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SPECTRUM OF THE LAPLACIAN ON QUATERNIONIC KÄHLER MANIFOLDS

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Abstract

Let M^{4n} be a complete quaternionic Kähler manifold with scalar curvature bounded below by $-16n(n+2)$. We get a sharp estimate for the first eigenvalue $\lambda_1(M)$ of the Laplacian, which is $\lambda_1(M) \le (2n+1)^2$. If the equality holds, then either M has only one end, or M is diffeomorphic to $\mathbb{R} \times N$ with N given by a compact manifold. Moreover, if M is of bounded curvature, M is covered by the quaterionic hyperbolic space $\mathbb{QH}^n$ and N is a compact quotient of the generalized Heisenberg group. When $\lambda_1(M) \ge \frac{8(n+2)}{3}$, we also prove that M must have only one end with infinite volume.

0. Introduction

Let M^n be a complete n-dimensional Riemannian manifold with Ricci curvature bounded below by $-(n-1)$. It is well known from Cheng [**Ch**] that the first eigenvalue $\lambda_1(M)$ satisfies

$$\lambda_1(M) \le \frac{(n-1)^2}{4}.$$

In [**LW3**], Li and Wang proved an analogous theorem for complete Kähler manifolds. They showed that if M^{2n} is a complete Kähler manifold of complex dimension n with holomorphic bisectional curvature BK_M bounded below by -1, then the first eigenvalue $\lambda_1(M)$ satisfies

$$\lambda_1(M) \le n^2.$$

Here $\mathrm{BK}_M \ge -1$ means that

$$R_{i\bar{i}j\bar{j}} \ge -(1+\delta_{ij})$$

for any unitary frame $e_1, \dots, e_n$.

In this paper, we prove the corresponding Laplacian comparison theorem for a quaterionic Kähler manifold M^{4n}. As an application we get

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the sharp estimate $\lambda_1(M)$ for a complete quaterionic Kähler manifold M^{4n} with scalar curvature bounded below by $-16n(n+2)$ as

$$\lambda_1(M) \leq (2n+1)^2.$$

It is an interesting question to ask what one can say about those manifolds when the above inequalities are realized as equalities. In works of Li and Wang [**LW1**] and [**LW2**], the authors obtained the following theorems. The first was a generalization of the theory of Witten-Yau [**WY**], Cai-Galloway [**CG**], and Wang [**W**] for conformally compact manifolds. The second was to answer the aforementioned question.

Theorem 0.1. *Let M^n be a complete Riemannian manifold of dimension $n \geq 3$ with Ricci curvature bounded below by $-(n-1)$. If $\lambda_1(M) \geq n-2$, then either*

(1) *M has only one infinite volume end; or*
(2) *$M = \mathbb{R} \times N$ with warped product metric of the form*

$$ds_M^2 = dt^2 + \cosh^2 t \, ds_N^2,$$

where N is an $(n-1)$-dimensional compact manifold of Ricci curvature bounded below by $\lambda_1(M)$.

Theorem 0.2. *Let M^n be a complete Riemannian manifold of dimension $n \geq 2$ with Ricci curvature bounded below by $-(n-1)$. If $\lambda_1(M) \geq \frac{(n-1)^2}{4}$, then either*

(1) *M has no finite volume end; or*
(2) *$M = \mathbb{R} \times N$ with warped product metric of the form*

$$ds_M^2 = dt^2 + e^{2t} \, ds_N^2,$$

where N is an $(n-1)$-dimensional compact manifold of nonnegative Ricci curvature.

In [**LW3**] and [**LW5**], Li and Wang also consider the Kähler case. They proved the following theorems.

Theorem 0.3. *Let M^n be a complete Kähler manifold of complex dimension $n \geq 1$ with Ricci curvature bounded below by*

$$\mathrm{Ric}_M \geq -2(n+1).$$

If $\lambda_1(M) > \frac{n+1}{2}$, then M must have only one infinite volume end.

Theorem 0.4. *Let M^n be a complete Kähler manifold of complex dimension $n \geq 2$ with holomorphic bisectional curvature bounded by*

$$\mathrm{BK}_M \geq -1.$$

If $\lambda_1(M) \geq n^2$, then either

(1) *M has only one end; or*

(2) *$M = \mathbb{R} \times N$ with N being a compact manifold. Moreover, the metric on M is of the form*

$$ds_M^2 = dt^2 + e^{4t}\,\omega_2^2 + e^{2t}\,\sum_{i=3}^{2n}\omega_i^2,$$

where $\{\omega_2, \omega_3, \ldots, \omega_{2n}\}$ are orthonormal coframe of N with $Jdt = \omega_2$.

