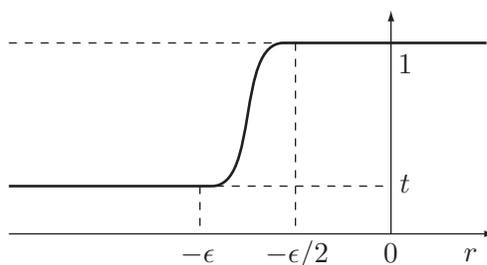
- In order to be nonsingular, the metric must have a cone angle of  $2\pi$  along the core, i.e., at the points  $r = r_0$  such that  $f(r_0) = 0$ . Bleiler and Hodgson computed that this cone angle is exactly  $f'(r_0)$ . Thus we need to ensure  $f'(r_0) = 2\pi$ .
- Bleiler and Hodgson computed that the sectional curvatures are all convex combinations of:

$$\kappa_{12} = -\frac{f''}{f}, \quad \kappa_{13} = -\frac{g''}{g}, \quad \kappa_{23} = -\frac{f' \cdot g'}{f \cdot g}.$$

To ensure they are all bounded above by  $-\zeta h(\ell_1)$ , we ensure that each of these quantities is bounded.

- The volume of  $V$  is given by  $\int_{r_0}^0 fg \, dr$ . For the volume estimate, we ensure this quantity is bounded below.

With these requirements in mind, we can begin to construct  $f$  and  $g$ . Basically, we construct both functions so that the curvature estimate will be automatically true, and show that the other conditions follow. Roughly, we would like to fix a value  $t > 0$  and define  $f$  by a differential equation  $f''/f = t$ , and  $g$  by  $f'g'/fg = t$ . This would imply that all curvatures are bounded above by  $-t$ . However, this simple definition will not give a smooth hyperbolic metric near  $\partial V$ . Thus we introduce smooth bump functions.



**Figure 4.** The bump function  $k_{t,\epsilon}(r)$ .

For  $\epsilon > 0$  and  $0 < t < 1$ , let  $k_{t,\epsilon}(r)$  be the smooth bump function defined as follows:  $k_{t,\epsilon}(r) = t$  if  $r \leq -\epsilon$ ,  $k_{t,\epsilon}(r) = 1$  if  $r \geq -\epsilon/2$ . For  $r$  between  $-\epsilon$  and  $-\epsilon/2$ ,  $k_{t,\epsilon}(r)$  is smooth and strictly increasing. See Figure 4 for a typical graph. We also extend the definition of  $k_{t,\epsilon}$  to  $\epsilon = 0$ , obtaining a step function:

$$k_{t,0}(r) := \lim_{\epsilon \rightarrow 0^+} k_{t,\epsilon}(r) = \begin{cases} t & \text{if } r < 0, \\ 1 & \text{if } r \geq 0. \end{cases}$$

Note that  $k$  is continuous in the three variables  $(r, t, \epsilon)$  for  $\epsilon > 0$ .

For any  $\epsilon \geq 0$  and  $t \in (0, 1)$ , define a function  $f_{t,\epsilon}$  according to the differential equation

$$(2) \quad f''_{t,\epsilon}(r) = k_{t,\epsilon}(r)f_{t,\epsilon}(r),$$

with initial conditions  $f_{t,\epsilon}(0) = \ell_1 = f'_{t,\epsilon}(0)$ . When  $\epsilon > 0$  (and  $k$  is continuous), the existence and uniqueness of the solution  $f_{t,\epsilon}$  is a standard result in differential equations (see for example [26]). When  $\epsilon = 0$ , the equation can be solved explicitly;  $f_{t,0}$  is a  $C^1$  function that satisfies (2) for all  $r \neq 0$ . (See equation (4) for the exact formula.)

In Section 5, we prove that the family of functions  $f_{t,\epsilon}$  has a number of nice properties. In particular, by Theorem 5.4,  $f_{t,\epsilon}(r)$  depends continuously and uniformly on the parameters  $t$  and  $\epsilon$ . (When  $\epsilon > 0$  and  $k_{t,\epsilon}(r)$  is continuous, this is a standard result in differential equations; when  $\epsilon \rightarrow 0$  and  $k_{t,\epsilon}$  becomes discontinuous, this takes some work.)

Given  $f_{t,\epsilon}$ , we define  $g_{t,\epsilon}$  according to the differential equation

$$(3) \quad \frac{g'_{t,\epsilon}(r)}{g_{t,\epsilon}(r)} := k_{t,\epsilon}(r) \frac{f_{t,\epsilon}(r)}{f'_{t,\epsilon}(r)},$$

with initial condition  $g_{t,\epsilon}(0) = \ell_2$ . Note that by Lemma 5.3(c),  $f'_{t,\epsilon}(r) > 0$  for all  $r$ , so the right-hand side is always well-defined. Because the left-hand side of (3) is merely the derivative of  $\ln g_{t,\epsilon}(r)$ , the existence and uniqueness of solutions follows immediately by integration.

Before we delve deeper into the properties of  $f_{t,\epsilon}$  and  $g_{t,\epsilon}$ , a roadmap is in order. By Lemma 5.5, we know that  $f_{t,\epsilon}$  has a unique root  $r_0 < 0$ . On the interval  $[r_0, -\epsilon]$ ,  $f_{t,\epsilon}$  and  $g_{t,\epsilon}$  will have the form

$$f_{t,\epsilon}(r) = a \sinh(\sqrt{t}(r - r_0)), \quad g_{t,\epsilon}(r) = b \cosh(\sqrt{t}(r - r_0)),$$

for constants  $a, b$  that depend on  $t$  and  $\epsilon$ . Qualitatively, this means that the metric defined by equation (1) realizes the inner part of the solid torus as a rescaled hyperbolic tube, with constant curvature  $-t$  and a cone angle of  $a\sqrt{t}$  along the core.

We will show that when  $t = h(\ell_1)$  and  $\epsilon = 0$ , the cone angle is exactly  $2\pi$ , and we get a non-singular tube of constant curvature  $-h(\ell_1)$ . Furthermore, the volume of this metric is exactly  $\frac{1}{2}\ell_1\ell_2$ . These values are certainly enough to satisfy conditions (b) and (c) of the theorem. However, because  $k_{t,0}$  is discontinuous, this metric fails to transition smoothly between curvature  $-h(\ell_1)$  and curvature  $-1$  (in fact,  $g_{t,0}(r)$  is not even differentiable at  $r = 0$ ). To address this issue, we will find values of  $t$  near  $h(\ell_1)$  and  $\epsilon$  near 0 where the metric is smooth and non-singular, and satisfies all the conditions of the theorem.

First, note for any  $\epsilon > 0$  and any  $0 < t < 1$ , the functions  $f_{t,\epsilon}$  and  $g_{t,\epsilon}$  define a hyperbolic metric near  $\partial V$ . On the interval  $(-\epsilon/2, 0]$ ,  $k_{t,\epsilon}(r)$  is identically 1, hence the differential equations satisfied by  $f_{t,\epsilon}$  and  $g_{t,\epsilon}$  are solved by  $f_{t,\epsilon}(r) = \ell_1 e^r$  and  $g_{t,\epsilon}(r) = \ell_2 e^r$ . Thus in the collar

neighborhood  $(-\epsilon/2, 0]$  of 0, setting  $f = f_{t,\epsilon}$  and  $g = g_{t,\epsilon}$  in (1) gives the metric desired near  $\partial V$ .

The regularity of  $f_{t,\epsilon}$  allows us to find a metric that is non-singular. Recall that the cone angle along the core of  $V$  will be  $2\pi$  whenever  $f'(r_0) = 2\pi$  (where  $r_0$  is the root of  $f_{t,\epsilon}(r)$ ).

**Lemma 2.2.** *The roots of  $f_{t,\epsilon}(r)$  have the following behavior:*

- (a) *For all  $t \in (0, 1)$  and  $\epsilon \geq 0$ ,  $f_{t,\epsilon}(r)$  has a unique root  $r_0(t, \epsilon)$ .*
- (b) *The function  $m(t, \epsilon) := f'_{t,\epsilon}(r_0(t, \epsilon))$  is continuous in  $t$  and  $\epsilon$ , and strictly decreasing in both variables.*
- (c) *For every  $t \in (0, h(\ell_1))$ , there is a unique value  $\epsilon(t) > 0$  such that  $m(t, \epsilon(t)) = 2\pi$ .*
- (d) *As  $t \rightarrow h(\ell_1)$ ,  $\epsilon(t) \rightarrow 0$ .*

*Proof.* Parts (a) and (b) are proved in Lemma 5.5. To prove part (c), we study the explicit solution to the equation for  $f_{t,\epsilon}(r)$  when  $\epsilon = 0$ . For all  $r < 0$ ,  $f_{t,0}$  is given by the simple differential equation

$$f''_{t,0}(r) = t f_{t,0}(r),$$

and the initial conditions  $f_{t,0}(0) = \ell_1 = f'_{t,0}(0)$ . This has solution:

$$\begin{aligned} (4) \quad f_{t,0}(r) &= \ell_1 \cosh(r\sqrt{t}) + \frac{\ell_1}{\sqrt{t}} \sinh(r\sqrt{t}) \\ &= \frac{\ell_1 \sqrt{1-t}}{\sqrt{t}} \sinh(\sqrt{t}(r - r_0(t, 0))), \end{aligned}$$

where  $r_0(t, 0) = -\tanh^{-1}(\sqrt{t})/\sqrt{t}$ . Thus, for all  $t \in (0, h(\ell_1)]$ ,

$$\begin{aligned} m(t, 0) &= f'_{t,0}(r_0(t, 0)) \\ &= \ell_1 \sqrt{1-t} \\ &\geq \ell_1 \sqrt{(2\pi/\ell_1)^2} \\ &= 2\pi, \end{aligned}$$

with equality if and only if  $t = h(\ell_1)$ .

On the other hand, for all  $\epsilon > 2 \ln(\ell_1/2\pi)$ ,  $f'_{t,\epsilon}(-\epsilon/2) = \ell_1 e^{-\epsilon/2} < 2\pi$ . By its defining equation,  $f_{t,\epsilon}(r)$  is concave up in  $r$  when  $f_{t,\epsilon}(r)$  is positive, and concave down when negative. Thus  $m(t, \epsilon)$  is the absolute minimum of  $f'_{t,\epsilon}(r)$  over  $\mathbb{R}$ . Therefore

$$m(t, \epsilon) < 2\pi \quad \text{whenever} \quad \epsilon > 2 \ln(\ell_1/2\pi).$$

By the intermediate value theorem, we can conclude that for all  $t \in (0, h(\ell_1))$ , there is a value  $\epsilon(t) > 0$  such that  $m(t, \epsilon(t)) = 2\pi$ . Furthermore, by part (b),  $m(t, \epsilon)$  is strictly decreasing in  $\epsilon$ , and therefore  $\epsilon(t)$  is unique.

By part (b), we know that  $m(t, \epsilon)$  depends continuously on  $t$  and  $\epsilon$ . Thus  $\epsilon(t)$  depends continuously on  $t$ . As a result, as  $t \rightarrow h(\ell_1)$ ,

$\epsilon(t) \rightarrow \epsilon(h(\ell_1))$ . Since we have already computed that  $m(h(\ell_1), 0) = 2\pi$ , it follows that  $\epsilon(h(\ell_1)) = 0$ , completing the proof. q.e.d.

From now on, we require that  $t \in (0, h(\ell_1))$ , and restrict our attention to the functions  $f_t := f_{t,\epsilon(t)}$  and  $g_t := g_{t,\epsilon(t)}$  that give a non-singular Riemannian metric  $\tau(t)$  on the solid torus  $V$ . It remains to check the curvature and volume estimates for this metric.