If M has bounded curvature, then we further conclude that M is covered by $\mathbb{CH}^n$ and N is a compact quotient of the Heisenberg group.

In [**LW5**], the authors pointed out that the assumption on the lower bound of $\lambda_1(M)$ in Theorem 0.3 is sharp, since one can construct M of the form $M = \Sigma \times N$ satisfying

$$\mathrm{Ric}_M \geq -2(n+1) \tag{0.1}$$

and

$$\lambda_1(M) = \frac{n+1}{2} \tag{0.2}$$

with N being a compact Kähler manifold and Σ being a complete surface with at least two infinite volume ends. However, it is still an open question to characterize all those complete Kähler manifolds satisfying conditions (0.1) and (0.2).

In Sections 4 and 5, we will prove the following quaternionic Kähler versions of the above theorems.

Theorem 0.5. *Let (M^{4n}, g) be a complete quaternionic Kähler manifold with scalar curvature satisfying*

$$\mathrm{S}_M \geq -16n(n+2).$$

If $\lambda_1(M) \geq \frac{8(n+2)}{3}$, then M must have only one infinite volume end.

Theorem 0.6. *Let (M^{4n}, g) be a complete quaternionic Kähler manifold with scalar curvature satisfying*

$$\mathrm{S}_M \geq -16n(n+2).$$

If $\lambda_1(M) \geq (2n+1)^2$, then either

(1) *M has only one end, or*

(2) *M is diffeomorphic to $\mathbb{R} \times N$, where N is a compact manifold. Moreover, the metric is given by the form*

$$ds_M^2 = dt^2 + e^{4t}\,\sum_{p=2}^{4}\omega_p^2 + e^{2t}\,\sum_{\alpha=5}^{4n}\omega_\alpha^2,$$

where $\{\omega_2, \ldots, \omega_{4n}\}$ are orthonormal coframes for N.

If M is of bounded curvature then we further conclude that M is covered by the quaterionic hyperbolic space $\mathbb{QH}^n$ and N is a compact quotient of the generalized Heisenberg group.

Remark 0.1. It is known that a horosphere in $\mathbb{QH}^n$ is isometric to a certain generalized Heisenberg group with three-dimensional center and left-invariant Riemannian metric. Such generalized Heisenberg groups have compact quotients. For an explicit construction, see for instance Example 2.6 in [**G**]. We don't have an example to show that the bounded curvature condition in Theorem 0.6 is necessary. If such an example exists, its curvature should decay exponentially in some directions.

Perhaps it is interesting to restrict our attention to the special case when $M^{4n} = \mathbb{QH}^n/\Gamma$ is given by the quotient of the quaternionic hyperbolic space $\mathbb{QH}^n$ with a discrete group of isometies Γ. In particular, it is instructional to compare with previous results by Corlette [**C2**] and Corlette-Iozzi [**CI**] where the Lie group theoretic approach was used in understanding these manifolds. For example, in [**CI**], the authors proved a Patterson-Sullivan type formula for $\lambda_1(M)$ in terms of the Hausdorff dimension $\delta(\Gamma)$ of the limit set of Γ. More specifically, they proved that if Γ is geometrically finite, then for $\delta(\Gamma) \geq 2n+1$ one has

$$\lambda_1(M) = \delta(\Gamma)((4n+2) - \delta(\Gamma)).$$

Hence in this case, the condition in Theorem 0.6 on $\lambda_1(M) = (2n+1)^2$ is equivalent to the condition $\delta(\Gamma) = 2n+1$.

In [**C2**] (Theorem 4.4), Corlette also pointed out that by a result of Kostant $\lambda_1(M) = 0$ or $\lambda_1(M) \geq 8n$. On the other hand, it was also shown in [**CI**] that if Γ is geometrically finite and torsion free, then $M = \mathbb{QH}^n/\Gamma$ must have at most one end with infinite volume. These two statements give an interesting comparison to Theorem 0.5 stated above.

We would also like to point out to the interested readers that in [**LW4**] and [**LW5**] Li and Wang considered a more general class of manifolds satisfying a weighted Poincaré inequality. However, since quaternionic Kähler manifolds are automatically Einstein, the same type of questions are not interesting for this class of manifolds.

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1. Preliminaries on quaternionic Kähler manifolds

In this section, we will recall basic properties of quaternionic Kähler manifolds that will be needed in the sequel. These properties were proved by Berger [**B**] and Ishihara [**I**] (also see [**Be**]).