**Lemma 2.3.** *Fix a value of  $t$  such that  $\zeta h(\ell_1) \leq t < h(\ell_1)$ . Then the Riemannian metric  $\tau(t)$  defined by  $f_t$  and  $g_t$  has all sectional curvatures bounded above by  $-\zeta h(\ell_1)$ .*

*Proof.* We will actually prove the sectional curvatures of  $\tau(t)$  are bounded above by  $-t$ . Bleiler and Hodgson computed that these sectional curvatures are convex combinations of

$$\frac{-f''}{f}, \quad \frac{-g''}{g}, \quad \text{and} \quad \frac{-f' \cdot g'}{f \cdot g}.$$

By equations (2) and (3), we have

$$-\frac{f_t''(r)}{f_t(r)} = -k_{t,\epsilon(t)}(r) \in [-1, -t], \quad -\frac{f_t'(r)g_t'(r)}{f_t(r)g_t(r)} = -k_{t,\epsilon(t)}(r) \in [-1, -t].$$

As for  $g_t''(r)/g_t(r)$ , we differentiate both sides of equation (3) to obtain

$$\frac{g_t''}{g_t} - \left(\frac{g_t'}{g_t}\right)^2 = k_{t,\epsilon(t)} - \left(\frac{f_t}{f_t'}\right)^2 \frac{f_t''}{f_t} k_{t,\epsilon(t)} + \frac{f_t}{f_t'} k_{t,\epsilon(t)}',$$

which simplifies, using equations (2) and (3), to

$$\frac{g_t''}{g_t} = k_{t,\epsilon(t)} + \frac{f_t}{f_t'} k_{t,\epsilon(t)}'.$$

Since  $1 \geq k_{t,\epsilon(t)} \geq t$  and all other terms are nonnegative (because  $f_t$  and  $k_{t,\epsilon(t)}$  are both increasing),  $-g_t''/g_t \leq -t$ . q.e.d.

**Lemma 2.4.** *Let  $t$  vary in the interval  $(\zeta h(\ell_1), h(\ell_1))$ , and define the Riemannian metric  $\tau(t)$  by the functions  $f_t$  and  $g_t$ . Then*

$$\lim_{t \rightarrow h(\ell_1)} \text{vol}(V, \tau(t)) = \frac{\ell_1 \ell_2}{2} = \frac{1}{2} \text{area}(\partial V).$$

*Proof.* By equation (1),

$$\text{vol}(V, \tau(t)) = \int_{r_0(t,\epsilon(t))}^0 f_{t,\epsilon(t)}(r) g_{t,\epsilon(t)}(r) dr.$$

Let  $t_{\text{lim}} := h(\ell_1)$ . By Lemma 2.2, as  $t \rightarrow t_{\text{lim}}$ ,  $\epsilon(t) \rightarrow 0$ . Furthermore, by Theorems 5.4 and 5.6, the functions  $f_{t,\epsilon(t)}$  and  $g_{t,\epsilon(t)}$  converge uniformly to  $f_{t_{\text{lim}},0}$  and  $g_{t_{\text{lim}},0}$ , respectively. Theorem 5.4 also implies that

$$r_0(t, \epsilon) := f_{t,\epsilon}^{-1}(0)$$

is continuous in  $t$  and  $\epsilon$ . Thus, as  $t \rightarrow t_{\text{lim}}$ ,  $r_0(t, \epsilon) \rightarrow r_0(t_{\text{lim}}, 0)$ .

By equation (4), we know that for  $r < 0$ ,

$$f_{t_{\text{lim}},0}(r) = \frac{\ell_1 \sqrt{1 - t_{\text{lim}}}}{\sqrt{t_{\text{lim}}}} \sinh(\sqrt{t_{\text{lim}}}(r - r_0)),$$

where  $r_0 = -\tanh^{-1}(\sqrt{t_{\text{lim}}})/\sqrt{t_{\text{lim}}}$ .

Similarly, when  $t = t_{\text{lim}}$  and  $\epsilon = 0$ , the differential equation for  $g_{t,\epsilon}$  has solution

$$g_{t_{\text{lim}},0}(r) = \ell_2 \sqrt{1 - t_{\text{lim}}} \cosh(\sqrt{t_{\text{lim}}}(r - r_0)).$$

Thus we may compute:

$$\begin{aligned} & \lim_{t \rightarrow t_{\text{lim}}} \text{vol}(V, \tau(t)) \\ &= \lim_{t \rightarrow t_{\text{lim}}} \int_{r_0(t, \epsilon(t))}^0 f_{t, \epsilon(t)}(r) g_{t, \epsilon(t)}(r) dr \\ &= \int_{r_0(t_{\text{lim}}, 0)}^0 f_{t_{\text{lim}}, 0}(r) g_{t_{\text{lim}}, 0}(r) dr \\ &= \int_{r_0}^0 \ell_1 \ell_2 \frac{(1 - t_{\text{lim}})}{\sqrt{t_{\text{lim}}}} \sinh(\sqrt{t_{\text{lim}}}(r - r_0)) \cosh(\sqrt{t_{\text{lim}}}(r - r_0)) dr \\ &= \left[ \ell_1 \ell_2 \frac{(1 - t_{\text{lim}})}{2 t_{\text{lim}}} \sinh^2(\sqrt{t_{\text{lim}}}(r - r_0)) \right]_{r_0}^0 \\ &= \frac{\ell_1 \ell_2}{2} \cdot \frac{1 - t_{\text{lim}}}{t_{\text{lim}}} \cdot \sinh^2(\tanh^{-1}(\sqrt{t_{\text{lim}}})) \\ &= \frac{\ell_1 \ell_2}{2} \cdot \frac{1 - t_{\text{lim}}}{t_{\text{lim}}} \cdot \frac{t_{\text{lim}}}{1 - t_{\text{lim}}} \\ &= \frac{\ell_1 \ell_2}{2}. \end{aligned}$$

This completes the proof of Lemma 2.4. q.e.d.

We are now ready to complete the proof of Theorem 2.1. By Lemma 2.2, if we select any  $t < h(\ell_1)$  and  $\epsilon = \epsilon(t) > 0$ , we get a non-singular metric satisfying conclusion (a) of the theorem. By Lemma 2.3, conclusion (b) is satisfied if we ensure that  $t$  is between  $\zeta h(\ell_1)$  and  $h(\ell_1)$ . Finally, by Lemma 2.4, if we select  $t$  near enough to  $h(\ell_1)$ , we will have  $\text{vol}(V) \geq \frac{\zeta}{2} \text{area}(\partial V)$ , satisfying conclusion (c). q.e.d.

**2.2. Negatively curved metrics on a 3-manifold.** By applying Theorem 2.1 to several cusps of a cusped manifold  $M$ , we obtain a negatively curved metric on a Dehn filling of  $M$ .

**Theorem 2.5.** *Let  $M$  be a complete, finite-volume hyperbolic manifold with cusps. Suppose  $C_1, \dots, C_k$  are disjoint horoball neighborhoods of some (possibly all) of the cusps. Let  $s_1, \dots, s_k$  be slopes on  $\partial C_1, \dots, \partial C_k$ , each with length greater than  $2\pi$ . Denote the minimal*

slope length by  $\ell_{\min}$ . Let  $S$  be the set of all Riemannian metrics on  $M(s_1, \dots, s_k)$  whose sectional curvatures lie in an interval  $[-a, -1]$  for some constant  $a \geq 1$ . Then  $S$  is non-empty, and

$$\sup_{\sigma \in S} \text{vol}(M(s_1, \dots, s_k), \sigma) \geq (h(\ell_{\min}))^{3/2} \text{vol}(M).$$

*Proof.* Fix an arbitrary constant  $\zeta \in (0, 1)$ . We will replace each cusp  $C_i$  by a solid torus  $V_i$  whose meridian is  $s_i$ . Theorem 2.1 guarantees the existence of a smooth Riemannian metric  $\tau_i$  on  $V_i$ , satisfying the following properties:

- The sectional curvatures on  $V_i$  are all at most

$$-\zeta h(\ell(s_i)) \leq -\zeta h(\ell_{\min}).$$

- $\text{vol}(V_i, \tau_i) \geq \frac{1}{2}\zeta \text{area}(\partial C_i) = \zeta \text{vol}(C_i)$ .

Furthermore, in a neighborhood of each torus  $\partial C_i$ , the metric  $\tau_i$  agrees with the hyperbolic metric on  $M$ . Thus we may cut out the cusps  $C_1, \dots, C_k$  and glue in the solid tori  $V_1, \dots, V_k$ , obtaining a smooth Riemannian metric  $\tau$  on the filled manifold  $M(s_1, \dots, s_k)$ , satisfying the following properties:

- The sectional curvatures of  $\tau$  are bounded above by  $-\zeta h(\ell_{\min})$ , and below by some constant. The lower bound comes from the fact that the solid tori  $V_1, \dots, V_k$  are compact, and  $\tau$  has constant curvature  $-1$  on  $M \setminus \cup_{i=1}^k V_i$ .
- $\text{vol}(M(s_1, \dots, s_k), \tau) \geq \text{vol}(M \setminus \cup_{i=1}^k C_i) + \zeta \sum_{i=1}^k \text{vol}(C_i) \geq \zeta \text{vol}(M)$ .

Now, we would like our metric to have sectional curvatures bounded above by  $-1$ . Note the definition of sectional curvature implies that if we rescale the metric  $\tau$  to be  $x\tau$ , then all sectional curvatures are multiplied by  $x^{-2}$ . Thus we rescale  $\tau$  to be  $\sigma = \sqrt{\zeta h(\ell_{\min})} \tau$ . This, in turn, rescales the volume by a factor  $x^3 = (\sqrt{\zeta h(\ell_{\min})})^3$ . Thus under the rescaled metric:

- The sectional curvatures of  $\sigma$  lie in  $[-a, -1]$  for some  $a \geq 1$ .
- $\text{vol}(M(s_1, \dots, s_k), \sigma) \geq \zeta^{5/2} (h(\ell_{\min}))^{3/2} \text{vol}(M)$ .

Thus we have found a metric  $\sigma$  that lies in the set  $S$ . Now, because  $\zeta \in (0, 1)$  was arbitrary, we can conclude that

$$\sup_{\sigma \in S} \text{vol}(M(s_1, \dots, s_k), \sigma) \geq (h(\ell_{\min}))^{3/2} \text{vol}(M).$$

q.e.d.

To complete the proof of Theorem 1.1, suppose that the manifold  $N = M(s_1, \dots, s_k)$  admits a complete hyperbolic metric  $\sigma_{\text{hyp}}$ . (Since we have already proved that  $N$  admits a negatively curved metric  $\sigma$ , the geometrization conjecture implies that  $N$  will indeed be hyperbolic.) Now, we compare the volumes of these metrics via the the following

theorem of Boland, Connell, and Souto [11], stated here in a special case.

**Theorem 2.6** ([11]). *Let  $\sigma$  and  $\sigma'$  be two complete, finite-volume Riemannian metrics on the same 3-manifold  $N$ . Suppose that all sectional curvatures of  $\sigma$  lie in the interval  $[-1, 1]$  and all sectional curvatures of  $\sigma'$  lie in the interval  $[-a, -1]$  for some constant  $a \geq 1$ . Then*

$$\text{vol}(N, \sigma) \geq \text{vol}(N, \sigma'),$$

*with equality if and only if both metrics are hyperbolic.*

**Remark.** When  $N$  is a closed manifold, this theorem was originally proved by Besson, Courtois, and Gallot [8]. In fact, it is quite likely that their proof would apply in our setting, because the negatively curved metrics that we construct all have constant curvature on the remaining cusps of  $N$ .

*Proof of Theorem 1.1.*

By Theorem 2.5, we know that  $N = M(s_1, \dots, s_k)$  admits a non-empty set  $S$  of Riemannian metrics whose sectional curvatures lie in an interval  $[-a, -1]$ . By Theorem 2.6, the hyperbolic metric  $\sigma_{\text{hyp}}$  uniquely maximizes volume over the set  $S$ . Thus, by putting together the statements of the two theorems, we get:

$$\text{vol}(N, \sigma_{\text{hyp}}) = \max_{\sigma \in S} \text{vol}(N, \sigma) \geq (h(\ell_{\min}))^{3/2} \text{vol}(M).$$

q.e.d.

**2.3. How sharp is Theorem 1.1?** We will attempt to answer this question in two ways. For long slopes, we compare the volume estimate of Theorem 1.1 to the asymptotic formula proved by Neumann and Zagier [33]. For medium-length slopes, we present the results of numerical experiments conducted using SnapPea.

To compare asymptotic estimates, we restrict our attention to the case when  $M$  has exactly one cusp. Let  $C$  be a maximal horoball neighborhood of the cusp, let  $s$  be a slope on  $\partial C$ , and let

$$\Delta V := \text{vol}(M) - \text{vol}(M(s)).$$

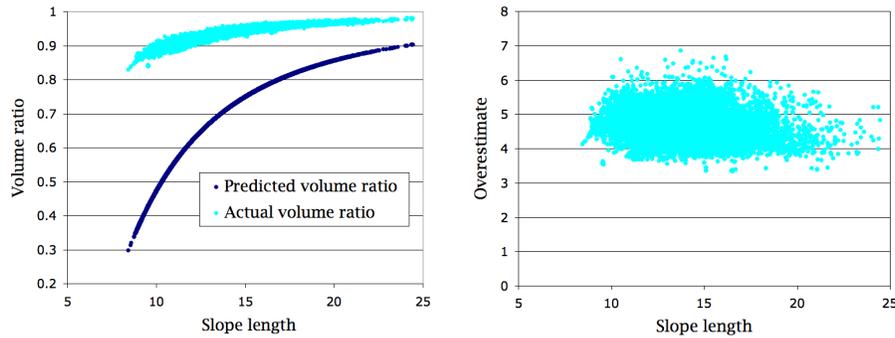
With this notation, Neumann and Zagier [33] proved that as  $\ell(s) \rightarrow \infty$ ,

$$\Delta V \approx \frac{\pi^2 \text{area}(\partial C)}{\ell(s)^2} = \frac{2\pi^2 \text{vol}(C)}{\ell(s)^2}.$$

Meanwhile, by expanding the Taylor series for  $(1-x)^{3/2}$ , we see that Theorem 1.1 implies

$$\Delta V \leq \frac{3}{2} \left( \frac{2\pi}{\ell(s)} \right)^2 \text{vol}(M) = \frac{6\pi^2 \text{vol}(M)}{\ell(s)^2}.$$

Thus, as  $\ell(s) \rightarrow \infty$ , Theorem 1.1 overestimates the change in volume by a factor of  $3\text{vol}(M)/\text{vol}(C)$ . The quantity  $\text{vol}(C)/\text{vol}(M)$  is known as the *cuspid density* of  $M$ . Böröczky [12] has proved that the cuspid density of a hyperbolic manifold is at most 0.8533. There is no known lower bound on the cuspid density; out of the approximately 5,000 orientable cusped manifolds in the SnapPea census, exactly six have density less than 0.45. These numbers suggest that for most small manifolds, Theorem 1.1 overestimates the asymptotic change in volume by a constant factor between 3.5 and 7.



**Figure 5.** The change in volume for medium-length slopes.

For medium-length slopes, we also tested the estimate of Theorem 1.1 on over 14,000 manifold-slope pairs from the SnapPea census. The results are plotted in Figure 5. In the graph on the left, the dark curve represents the estimate of Theorem 1.1, while the lighter point cloud represents the actual ratio  $\text{vol}(M(s))/\text{vol}(M)$ . In the graph on the right, one can see that for all the manifolds and slopes tested, Theorem 1.1 overestimates the change in volume by a factor between 3 and 7.

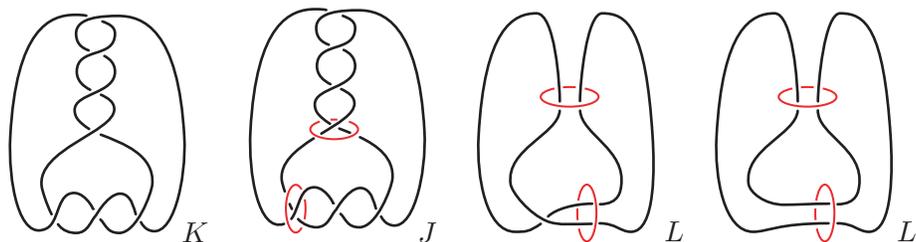
### 3. Volumes of knots, links, and their fillings

In this section, we apply Theorem 1.1 to hyperbolic link complements in  $S^3$ , their Dehn fillings, and branched covers of  $S^3$  over hyperbolic links.

**3.1. Volumes of link complements.** To prove Theorem 1.2, we express a link  $K$  as a Dehn filling of another link  $L$ .

Let  $D(K)$  be a prime, twist-reduced diagram of a link  $K$  (see Section 1.2 for definitions). For every twist region of  $D(K)$ , we add an extra link component, called a *crossing circle*, that wraps around the two strands of the twist region. The result is a new link  $J$ . (See Figure 6.) Now, the manifold  $S^3 \setminus J$  is homeomorphic to  $S^3 \setminus L$ , where  $L$  is obtained by removing all full twists (pairs of crossings) from the twist regions of  $J$ . This *augmented link*  $L$  has the property that  $K$  can be recovered by

Dehn filling the crossing circles of  $L$ . Similarly, every Dehn filling of  $K$  can be expressed as a filling of  $L$ .



**Figure 6.** An augmented link  $L$  is constructed by adding a *crossing circle* around each twist region of  $D(K)$ , then removing full twists.

The advantage of this construction is that the augmented link  $L$  has a simple geometry that allows for very explicit estimates.

An estimate on volumes given by an estimate of cusp volume was given in [37]. Here we are able to improve that estimate.

**Proposition 3.1.** *Let  $D(K)$  be a prime, twist-reduced diagram with at least two twist regions. Then the corresponding augmented link  $L$  is hyperbolic, and*

$$\text{vol}(S^3 \setminus L) \geq 2v_8(\text{tw}(D) - 1),$$

where  $v_8 = 3.66386\dots$  is the volume of a regular ideal octahedron. If  $K$  is a two-bridge link, this inequality is an equality.

*Proof.* The hyperbolicity of  $S^3 \setminus L$  is a consequence of work of Adams [2]. See also Purcell [36].

To estimate the volume of  $S^3 \setminus L$ , we simplify the link  $L$  even further, by removing all remaining single crossings from the twist regions of  $L$ . The resulting *flat augmented link*  $L'$  has the same volume as  $L$ , by the work of Adams [1]. (See Figure 6.) This link  $L'$  is preserved by a reflection in the projection plane. Thus the projection plane is isotopic to a totally geodesic surface in  $S^3 \setminus L'$ .

Cut the manifold  $S^3 \setminus L'$  along the projection plane. The result is two hyperbolic manifolds  $M$  and  $M'$  with totally geodesic boundary. Since  $M$  and  $M'$  are interchanged by the reflection of  $S^3 \setminus L'$ , they have the same volume. Moreover, the volume of  $S^3 \setminus L$  is given by the sum of the volumes of  $M$  and  $M'$ .

Note that the manifold  $M$  is a ball with a tube drilled out for each crossing circle. Hence it is topologically a handlebody with genus  $\text{tw}(D)$ . Miyamoto showed that if  $N$  is a hyperbolic 3-manifold with totally geodesic boundary, then  $\text{vol}(N) \geq -v_8\chi(N)$ . (See [31, Proposition 1.1

and Lemma 4.1].) We apply this result to  $M$ , and find

$$\text{vol}(S^3 \setminus L) = 2 \text{vol}(M) \geq -2 v_8 \chi(M) = 2 v_8 (\text{tw}(D) - 1).$$

Finally, when  $D(K)$  is a standard diagram of a two-bridge link, it is well-known that the augmented link  $L$  is obtained by gluing together  $(\text{tw}(D) - 1)$  copies of the Borromean rings, each of which has volume  $2v_8$ . (See, for example, Futer and Guéritaud [24, Theorem B.3].) Thus, for two-bridge links,  $\text{vol}(S^3 \setminus L) = 2v_8 (\text{tw}(D) - 1)$ , making our estimate sharp. q.e.d.

In fact, Proposition 3.1 is sharp for many additional large classes of knot and link diagrams.

To recover  $K$  from  $L$ , one must perform Dehn filling along the crossing circles. Thus we need to estimate the lengths of those slopes. To obtain information about Dehn fillings of  $K$ , we also estimate the lengths of non-trivial (that is, non-meridional) slopes on the components of  $L$  that come from strands of  $K$ .

**Proposition 3.2** (Theorem 3.10 of [23]). *Let  $K = \cup_{j=1}^m K_j$  be a link in  $S^3$  with a prime, twist-reduced diagram  $D(K)$ . Suppose that  $D(K)$  contains twist regions  $R_1, \dots, R_n$  ( $n \geq 2$ ) and that twist region  $R_i$  contains  $a_i$  crossings. For each component  $K_j$ , let  $n_j$  be the number of twist regions visited by  $K_j$ , counted with multiplicity; and let  $s_j$  be a non-trivial Dehn filling slope.*

*Then the Dehn filling on  $K$  with these slopes corresponds to a filling on the augmented link  $L$ . Furthermore, then there exists a choice of disjoint cusp neighborhoods in  $S^3 \setminus L$ , such that the slopes have the following lengths:*

- 1) *For a component  $K_j$  of  $K$ , the slope  $s_j$  has length at least  $n_j$ .*
- 2) *For a crossing circle  $C_i$ , the slope has length at least  $\sqrt{a_i^2 + 1}$ .*

We now have enough information to prove Theorem 1.2.

**Theorem 1.2.** *Let  $K \subset S^3$  be a link with a prime, twist-reduced diagram  $D(K)$ . Assume that  $D(K)$  has  $\text{tw}(D) \geq 2$  twist regions, and that each region contains at least 7 crossings. Then  $K$  is a hyperbolic link satisfying*

$$0.70735 (\text{tw}(D) - 1) < \text{vol}(S^3 \setminus K) < 10 v_3 (\text{tw}(D) - 1),$$

where  $v_3 \approx 1.0149$  is the volume of a regular ideal tetrahedron.

*Proof.* The conclusion that  $K$  is hyperbolic was proved by Futer and Purcell [23, Theorem 1.4], relying on W. Thurston’s hyperbolization theorem [40]. (In fact, 6 crossings per twist region suffice to show  $K$  is hyperbolic.) The upper bound on volume is due to Agol and D. Thurston [29].

To prove the lower bound, we apply Theorem 1.1 to the augmented link  $L$ . Since every twist region has at least 7 crossings, by Proposition 3.2 the slope on each crossing circle will be at least  $\sqrt{7^2 + 1} = 5\sqrt{2} > 2\pi$ .

Thus, by Theorem 1.1,

$$\begin{aligned} \text{vol}(S^3 \setminus K) &\geq \left(1 - \left(\frac{2\pi}{5\sqrt{2}}\right)^2\right)^{3/2} 2v_8(\text{tw}(D) - 1) \\ &= 0.70735\dots (\text{tw}(D) - 1). \end{aligned}$$

q.e.d.

**Remark.** In the proof of Theorem 1.2, we used the fact that to insert 7 crossings into a twist region, one fills along a slope of length at least  $\sqrt{7^2 + 1} = 5\sqrt{2}$ . In fact, if we require 8 crossings per twist region, we may replace  $5\sqrt{2}$  with  $\sqrt{8^2 + 1} = \sqrt{65}$ , and the lower bound improves to  $1.8028(\text{tw}(D) - 1)$ . As the number of required crossings increases, the estimate becomes better still. In the case of 8 crossings, our estimate is similar to the lower bound for alternating links due to Lackenby [29] and Agol, Storm, and Thurston [7], which is  $1.83(\text{tw}(D) - 2)$ . Their estimate is known to be sharp for the Borromean rings.

**3.2. Dehn fillings and branched covers.** Under a slightly stronger diagrammatic condition than that of Theorem 1.2, we can show that the combinatorics of a link  $K$  determines the volumes of all of its non-trivial fillings, up to an explicit and bounded constant.

**Theorem 3.3.** *Let  $K$  be a link in  $S^3$  with a prime, twist-reduced diagram  $D(K)$ . Suppose that every twist region of  $D(K)$  contains at least 7 crossings and each component of  $K$  passes through at least 7 twist regions (counted with multiplicity). Let  $N$  be a manifold obtained by a non-trivial Dehn filling of some (possibly all) components of  $K$ , which satisfies geometrization. Then  $N$  is hyperbolic, and*

$$0.62768(\text{tw}(D) - 1) < \text{vol}(N) < 10v_3(\text{tw}(D) - 1).$$

Note that if  $K$  is a knot, a diagram with 4 or more twist regions and 7 or more crossings per region satisfies the hypotheses of Theorem 3.3.

The conclusion that every non-trivial filling of  $K$  is hyperbolic was first proved by Futer and Purcell [23, Theorem 1.7], modulo the geometrization conjecture. In fact, 6 crossings per twist region suffice.

*Proof.* To prove that  $N$  is hyperbolic and compute the lower bound on volume, we once again apply Theorem 1.1 to the augmented link  $L$ . We know that every non-trivial filling of  $K$  can be realized as a filling of  $L$ . By Proposition 3.2, every slope on a strand of  $K$  will have length at least 7, and every slope on a crossing circle will have length at least

$5\sqrt{2}$ . Thus, by Theorem 1.1,  $N$  is hyperbolic and

$$\begin{aligned} \text{vol}(N) &\geq \left(1 - \left(\frac{2\pi}{7}\right)^2\right)^{3/2} 2 v_8 (\text{tw}(D) - 1) \\ &= 0.62768\dots (\text{tw}(D) - 1) \end{aligned}$$

For the upper bound, note that volume goes down under Dehn filling (see Thurston [39]). Thus, by Theorem 1.2,  $\text{vol}(N) < 10 v_3 (\text{tw}(D) - 1)$ .  
q.e.d.

Theorem 1.1 also applies to Dehn fillings of arbitrary hyperbolic knots.

**Theorem 3.4.** *Let  $N$  be a hyperbolic manifold obtained by  $p/q$ -Dehn surgery along a hyperbolic knot  $K$  in  $S^3$ , where  $|q| \geq 12$ . Then*

$$\text{vol}(N) > \left(1 - \frac{127}{q^2}\right)^{3/2} \text{vol}(S^3 \setminus K).$$

*Proof.* Let  $C$  be a maximal cusp of  $S^3 \setminus K$ . Let  $m$  and  $s$  be Euclidean geodesics on  $\partial C$  that represent the meridian of  $K$  and the slope  $p/q$ , respectively. Let  $\theta$  be the angle between these geodesics. Then

$$(5) \quad |q| \cdot \text{area}(\partial C) = \ell(m) \ell(s) \sin(\theta) \leq \ell(m) \ell(s).$$

We can use equation (5) to estimate  $\ell(s)$ . By a theorem of Cao and Meyerhoff [14, Theorem 5.9],  $\text{area}(\partial C) \geq 3.35$ . Furthermore, by the 6-Theorem of Agol and Lackenby [5, 28], surgery along a slope of length more than 6 yields a manifold with infinite fundamental group, which cannot be  $S^3$ . Thus  $\ell(m) \leq 6$ . Combining these results with equation (5) gives

$$(6) \quad \ell(s) \geq |q| \cdot 3.35/6.$$

In particular, when  $|q| \geq 12$ ,  $\ell(s) > 2\pi$ . Plugging inequality (6) into Theorem 1.1 gives

$$\text{vol}(N) \geq \left(1 - \left(\frac{6 \cdot 2\pi}{3.35q}\right)^2\right)^{3/2} \text{vol}(S^3 \setminus K) > \left(1 - \frac{127}{q^2}\right)^{3/2} \text{vol}(S^3 \setminus K).$$

q.e.d.

We conclude the section with an application to branched covers. Recall that the cyclic  $p$ -fold cover of a hyperbolic knot complement  $S^3 \setminus K$  is a hyperbolic 3-manifold  $X_p$  with torus boundary. The meridian  $m$  of  $K$  lifts to a slope  $m_p$  on  $\partial X_p$ . Then the  $p$ -fold branched cover of  $S^3$  over  $K$ , denoted  $M_p$ , is obtained by Dehn filling  $\partial X_p$  along the slope  $m_p$ .

**Theorem 3.5.** *If  $p \geq 4$ , the branched cover  $M_p$  is hyperbolic. For all  $p \geq 7$ , we have*

$$(7) \quad \left(1 - \frac{4\pi^2}{p^2}\right)^{3/2} \operatorname{vol}(S^3 \setminus K) \leq \frac{\operatorname{vol}(M_p)}{p} < \operatorname{vol}(S^3 \setminus K).$$

*If  $K$  is not the figure-8 or  $5_2$  knot and  $p \geq 6$ , the estimate improves to*

$$(8) \quad \left(1 - \frac{2\sqrt{2}\pi^2}{p^2}\right)^{3/2} \operatorname{vol}(S^3 \setminus K) \leq \frac{\operatorname{vol}(M_p)}{p} < \operatorname{vol}(S^3 \setminus K).$$

*Proof.* The fact that  $M_p$  is hyperbolic for  $p \geq 4$  is a well-known consequence of the orbifold theorem (see e.g. [16, Corollary 1.26]).

As for the volume estimate, the hyperbolic metric of  $S^3 \setminus K$  lifts to a hyperbolic metric on  $X_p$ , implying that  $\operatorname{vol}(X_p) = p \operatorname{vol}(S^3 \setminus K)$ . Furthermore, because a maximal cusp of  $S^3 \setminus K$  lifts to a maximal cusp of  $X_p$ , we have  $\ell(m_p) = p \ell(m)$ . To estimate the volume of  $M_p$ , we need to estimate the length of  $m$ .

Adams has shown that every hyperbolic knot in  $S^3$  has meridian of length at least 1 [3]. Thus, in the cyclic cover  $X_p$ ,  $\ell(m_p) \geq p$ . In particular, when  $p \geq 7$ , we have  $\ell(m_p) > 2\pi$ . Plugging  $\ell(m_p) \geq p$  into Theorem 1.1 proves the estimate of equation (7).

Adams has also proved that apart from the figure-8 and  $5_2$  knots, every hyperbolic knot in  $S^3$  has meridian of length at least  $2^{1/4}$  [4]. Thus, in the cyclic cover  $X_p$ , we have  $\ell(m_p) \geq 2^{1/4}p$ , proving equation (8). q.e.d.

**Remark.** Numerical experiments with SnapPea confirm that the  $p$ -fold branched covers over the figure-8 and  $5_2$  knots also satisfy equation (8) when  $6 \leq p \leq 1000$ . The complements of these knots admit ideal triangulations consisting (respectively) of two and three tetrahedra, with simple gluing equations. Thus one can probably employ the methods of Neumann and Zagier [33] to rigorously prove equation (8) for these two knots. Most of the details of the figure-8 case are worked out in [33, Section 6].

#### 4. Twist number and the Jones polynomial

In this section, we prove Theorem 1.5. The proof has three main steps. The first step, due to Stoimenow [38], expresses the coefficients of the Jones polynomial in terms of the combinatorics of the graphs  $\mathbb{G}_A$  and  $\mathbb{G}_B$ , defined in Definition 1.4. The second and third steps relate the combinatorics of the graphs to upper and lower bounds on the twist number of a diagram.

**4.1. Reduced graphs and polynomial coefficients.**

**Definition 4.1.** Let  $D$  be an connected link diagram, with associated graphs  $\mathbb{G}_A, \mathbb{G}_B$ , as in Definition 1.4. The multiplicity of an edge  $e$  of  $\mathbb{G}_A$  or  $\mathbb{G}_B$  is the number of edges that have their endpoints on the same pair of vertices as  $e$ . Let  $\mathbb{G}'_A$  denote the graph obtained from  $\mathbb{G}_A$  by removing multiple edges connected to the same pair of vertices. We will refer to  $\mathbb{G}'_A$  as the *reduced A-graph* associated to  $D$ . Similarly, the reduced  $B$ -graph  $\mathbb{G}'_B$  is obtained by removing multiple edges connected to the same pair of vertices.

Let  $v_A(D), e'_A(D)$  (resp.  $v_B(D), e'_B(D)$ ) denote the number of vertices and edges of  $\mathbb{G}'_A$  (resp.  $\mathbb{G}'_B$ ). When there is no danger of confusion we will omit  $D$  from the notation above to write  $v_A := v_A(D), v_B := v_B(D), e'_A := e'_A(D)$  and  $e'_B := e'_B(D)$ , and so on.

**Proposition 4.2** (Stoimenow). *For a link diagram  $D$ , let*

$$\langle D \rangle = \alpha A^m + \beta A^{m-4} + \gamma A^{m-8} + \dots + \gamma' A^{k+8} + \beta' A^{k+4} + \alpha' A^k$$

*denote the Kauffman bracket of  $D$ , so that  $m$  (resp.  $k$ ) is the highest (resp. lowest) power in  $A$ . If  $D$  is connected and  $A$ -adequate, then*

$$|\beta| = e'_A(D) - v_A(D) + 1.$$

*Similarly, if  $D$  is connected and  $B$ -adequate, then*

$$|\beta'| = e'_B(D) - v_B(D) + 1.$$

Note it is well known that for an  $A$ -adequate diagram,  $|\alpha| = 1$ , and for a  $B$ -adequate diagram,  $|\alpha'| = 1$ .

*Proof.* For proofs of these statements, see the papers of Stoimenow [38, Proposition 3.1] or Dasbach and Lin [21, Theorem 2.4]. q.e.d.

To obtain the Jones polynomial  $J_K(t)$  from the Kauffman bracket  $\langle D \rangle$ , one multiplies  $\langle D \rangle$  by a power of  $-A$  and sets  $t := A^4$ . Thus the absolute values of the coefficients remain the same. This gives the following immediate corollary:

**Corollary 4.3.** *Let  $D$  be an adequate diagram of a link  $K$ . Let  $\beta$  and  $\beta'$  be the second and next-to-last coefficients of  $J_K(t)$ . Then*

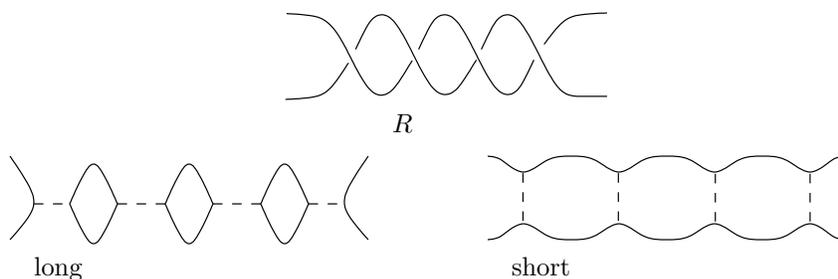
$$|\beta| + |\beta'| = e'_A + e'_B - v_A - v_B + 2.$$

Given Corollary 4.3, we can complete the proof of Theorem 1.5 by estimating the quantity  $e'_A(D) + e'_B(D) - v_A(D) - v_B(D) + 2$  in terms of  $\text{tw}(D)$ .

**4.2. Long and short resolutions.**

**Definition 4.4.** Let  $D$  be a diagram, and let  $R$  be a twist region of  $D$  containing  $c_R > 1$  crossings. One of the graphs associated to  $D$ , say  $\mathbb{G}_A$ , will inherit  $c_R - 1$  vertices from the  $c_R - 1$  bigons contained in  $R$ . We say that this is the *long resolution* of the twist region  $R$ . The other graph, say  $\mathbb{G}_B$ , contains  $c_R$  parallel edges (only one of which survives in  $\mathbb{G}'_B$ ). This is the *short resolution* of  $R$ . See Figure 7.

When a twist region  $R$  contains a single crossing, there is no natural way to choose the short and long resolutions. For such a twist region, we say that both resolutions are short.



**Figure 7.** Resolutions of a twist region  $R$ .

In order to count the vertices and edges of  $\mathbb{G}'_A$  and  $\mathbb{G}'_B$ , we regroup them into short and long resolutions.

**Definition 4.5.** Recall from Definition 1.4 that every vertex of  $\mathbb{G}_A$  and  $\mathbb{G}_B$  (and thus of  $\mathbb{G}'_A$  and  $\mathbb{G}'_B$ ) comes from a component of one of the  $A$ - or  $B$ -resolutions of the diagram  $D$ . We say that a vertex adjacent to exactly 2 edges of  $\mathbb{G}_A$  or  $\mathbb{G}_B$  is a *bigon vertex*; these vertices correspond to bigons in twist regions of  $D$ . Let  $v_{\text{bigon}}$  be the total number of bigon vertices in  $\mathbb{G}_A$  and  $\mathbb{G}_B$ , and let  $v_{\text{n-gon}}$  be the total number of remaining, non-bigon vertices of  $\mathbb{G}_A$  and  $\mathbb{G}_B$ .

In a similar vein, let  $e_{\text{short}}$  (resp.  $e_{\text{long}}$ ) be the total number of edges of  $\mathbb{G}_A$  and  $\mathbb{G}_B$  coming from short (resp. long) resolutions of twist regions. Observe that an edge comes from a long resolution if and only if it is adjacent to at least one bigon vertex. Thus, when a pair of vertices is connected by multiple edges, if neither vertex is a bigon vertex, those edges all short. In any other case, those edges are all long. As a result, we can think of every edge of  $\mathbb{G}'_A$  and  $\mathbb{G}'_B$  as either short or long, and define  $e'_{\text{short}}$  and  $e'_{\text{long}}$  accordingly.

An immediate consequence of this definition is that

$$v_A + v_B = v_{\text{bigon}} + v_{\text{n-gon}} \quad \text{and} \quad e'_A + e'_B = e'_{\text{long}} + e'_{\text{short}}.$$

We are now ready to prove one direction of Theorem 1.5.

**Proposition 4.6.** *Let  $D$  be an adequate diagram of a link  $K$ . Let  $\beta$  and  $\beta'$  be the second and next-to-last coefficients of the Jones polynomial  $J_K$ . Then*

$$|\beta| + |\beta'| = e'_A + e'_B - v_A - v_B + 2 \leq 2 \operatorname{tw}(D).$$

*Proof.* Suppose that an adequate diagram  $D$  has  $c := c(D)$  crossings and  $t := \operatorname{tw}(D)$  twist regions. Given Corollary 4.3 and Definition 4.5, it suffices to estimate the quantities  $v_{\text{bigon}}$ ,  $v_{\text{n-gon}}$ ,  $e'_{\text{long}}$ , and  $e'_{\text{short}}$  in terms of  $c$  and  $t$ .

In a twist region  $R$  containing  $c_R$  crossings, there are  $c_R - 1$  bigons. Thus  $v_{\text{bigon}} = c - t$ . Notice that in both the  $A$ - and  $B$ -resolutions of  $D$ , at least one circle passes through multiple twist regions. Thus each of  $\mathbb{G}_A$  and  $\mathbb{G}_B$  contains at least one non-bigon vertex, and  $v_{\text{n-gon}} \geq 2$ . In every twist region, all the edges of the short resolution get identified to a single edge in either  $\mathbb{G}'_A$  or  $\mathbb{G}'_B$ . Thus  $e'_{\text{short}} \leq t$ . Meanwhile, since each crossing has at most one long resolution,  $e'_{\text{long}} \leq e_{\text{long}} \leq c$ . Putting these facts together, we get

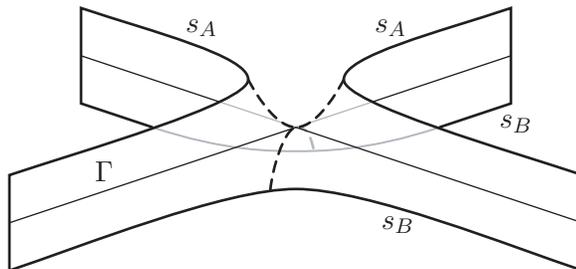
$$\begin{aligned} |\beta| + |\beta'| &= e'_A + e'_B - v_A - v_B + 2, && \text{(Corollary 4.3)} \\ &= e'_{\text{short}} + e'_{\text{long}} - v_{\text{bigon}} - v_{\text{n-gon}} + 2, && \text{(Definition 4.5)} \\ &\leq t + c - (c - t) - 2 + 2 \\ &= 2t. \end{aligned}$$

We note that the adequacy of  $D$  is only needed to apply Corollary 4.3. The remainder of the proof works for any connected diagram  $D$ . q.e.d.

**4.3. Estimates from Turaev surfaces.** To obtain a lower bound on  $|\beta| + |\beta'|$ , we engage in the detailed study of a *Turaev surface* associated to the  $A$ - and  $B$ -resolutions of a diagram. The construction of this surface was first described by Cromwell [19], building on work of Turaev [41].

Let  $\Gamma \subset S^2$  be the planar, 4-valent graph of the link diagram  $D$ . Thicken the projection plane to a slab  $S^2 \times [-1, 1]$ , so that  $\Gamma$  lies in  $S^2 \times \{0\}$ . Outside a neighborhood of the vertices (crossings), our surface will intersect this slab in  $\Gamma \times [-1, 1]$ . In the neighborhood of each vertex, we insert a saddle, positioned so that the boundary circles on  $S^2 \times \{1\}$  are the components of the  $A$ -resolution  $s_A(D)$ , and the boundary circles on  $S^2 \times \{-1\}$  are the components of  $s_B(D)$ . (See Figure 8.) Then, we cap off each circle with a disk, obtaining an unknotted closed surface  $F(D)$ .

In the special case when  $D$  is an alternating diagram, each circle of  $s_A(D)$  or  $s_B(D)$  follows the boundary of a region in the projection plane. Thus, for alternating diagrams, the surface  $F(D)$  is exactly the

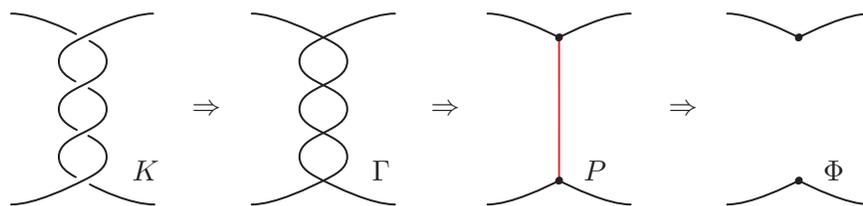


**Figure 8.** Near each crossing of the diagram, a saddle surface interpolates between circles of  $s_A(D)$  and circles of  $s_B(D)$ . The edges of  $\mathbb{G}_A$  and  $\mathbb{G}_B$  can be seen as gradient lines at the saddle.

projection sphere  $S^2$ . For general diagrams, it is still the case that the knot or link has an alternating projection to  $F(D)$  [20, Lemma 4.4].

Furthermore, the construction of  $F(D)$  endows it with a natural cellulation, whose 1-skeleton is the graph  $\Gamma$  and whose 2-cells correspond to circles of  $s_A(D)$  or  $s_B(D)$ , hence to vertices of  $\mathbb{G}_A$  or  $\mathbb{G}_B$ . These 2-cells admit a natural checkerboard coloring, in which the regions corresponding to the vertices of  $\mathbb{G}_A$  are white and the regions corresponding to  $\mathbb{G}_B$  are shaded. The graph  $\mathbb{G}_A$  (resp.  $\mathbb{G}_B$ ) can be embedded in  $F(D)$  as the adjacency graph of white (resp. shaded) regions.

**Definition 4.7.** Let  $D$  be a diagram in which every twist region has at least 2 crossings (hence, at least one bigon). Then we may modify the 4-valent graph  $\Gamma \subset F(D)$ , by collapsing the chain of bigons in each twist region to a single *red edge*. The result is a tri-valent graph  $P \subset F(D)$ , in which exactly one edge at each vertex is colored red. (See Figure 9.)



**Figure 9.** The construction of the graphs  $P$  and  $\Phi$  from a knot diagram. The entire construction takes place on the surface  $F(D)$ .

If we remove all the red edges of  $P$ , we obtain a di-valent graph  $\Phi$ . In other words,  $\Phi$  is a union of disjoint simple closed curves. We call the closures of regions in the complement of  $P$  the *provinces* of  $F(D)$ , and the closures of regions in the complement of  $\Phi$  the *countries* of  $F(D)$ .

Every province of  $F(D)$  comes from a non-trigon region of  $F(D) \setminus \Gamma$ , and corresponds to an  $n$ -gon vertex of  $\mathbb{G}_A$  or  $\mathbb{G}_B$ . Thus each province is a disk. On the other hand, the countries of  $F(D)$  need not be simply connected.

The subdivision of  $F(D)$  into countries allows us to partition the twist regions of  $D$  and the short edges of  $\mathbb{G}_A$  and  $\mathbb{G}_B$ . Every twist region of  $D$  corresponds to a red edge that lies in some country  $N$  of  $F(D)$ . Similarly, every short edge of  $\mathbb{G}_A$  or  $\mathbb{G}_B$  connects two vertices that belong to the same country  $N$ . (Compare Figure 7 with Figure 9.) Thus we may define  $\text{tw}(N)$  to be the number of twist regions belonging to  $N$ , and  $e_{\text{short}}(N)$  to be the number of short edges belonging to  $N$ . In a similar fashion, we may define  $e'_{\text{short}}(N)$  by removing the duplicate edges of  $\mathbb{G}_A$  or  $\mathbb{G}_B$  that belong to  $N$ .

**Lemma 4.8.** *Let  $N$  be a country of  $F(D)$ . Then*

$$e'_{\text{short}}(N) \geq \text{tw}(N) + \chi(N) - 1.$$

*Proof.* The country  $N$  is constructed by taking a number of contractible provinces and gluing them along disjoint segments (red edges) on the boundary. We claim if we cut  $N$  along a well-chosen set of  $1 - \chi(N)$  red edges, it becomes a disk. This can be seen by considering the dual graph to the red edges. Note the country deformation retracts to this dual graph. A maximal spanning tree is obtained by removing  $1 - \chi(N)$  edges, which correspond to red edges in  $N$ .

After this operation, there remain  $\text{tw}(N) + \chi(N) - 1$  red edges along which we did not cut. Call these the *remnant red edges*. The remnant edges are in one-to-one correspondence with a subset of elements of  $e'_{\text{short}}(N)$ , given by selecting a short edge of  $\mathbb{G}_A$  or  $\mathbb{G}_B$  from the corresponding twist region. So  $e'_{\text{short}}(N) \geq \text{tw}(N) + \chi(N) - 1$ . q.e.d.

To estimate  $e'_{\text{short}}(D)$  more globally, we need a bound on the number of countries.

**Lemma 4.9.** *Let  $D$  be an adequate diagram, in which every twist region contains at least 2 crossings. Let  $n(D)$  be the number of countries in the surface  $F(D)$ . Then*

$$n(D) \leq \frac{2}{3} \text{tw}(D) + 1.$$

*Proof.* Recall, from Definition 4.7, that every national border is a component of the graph  $\Phi$ . Let  $|\Phi|$  denote the number of components of the graph  $\Phi$ . Thus  $n(D) \leq |\Phi| + 1$ . Observe as well that the graph  $P$  has exactly two vertices for every twist region of  $D$  (these are the two endpoints of the red edge constructed from the twist region). Thus we compare the number of components of  $\Phi$  to the number of vertices of  $P$ .

Let  $\varphi$  be one simple closed curve contained in  $\Phi$ . We will count the number of vertices of  $P$  that lie on  $\varphi$ . There are two straightforward cases:

*Case 1 :  $\varphi$  bounds a one-province country.* This province cannot be a monogon, since monogons cannot occur in an adequate diagram. It also cannot be a bigon, because the bigon would have been collapsed in the construction of the graph  $P$ . Thus  $\varphi$  contains at least 3 vertices of  $P$ .

*Case 2 :  $\varphi$  does not bound a one-province country.* Then consider the provinces that adjoin  $\varphi$ . The provinces of  $F(D)$  are simply connected, so each side of  $\varphi$  must meet at least one provincial border (red edge). In fact, the hypothesis that  $D$  is adequate implies that a province cannot border on itself along a red edge (otherwise, an edge of  $\mathbb{G}_A$  or  $\mathbb{G}_B$  dual to this red edge would form a loop, violating Definition 1.4). Thus each side of  $\varphi$  must meet at least two provinces, so  $\varphi$  must contain at least 4 vertices of  $P$ .

In either case, each curve  $\varphi \subset \Phi$  contains at least 3 vertices from twist regions. Since each twist region gives rise to two such vertices,  $\text{tw}(D) \geq \frac{3}{2}|\Phi|$ . We can conclude that

$$n(D) \leq |\Phi| + 1 \leq \frac{2}{3} \text{tw}(D) + 1.$$

q.e.d.

We can now prove the remaining direction of Theorem 1.5.

**Theorem 4.10.** *Let  $D$  be an adequate diagram of a link  $K$ , in which every twist region contains at least 3 crossings. Let  $\beta$  and  $\beta'$  be the second and next-to-last coefficients of the Jones polynomial  $J_K$ . Then*

$$|\beta| + |\beta'| = e'_A + e'_B - v_A - v_B + 2 \geq \frac{\text{tw}(D)}{3} + 1.$$

*Proof.* Suppose that the diagram  $D$  has  $c := c(D)$  crossings and  $t := \text{tw}(D)$  twist regions. Given Corollary 4.3 and Definition 4.5, it suffices to estimate the quantities  $v_{\text{bigon}}$ ,  $v_{\text{n-gon}}$ ,  $e'_{\text{long}}$ , and  $e'_{\text{short}}$  in terms of  $c$  and  $t$ .

In a twist region  $R$  containing  $c_R$  crossings, there are  $c_R - 1$  bigons and  $c_R$  long edges. Thus  $v_{\text{bigon}} = c - t$  and  $e_{\text{long}} = c$ . When every twist region contains at least 3 crossings, it is evident from Figure 7 that all long edges of  $\mathbb{G}_A$  and  $\mathbb{G}_B$  will survive in  $\mathbb{G}'_A$  and  $\mathbb{G}'_B$ . Thus we can conclude that  $e'_{\text{long}} = c$ , giving us

$$(9) \quad e'_{\text{long}} - v_{\text{bigon}} = \text{tw}(D).$$

To estimate  $v_{\text{n-gon}}$  and  $e'_{\text{short}}$ , we compute the Euler characteristic of  $F(D)$ . Recall that the tri-valent graph  $P$  has  $2t$  vertices (two for every red edge) and  $3t$  edges (since every third edge is red). The 2-cells in

the complement of  $P$  are provinces, one for every  $n$ -gon vertex of  $\mathbb{G}_A$  and  $\mathbb{G}_B$ . Thus

$$(10) \quad \chi(F(D)) = v_{n\text{-gon}} - 3\text{tw}(D) + 2\text{tw}(D) = v_{n\text{-gon}} - \text{tw}(D).$$

Lemma 4.8 tells us that  $e'_{\text{short}}(N) \geq \text{tw}(N) + \chi(N) - 1$  for every country  $N$  of  $F(D)$ . By summing this over all countries, we get

$$\begin{aligned} e'_{\text{short}} &\geq \text{tw}(D) + \chi(F(D)) - n(D) \\ &= v_{n\text{-gon}} - n(D), && \text{by Equation (10)} \\ &\geq v_{n\text{-gon}} - \frac{2}{3}\text{tw}(D) - 1 && \text{by Lemma 4.9.} \end{aligned}$$

Putting all of these results together gives

$$\begin{aligned} |\beta| + |\beta'| &= e'_A + e'_B - v_A + v_B + 2, && \text{by Corollary 4.3} \\ &= (e'_{\text{long}} - v_{\text{bigon}}) + (e'_{\text{short}} - v_{n\text{-gon}}) + 2, && \text{by Definition 4.5} \\ &\geq \text{tw}(D) + \left(-\frac{2}{3}\text{tw}(D) - 1\right) + 2, \\ &= \frac{\text{tw}(D)}{3} + 1. \end{aligned}$$

q.e.d.

### 5. Families of differential equations

In the proof of Theorem 2.1, we defined a family of functions  $f_{t,\epsilon}$  and  $g_{t,\epsilon}$ . Our goal in this section is to prove that  $f_{t,\epsilon}$ ,  $f'_{t,\epsilon}$ , and  $g_{t,\epsilon}$  depend continuously and uniformly on  $t$  and  $\epsilon$ .

Let us recap the definitions. For parameters  $\epsilon > 0$  and  $0 < t < 1$ , we began with a smooth bump function  $k_{t,\epsilon}(r)$ . This function has a precise definition, as follows:

$$k_{t,\epsilon}(r) := \begin{cases} t & \text{if } r \leq -\epsilon, \\ t + (1-t) \frac{\int_0^{2+2r/\epsilon} z(u) du}{\int_0^1 z(u) du} & \text{if } -\epsilon < r < -\epsilon/2, \\ 1 & \text{if } r \geq -\epsilon/2, \end{cases}$$

where  $z(u) = \exp\left(-\frac{1}{u^2} - \frac{1}{(u-1)^2}\right)$ . (See Figure 4 for a typical graph.)

By extension to  $\epsilon = 0$ , we defined  $k_{t,0}(r)$  as a step function whose value is  $t$  for  $r < 0$  and  $1$  for  $r \geq 0$ .

Given  $k_{t,\epsilon}$ , we defined  $f_{t,\epsilon}$  and  $g_{t,\epsilon}$  according to the differential equations

$$(11) \quad f''_{t,\epsilon}(r) = k_{t,\epsilon}(r) f_{t,\epsilon}(r), \quad \frac{g'_{t,\epsilon}(r)}{g_{t,\epsilon}(r)} = k_{t,\epsilon}(r) \frac{f'_{t,\epsilon}(r)}{f_{t,\epsilon}(r)},$$

with initial conditions  $f_{t,\epsilon}(0) = f'_{t,\epsilon}(0) = \ell_1$  and  $g_{t,\epsilon}(0) = \ell_2$ .

We will prove that  $f_{t,\epsilon}$ ,  $f'_{t,\epsilon}$ , and  $g_{t,\epsilon}$  depend continuously and uniformly on  $t$  and  $\epsilon$ , even as  $\epsilon$  goes to 0, when  $k_{t,\epsilon}$  becomes discontinuous. Before we prove that statement, we need a monotonicity result.

**Definition 5.1.** For any  $t \in (0, 1)$  and  $\epsilon \geq 0$ , define

$$r_0(t, \epsilon) := \inf \{s \in \mathbb{R} : f_{t,\epsilon}(r) > 0 \text{ for all } r > s\}.$$

In other words,  $r_0$  is either the largest root of  $f_{t,\epsilon}$ , or  $-\infty$  if  $f_{t,\epsilon}$  has no root.

**Lemma 5.2.** For any  $t \in (0, 1)$  and  $\epsilon \geq 0$ ,  $r_0(t, \epsilon) < \min\{-\epsilon/2, -1\}$ .

*Proof.* If  $r_0(t, \epsilon) = -\infty$ , the result is trivially true. Thus we may assume that  $r_0(t, \epsilon)$  is a root of  $f_{t,\epsilon}$ . Note that for  $r \geq -\epsilon/2$ ,  $f_{t,\epsilon}(r) = \ell_1 e^r > 0$ , and thus  $r_0(t, \epsilon) < -\epsilon/2 \leq 0$ .

To prove that  $r_0(t, \epsilon) < -1$ , observe that equation (11) implies  $f_{t,\epsilon}(r)$  is concave up on  $(r_0(t, \epsilon), 0]$ . Thus, for all  $r \in (r_0(t, \epsilon), 0)$  we have

$$f'_{t,\epsilon}(r) < f'_{t,\epsilon}(0) = \ell_1.$$

Since the function  $f_{t,\epsilon}$  must climb from height 0 to height  $\ell_1$  with slope less than  $\ell_1$ , it follows  $r_0(t, \epsilon) < -1$ . q.e.d.

**Lemma 5.3.** The functions  $f_{t,\epsilon}$  and  $f'_{t,\epsilon}$  are monotonic in the parameters  $t, \epsilon$ ; and  $f_{t,\epsilon}$  is also monotonic in  $r$ . More precisely:

- (a) If  $0 \leq \epsilon_1 < \epsilon_2$  and  $r \in [r_0(t, \epsilon_2), 0]$ , then  $f_{t,\epsilon_1}(r) \leq f_{t,\epsilon_2}(r)$  and  $f'_{t,\epsilon_1}(r) \geq f'_{t,\epsilon_2}(r)$ , with strict inequalities on  $[r_0(t, \epsilon_2), -\epsilon_1/2)$ .
- (b) If  $0 < t_1 < t_2 < 1$  and  $r \in [r_0(t_2, \epsilon), 0]$ , then  $f_{t_1,\epsilon}(r) \leq f_{t_2,\epsilon}(r)$  and  $f'_{t_1,\epsilon}(r) \geq f'_{t_2,\epsilon}(r)$ , with strict inequalities on  $[r_0(t_2, \epsilon), -\epsilon/2)$ .
- (c) For any  $r \in \mathbb{R}$ ,  $\epsilon \geq 0$ , and  $t \in (0, 1)$ ,  $f'_{t,\epsilon}(r) > 0$ .

*Proof.* The key observation for this proof is that the bump function  $k_{t,\epsilon}(r)$  is increasing in both  $t$  and  $\epsilon$ . See Figure 4.

For part (a), suppose that  $0 \leq \epsilon_1 < \epsilon_2$ . To compare  $f$  and  $f'$  for these two values of  $\epsilon$ , define a function  $\varphi(r) := f_{t,\epsilon_2}(r) - f_{t,\epsilon_1}(r)$ . Then

$$\begin{aligned} \varphi''(r) &= f''_{t,\epsilon_2}(r) - f''_{t,\epsilon_1}(r) \\ &= k_{t,\epsilon_2}(r) f_{t,\epsilon_2}(r) - k_{t,\epsilon_1}(r) f_{t,\epsilon_1}(r) \\ &\geq k_{t,\epsilon_1}(r) f_{t,\epsilon_2}(r) - k_{t,\epsilon_1}(r) f_{t,\epsilon_1}(r) \quad \text{when } r \in [r_0(t, \epsilon_2), 0], \\ &\quad \text{with a strict inequality for } r \in [r_0(t, \epsilon_2), 0] \cap (-\epsilon_2, -\epsilon_1/2) \\ &= k_{t,\epsilon_1}(r) \varphi(r). \end{aligned}$$

By definition,  $0 \leq k_{t,\epsilon_1}(r) \leq 1$ . Thus we obtain a differential inequality with certain nice properties. By a result from analysis, whose proof we include as Theorem A.1 in the Appendix,  $\varphi(r) \geq 0$  and  $\varphi'(r) \leq 0$  for all  $r \in [r_0(t, \epsilon_2), 0]$ , with strict inequalities on  $[r_0(t, \epsilon_2), -\epsilon_1/2)$ . Note that this interval is non-empty, because by Lemma 5.2,

$$r_0(t, \epsilon_2) < -\epsilon_2/2 < -\epsilon_1/2.$$

This proves (a).

The proof of part (b) is very similar to (a), except this time we define  $\varphi(r) := f_{t_2, \epsilon}(r) - f_{t_1, \epsilon}(r)$ . An analogous calculation then goes through.

For part (c), fix values of  $\epsilon \geq 0$  and  $t \in (0, 1)$ . We want to prove that  $f'_{t, \epsilon}(r) > 0$  for all  $r$ . If  $r \geq -\epsilon/2$ , we have already seen that  $f'_{t, \epsilon}(r) = \ell_1 e^r > 0$ .

If  $r_0(t, \epsilon) \leq r < -\epsilon/2$ , we rely on part (a). That is, set  $\epsilon_1 = \epsilon$  and  $\epsilon_2 = -2r$ . Then provided we can show  $r \in [r_0(t, -2r), 0]$ , part (a) implies

$$f'_{t, \epsilon}(r) \geq f'_{t, -2r}(r) = \ell_1 e^r > 0.$$

If  $r_0(t, -2r) = -\infty$ , then certainly  $r$  is in the correct range. Otherwise, we know  $r \geq r_0(t, \epsilon)$  by assumption. For any  $s \in [r_0(t, -2r), 0]$ , part (a) implies  $f_{t, \epsilon}(s) \leq f_{t, -2r}(s)$ , so in particular,  $f_{t, \epsilon}(r_0(t, -2r)) \leq f_{t, -2r}(r_0(t, -2r)) = 0$ . Thus  $r_0(t, \epsilon) \geq r_0(t, -2r)$ . So  $r \geq r_0(t, -2r)$  as desired.

Finally, for  $r < r_0(t, \epsilon)$ , we note that when  $f_{t, \epsilon}(r) < 0$ , the function must also be concave down. Thus, since  $f'_{t, \epsilon}(r) > 0$  at the root  $r = r_0(t, \epsilon)$ ,  $f'_{t, \epsilon}(r)$  can only become more positive as  $r$  moves further to the left. q.e.d.

**Theorem 5.4.** *Fix constants  $t_{\text{lim}} \in (0, 1)$  and  $\epsilon_{\text{lim}} \geq 0$ . Then, as  $(t, \epsilon) \rightarrow (t_{\text{lim}}, \epsilon_{\text{lim}})$ ,*

$$f_{t, \epsilon}(r) \rightarrow f_{t_{\text{lim}}, \epsilon_{\text{lim}}}(r) \quad \text{and} \quad f'_{t, \epsilon}(r) \rightarrow f'_{t_{\text{lim}}, \epsilon_{\text{lim}}}(r),$$

*uniformly on compact sets. In particular, the functions  $f$  and  $f'$  are continuous in the three variables  $(t, \epsilon, r)$ .*

*Proof.* Observe that when  $\epsilon > 0$ , the function  $k_{t, \epsilon}(r)$  is continuous in all three variables  $(t, \epsilon, r)$ . Thus, when  $\epsilon_{\text{lim}} > 0$ , the conclusion of the theorem is a standard result in ODE theory (see, for example, [26]). We will therefore restrict our attention to the case when  $\epsilon_{\text{lim}} = 0$ .

Fix an integer  $n$  such that  $t_{\text{lim}} \in (\frac{1}{n}, \frac{n-1}{n})$ . Now, suppose that  $(t, \epsilon)$  varies in the compact domain  $[\frac{1}{n}, \frac{n-1}{n}] \times [0, 1]$ , and that  $r$  varies in the compact interval  $[-n, n]$ . We begin the argument by showing that the values of  $f_{t, \epsilon}(r)$  are uniformly bounded on this domain. By Lemma 5.3(c),  $f_{t, \epsilon}(r)$  is strictly increasing, and so attains its maximum value at  $r = n$ . Thus  $f_{t, \epsilon}(n) = \ell_1 e^n$  is a uniform upper bound.

For a lower bound on  $f_{t, \epsilon}(r)$ , we take a closer look at equation (11). When  $r \leq -\epsilon$ , the equation has the explicit solution

$$(12) \quad f_{t, \epsilon}(r) = c_1(t, \epsilon)e^{r\sqrt{t}} + c_2(t, \epsilon)e^{-r\sqrt{t}},$$

where

$$(13) \quad c_1(t, \epsilon) = \frac{f_{t, \epsilon}(-\epsilon)}{2e^{-\epsilon\sqrt{t}}} + \frac{f'_{t, \epsilon}(-\epsilon)}{2\sqrt{t}e^{-\epsilon\sqrt{t}}}, \quad c_2(t, \epsilon) = \frac{f_{t, \epsilon}(-\epsilon)}{2e^{\epsilon\sqrt{t}}} - \frac{f'_{t, \epsilon}(-\epsilon)}{2\sqrt{t}e^{\epsilon\sqrt{t}}}.$$

Now, when  $\epsilon \leq 1$ , Lemma 5.2 says that  $r_0(t, \epsilon) < -\epsilon$ . Thus  $f_{t,\epsilon}$  and  $f'_{t,\epsilon}$  are positive and increasing on  $[-\epsilon, 0]$ , and both are bounded above by  $\ell_1$ . Thus both  $f_{t,\epsilon}(-\epsilon)$  and  $f'_{t,\epsilon}(-\epsilon)$  must be in the interval  $(0, \ell_1]$ . In particular, this implies that

$$(14) \quad c_1(t, \epsilon) > 0, \quad c_2(t, \epsilon) > \frac{-\ell_1}{2\sqrt{1/n}}.$$

By putting together (12) and (14), we see that on the interval  $[-n, n]$ ,

$$f_{t,\epsilon}(r) \geq f_{t,\epsilon}(-n) > c_2(t, \epsilon) e^{n\sqrt{t}} \geq \frac{-\ell_1\sqrt{n}}{2} e^n.$$

We can conclude that when  $t \in [\frac{1}{n}, \frac{n-1}{n}]$ ,  $\epsilon \in [0, 1]$ , and  $r \in [-n, n]$ , the family of functions  $f_{t,\epsilon}(r)$  is uniformly bounded. Because  $k_{t,\epsilon}(r)$  is also uniformly bounded (by 0 and 1), it follows that  $f''_{t,\epsilon}(r)$  is uniformly bounded. By integration, it follows that  $f'_{t,\epsilon}(r)$  is uniformly bounded and equicontinuous. Integrating again, we see that  $f_{t,\epsilon}(r)$  is equicontinuous. Finally, for any  $\delta \in (0, n)$ ,  $k_{t,\epsilon}(r)$  has uniformly bounded derivative on  $[-n, -\delta]$ , which implies that  $f''_{t,\epsilon}(r)$  is equicontinuous on that interval.

Fix a number  $\delta \in (0, n)$ , and let  $(t_i, \epsilon_i)$  be a sequence that converges to  $(t_{\text{lim}}, 0)$ . Then the Arzela–Ascoli theorem implies that there is a continuous function  $f_{\text{lim}}$  on  $[-n, n]$ , twice differentiable on  $[-n, -\delta]$ , such that

$$f_{t_i, \epsilon_i}(r) \rightarrow f_{\text{lim}}(r), \quad f'_{t_i, \epsilon_i}(r) \rightarrow f'_{\text{lim}}(r), \quad f''_{t_i, \epsilon_i}(r) \rightarrow f''_{\text{lim}}(r),$$

uniformly on  $[-n, -\delta]$ . In fact,  $f_{t_i, \epsilon_i}$  and  $f'_{t_i, \epsilon_i}$  converge uniformly on  $[-n, n]$ . Furthermore, for all  $\epsilon_i < \delta$ ,  $k_{t_i, \epsilon_i}(r) = t_i$  on  $[-n, -\delta]$ , and thus  $k_{t_i, \epsilon_i}(r)$  converges uniformly to  $t_{\text{lim}}$ . Thus  $f_{\text{lim}}$  satisfies the differential equation

$$f''_{\text{lim}}(r) = k_{t_{\text{lim}}, 0}(r) f_{\text{lim}}(r),$$

for all  $r \in [-n, -\delta]$ . Since  $\delta$  was arbitrary, this equation is satisfied for all  $r \in [-n, 0)$ . Since  $f_{\text{lim}}(0) = f'_{\text{lim}}(0) = \ell_1$ ,  $f_{\text{lim}}$  is a solution to equation (11), for  $t = t_{\text{lim}}$  and  $\epsilon = 0$ . Therefore, by the uniqueness of solutions, we can conclude that  $f_{\text{lim}}(r) = f_{t_{\text{lim}}, 0}(r)$ , for all  $r \in [-n, n]$ .

q.e.d.

**Lemma 5.5.** *The roots of  $f_{t,\epsilon}(r)$  have the following behavior:*

- (a) *For all  $t \in (0, 1)$  and  $\epsilon \geq 0$ ,  $f_{t,\epsilon}(r)$  has a unique root, equal to  $r_0(t, \epsilon)$ .*
- (b) *The function  $m(t, \epsilon) := f'_{t,\epsilon}(r_0(t, \epsilon))$  is continuous in  $t$  and  $\epsilon$ , and strictly decreasing in both variables.*

*Proof.* By Lemma 5.3(c),  $f_{t,\epsilon}(r)$  is strictly increasing on  $\mathbb{R}$ . Thus if a root exists, it will be unique. To prove the existence of a root, we study the explicit formula for  $f_{t,\epsilon}(r)$  on the interval  $(-\infty, -\epsilon]$ , given in equation (12). As  $r \rightarrow -\infty$ , this equation is dominated by the term

$c_2(t, \epsilon) e^{-r\sqrt{t}}$ . In particular, since  $f_{t,\epsilon}$  is increasing on  $\mathbb{R}$ , we must have  $c_2(t, \epsilon) \leq 0$ . We will show that, in fact,  $c_2(t, \epsilon) < 0$ .

Suppose, for a contradiction, that  $c_2(t, \epsilon_1) = 0$  for some value  $\epsilon_1$ . Then, by equation (12),

$$f_{t,\epsilon_1}(r) = c_1(t, \epsilon_1) e^{r\sqrt{t}} \quad \text{on } (-\infty, -\epsilon_1],$$

for a positive constant  $c_1(t, \epsilon_1)$ . Now, choose a larger value  $\epsilon_2$ . As  $\epsilon_2 \rightarrow \infty$ , we have larger and larger subsets of  $(-\infty, 0]$  on which

$$f_{t,\epsilon_2}(r) = \ell_1 e^r.$$

Because  $t < 1$  and  $e^r$  decays faster than  $e^{r\sqrt{t}}$  as  $r \rightarrow -\infty$ , there will be an  $\epsilon_2 \gg \epsilon_1$  and an  $r \ll 0$  such that

$$f_{t,\epsilon_2}(r) = \ell_1 e^r < c_1(t, \epsilon_1) e^{r\sqrt{t}} = f_{t,\epsilon_1}(r),$$

contradicting Lemma 5.3(a). Thus  $c_2(t, \epsilon) < 0$  for all  $t, \epsilon$ .

As a result,  $f_{t,\epsilon}(r)$  approaches  $-\infty$  as  $r \rightarrow -\infty$ , and therefore has a root. By Definition 5.1, this unique root is equal to  $r_0(t, \epsilon)$ .

To prove part (b), we once again use the fact that  $f_{t,\epsilon}(r)$  is continuous and strictly increasing. Thus it has a continuous inverse  $f_{t,\epsilon}^{-1}$ , such that  $f_{t,\epsilon}^{-1}(0) = r_0(t, \epsilon)$ . This allows us to write

$$m(t, \epsilon) = f'_{t,\epsilon} \circ f_{t,\epsilon}^{-1}(0).$$

By Theorem 5.4, both  $f'_{t,\epsilon}$  and  $f_{t,\epsilon}^{-1}$  are continuous in  $t$  and  $\epsilon$ ; therefore,  $m(t, \epsilon)$  is continuous as well.

Now, fix starting values  $\epsilon_1, t_1$  of  $\epsilon$  and  $t$ . Then, for any  $\epsilon_2 > \epsilon_1$ , Lemma 5.3(a) implies that

$$m(t_1, \epsilon_1) = f'_{t_1,\epsilon_1}(r_0(t_1, \epsilon_1)) > f'_{t_1,\epsilon_2}(r_0(t_1, \epsilon_1)).$$

Since  $m(t_1, \epsilon_2)$  is the absolute minimum of  $f'_{t_1,\epsilon_2}(r)$  over all of  $\mathbb{R}$  (because  $f_{t,\epsilon}$  is concave up whenever  $f_{t,\epsilon}$  is positive, concave down when negative due to its defining equation (11)), we have

$$f'_{t_1,\epsilon_2}(r_0(t_1, \epsilon_1)) \geq m(t_1, \epsilon_2).$$

So  $m(t_1, \epsilon_1) \geq m(t_1, \epsilon_2)$ .

Similarly, by Lemma 5.3(b),  $m(t_1, \epsilon_1) > m(t_2, \epsilon_1)$  for  $t_2 > t_1$ . Thus  $m(t, \epsilon)$  is strictly decreasing in both  $t$  and  $\epsilon$ . q.e.d.

We now turn our attention to the function  $g_{t,\epsilon}(r)$ . Its defining equation (11) can be written as

$$(15) \quad \frac{d}{dr} (\ln g_{t,\epsilon}(r)) = k_{t,\epsilon}(r) \frac{f_{t,\epsilon}(r)}{f'_{t,\epsilon}(r)},$$

with initial condition  $g_{t,\epsilon}(0) = \ell_2$ . Note that by Lemma 5.3(c),  $f'_{t,\epsilon}(r) > 0$  for all  $r$ , so the right-hand side is always well-defined.

**Theorem 5.6.** *Fix constants  $t_{\lim} \in (0, 1)$  and  $\epsilon_{\lim} \geq 0$ . Then, as  $(t, \epsilon) \rightarrow (t_{\lim}, \epsilon_{\lim})$ ,*

$$g_{t,\epsilon}(r) \rightarrow g_{t_{\lim},\epsilon_{\lim}}(r),$$

*uniformly on compact sets.*

*Proof.* This proof follows the same outline as the proof of Theorem 5.4. As in that proof, we restrict our attention to the case when  $\epsilon_{\lim} = 0$ , because the conclusion of the theorem is a standard result for  $\epsilon_{\lim} > 0$ .

Fix an integer  $n$  such that  $t_{\lim} \in (\frac{1}{n}, \frac{n-1}{n})$ . Now, suppose that  $(t, \epsilon)$  varies in the compact domain  $[\frac{1}{n}, \frac{n-1}{n}] \times [0, 1]$ , and that  $r$  varies in the compact interval  $[-n, n]$ . We begin the argument by showing that the right-hand side of equation (15) is uniformly bounded on this domain. In the proof of Theorem 5.4, we have already shown that on this domain,

$$(16) \quad |k_{t,\epsilon}(r) f_{t,\epsilon}(r)| \leq |f_{t,\epsilon}(r)| \leq \ell_1 \sqrt{n} e^n.$$

Also, because  $m(t, \epsilon)$  is the absolute minimum value of  $f'_{t,\epsilon}(r)$  over all of  $\mathbb{R}$ , and by Lemma 5.5(b),

$$(17) \quad f'_{t,\epsilon}(r) \geq m(t, \epsilon) \geq m(\frac{n-1}{n}, 1) > 0.$$

Putting inequalities (16) and (17) together, we get

$$(18) \quad \left| \frac{d}{dr} (\ln g_{t,\epsilon}(r)) \right| \leq \frac{\ell_1 \sqrt{n} e^n}{m(\frac{n-1}{n}, 1)}.$$

By integrating (18), we conclude that the family of functions  $\ln g_{t,\epsilon}(r)$  is uniformly bounded and equicontinuous. Also, for any  $\delta \in (0, n)$ ,  $\frac{d}{dr} \ln g_{t,\epsilon}(r)$  is equicontinuous on  $[-n, -\delta]$ . This follows by differentiating the right-hand side of (15), because  $f_{t,\epsilon}(r)$ ,  $f'_{t,\epsilon}(r)$ ,  $f''_{t,\epsilon}(r)$ ,  $k_{t,\epsilon}(r)$ , and  $k'_{t,\epsilon}(r)$  are all uniformly bounded on that interval, with  $f'_{t,\epsilon}(r)$  bounded away from 0.

Fix a number  $\delta \in (0, n)$ , and let  $(t_i, \epsilon_i)$  be a sequence that converges to  $(t_{\lim}, 0)$ . Then the Arzela–Ascoli theorem implies that there is a continuous function  $g_{\lim}$ , differentiable on  $[-n, -\delta]$ , such that

$$\ln g_{t_i,\epsilon_i}(r) \rightarrow \ln g_{\lim}(r), \quad \frac{d}{dr} \ln g_{t_i,\epsilon_i}(r) \rightarrow \frac{d}{dr} \ln g_{\lim}(r),$$

uniformly on  $[-n, -\delta]$ . In fact,  $\ln g_{t_i,\epsilon_i}(r)$  converges uniformly on  $[-n, n]$ ; since this is a compact set,  $g_{t_i,\epsilon_i}(r)$  also converges uniformly to  $g_{\lim}(r)$ .

By letting  $\delta$  approach 0, we see that the function  $g_{\lim}(r)$  satisfies the differential equation (15) for  $t = t_{\lim}$  and  $\epsilon = 0$ . Thus, by the uniqueness of solutions,  $g_{\lim}(r) = g_{t_{\lim},0}(r)$ , as desired. q.e.d.

## Appendix A. Differential inequalities

The following elementary result from real analysis is probably well-known. However, since we could not find a reference, we include a proof here.

**Theorem A.1.** *Let  $I \subset \mathbb{R}$  be a closed interval that includes 0. Let  $\varphi : I \rightarrow \mathbb{R}$  be a  $C^1$  function, such that  $\varphi''(x)$  exists for all  $x \neq 0$ . Suppose that  $\varphi$  satisfies the differential inequality*

$$\varphi''(x) \geq k(x) \varphi(x) \quad \text{for all } x \neq 0,$$

where  $0 \leq k(x) \leq 1$ . Assume as well that  $\varphi(0) \geq 0$  and  $\varphi'(0) = 0$ . Then

- (a)  $\varphi'(x) \geq 0$  for  $x \geq 0$  and  $\varphi'(x) \leq 0$  for  $x \leq 0$ ,
- (b)  $\varphi(x) \geq 0$  and  $\varphi''(x) \geq 0$  for all  $x$ .

Furthermore,

- (c) If  $\varphi''(x_0) > 0$  and  $x_0 < 0$ , then  $\varphi(x) > 0$  and  $\varphi'(x) < 0$  for  $x < x_0$ .
- (d) If  $\varphi''(x_0) > 0$  and  $x_0 > 0$ , then  $\varphi(x) > 0$  and  $\varphi'(x) > 0$  for  $x > x_0$ .

A key step of the proof is the following, slightly weaker statement.

**Lemma A.2.** *Let  $I \subset \mathbb{R}$  be a closed interval that includes 0. Let  $\psi : I \rightarrow \mathbb{R}$  be a  $C^1$  function, such that  $\psi''(x)$  exists for all  $x \neq 0$ . Suppose that  $\psi$  satisfies the differential inequality*

$$\psi''(x) \geq k(x) \psi(x) \quad \text{for all } x \neq 0,$$

where  $0 \leq k(x) \leq 1$ . Assume as well that  $\psi(0) \geq 0$  and  $\psi'(0) \geq 0$ . Then

- (a)  $\psi(x) \geq 0$ ,  $\psi'(x) \geq 0$ , and  $\psi''(x) \geq 0$  on  $[0, 1] \cap I$ .
- (b) If  $\psi''(x_0) > 0$  or  $\psi'(x_0) > 0$  for some  $x_0 \in [0, 1]$ , then  $\psi(x) > 0$  and  $\psi'(x) > 0$  on  $(x_0, 1] \cap I$ .

*Proof.* To prove (a), let  $m = \min\{\psi(x) : x \in [0, 1] \cap I\}$ . Assume, for a contradiction, that  $m < 0$ . Then, because  $k(x) \leq 1$  for all  $x$ , we have  $\psi''(x) \geq m$ , for all  $x \in [0, 1] \cap I$ . Now, Taylor's theorem allows us to write

$$\psi(x) = \psi(0) + \psi'(0)x + \frac{1}{2} \psi''(x_0)x^2, \quad \text{for some } x_0 \in (0, x).$$

We can estimate each of these terms. We have  $\psi(0) \geq 0$  by hypothesis,  $\psi'(0)x \geq 0$  because both parts of the product are non-negative, and  $\psi''(x_0)x^2 \geq m$  because  $x^2 \leq 1$ . (Recall that we have assumed  $m < 0$ .) Putting all of this together gives

$$\psi(x) \geq 0 + 0 + \frac{1}{2} m > m \quad \text{for all } x \in [0, 1] \cap I,$$

contradicting the assumption that  $m$  was the minimum.

As a result of this contradiction,  $\psi(x) \geq 0$  on  $[0, 1] \cap I$ . Thus, since  $k(x) \geq 0$ , we have  $\psi''(x) \geq 0$  as well. Integration gives  $\psi'(x) \geq 0$ , completing the proof of (a).

To prove (b), suppose first that  $\psi''(x_0) > 0$  for some  $x_0 \in [0, 1]$ . Then  $\psi'(x)$  is strictly increasing in a neighborhood of  $x_0$ . Since we have already shown that  $\psi''(x) \geq 0$  for all  $x \in [0, 1] \cap I$ , we have

$$\psi'(x) > \psi'(x_0) \geq 0, \quad \text{for all } x \in (x_0, 1] \cap I.$$

By integration, we also have  $\psi(x) > 0$  on  $(x_0, 1] \cap I$ .

Now, suppose that  $\psi'(x_0) > 0$  for some  $x_0 \in [0, 1)$ . Since  $\psi''(x) \geq 0$  for all  $x$ , this implies that  $\psi'(x) > 0$  on  $[x_0, 1] \cap I$ . Then, integration gives  $\psi(x) > 0$  on  $(x_0, 1] \cap I$ , completing the proof. q.e.d.

*Proof of Theorem A.1.* We will apply Lemma A.2 inductively, many times. We assume without loss of generality that  $I = \mathbb{R}$ ; if  $I \subsetneq \mathbb{R}$ , the only alteration required is to stop the inductive process once we get to the boundary of  $I$ .

Applying the lemma to  $\psi(x) := \varphi(x)$  gives the conclusion of the theorem on the interval  $[0, 1]$ . That is,  $\varphi(x)$  satisfies the differential inequality of the lemma, and  $\varphi(0) \geq 0$ , and  $\varphi'(0) = 0$  so the lemma applies immediately. This will imply that  $\varphi(1) \geq 0$  and  $\varphi'(1) \geq 0$ , with strict inequalities if  $\varphi''(x_0) > 0$  for some  $x_0 \in [0, 1)$ .

Now, apply Lemma A.2 to the function  $\psi(x) := \varphi(x+1)$ . The lemma applies because  $\psi(0) = \varphi(1) \geq 0$ , and  $\psi'(0) = \varphi'(1) \geq 0$  (possibly with strict inequalities). This gives the conclusion of the theorem on the interval  $[1, 2]$ . Repeatedly applying the lemma in this way proves the theorem for all  $x \geq 0$ .

To prove the theorem for  $x \leq 0$ , we first apply Lemma A.2 to the function  $\psi(x) := \varphi(-x)$ . Note that  $\psi(0) = \varphi(0) \geq 0$ , and  $\psi'(0) = -\varphi'(0) = 0$ , so the lemma applies to this function. Then we obtain the conclusion of the theorem on  $[-1, 0]$ . Note, in particular, that we now have  $\varphi(-1) \geq 0$  and  $\varphi'(-1) \leq 0$  (with strict inequalities if  $\varphi''(x_0) > 0$  for some  $x_0 \in (-1, 0]$ ). Now, apply Lemma A.2 to  $\psi(x) := \varphi(-x-1)$ , etc., to obtain the conclusion of the theorem for all  $x \leq 0$ . q.e.d.

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