Let (M^n, g) be a Riemannian manifold, TM the tangent space of M and ∇ the Levi-Civita connection. The Riemannian curvature $R : TM \otimes TM \otimes TM \longrightarrow TM$ is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

If $\{e_1, \ldots, e_n\}$ is an orthonormal basis of TM, the components of curvature tensor is defined by

$$R_{ijkl} = \langle R(e_i, e_j)e_l, e_k \rangle,$$

the Ricci curvature is defined by

$$\mathrm{Ric}_M(X,Y) = \sum_{i=1}^{n} \langle R(X, e_i)e_i, Y \rangle,$$

and the scalar curvature is defined by

$$\mathrm{S}_M = \sum_{i,j=1}^{n} \langle R(e_i, e_j)e_j, e_i \rangle.$$

Definition 1.1. A quaternionic Kähler manifold (M, g) is a Riemannian manifold with a rank 3 vector bundle $V \subset End(TM)$ satisfying

(a) In any coordinate neighborhood U of M, there exists a local basis $\{I, J, K\}$ of V such that

$$\begin{aligned} &I^2 = J^2 = K^2 = -1 \\ &IJ = -JI = K \\ &JK = -KJ = I \\ &KI = -IK = J \end{aligned}$$

and

$$\langle IX, IY \rangle = \langle JX, JY \rangle = \langle KX, KY \rangle = \langle X, Y \rangle$$

for all $X, Y \in TM$.

(b) If $\phi \in \Gamma(V)$, then $\nabla_X \phi \in \Gamma(V)$ for all $X \in TM$.

Remark 1.1. It follows from (a) that $\dim M = 4n$. A well known fact about $4n$-dimensional Riemannian manifold is that it is quaternionic Kähler if and only if its restricted holonomy group is contained in $Sp(n)Sp(1)$.

The 4-dimensional Riemannian manifolds with holonomy $Sp(1)Sp(1)$ are simply the oriented Riemanian manifolds; naturally we only consider those when $n \geq 2$.

Notice that, in general, I, J, K are not defined everywhere on M. For example, the canonical quaternionic projective space QP^n admits no almost complex structure.

On the other hand, the vector space generated by I, J, K is well defined at each point of M and this 3-dimensional subbundle V of $\mathrm{End}(TM)$ is in fact "globally parallel" under the Levi-Civita connection ∇ of g. A basic fact about the connection is the following lemma.

Lemma 1.1. *The condition* (b) *is equivalent to the following condition:*

$$\begin{aligned}\nabla_X I &= c(X)J - b(X)K,\\ \nabla_X J &= -c(X)I + a(X)K,\\ \nabla_X K &= b(X)I - a(X)J,\end{aligned}$$

where a, b, c are local 1*-forms.*

Definition 1.2. Let (M, g) be a quaternionic Kähler manifold. We can define a 4-form by

$$\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3,$$

where

$$\begin{aligned}\omega_1 &= \langle \cdot, I\cdot\rangle,\\ \omega_2 &= \langle \cdot, J\cdot\rangle,\\ \omega_3 &= \langle \cdot, K\cdot\rangle.\end{aligned}$$

Let $\{e_1, Ie_1, Je_1, Ke_1, \dots, e_n, Ie_n, Je_n, Ke_n\}$ be an orthonormal basis of TM and $\{\theta^1, I\theta^1, J\theta^1, K\theta^1, \dots, \theta^n, I\theta^n, J\theta^n, K\theta^n\}$ the dual basis. It follows that

$$\begin{aligned}\omega_1 &= \sum_{i=1}^n \left(\theta^i \wedge I\theta^i + J\theta^i \wedge K\theta^i\right),\\ \omega_2 &= \sum_{i=1}^n \left(\theta^i \wedge J\theta^i + K\theta^i \wedge I\theta^i\right),\\ \omega_3 &= \sum_{i=1}^n \left(\theta^i \wedge K\theta^i + I\theta^i \wedge J\theta^i\right),\end{aligned}$$

and

$$\begin{aligned}\Omega =& \sum_{i,j} \left(\theta^i \wedge I\theta^i \wedge \theta^j \wedge I\theta^j + \theta^i \wedge J\theta^i \wedge \theta^j \wedge J\theta^j + \theta^i \wedge K\theta^i \wedge \theta^j \wedge K\theta^j\right)\\ &+ \sum_{i,j} \big(J\theta^i \wedge K\theta^i \wedge J\theta^j \wedge K\theta^j + K\theta^i \wedge I\theta^i \wedge K\theta^j \wedge I\theta^j\\ &+ I\theta^i \wedge J\theta^i \wedge I\theta^j \wedge J\theta^j\big)\\ &+ 2\sum_{i,j} \big(\theta^i \wedge I\theta^i \wedge J\theta^j \wedge K\theta^j + \theta^i \wedge J\theta^i \wedge K\theta^j \wedge I\theta^j\\ &+ \theta^i \wedge K\theta^i \wedge I\theta^j \wedge J\theta^j\big).\end{aligned}$$

Lemma 1.2. *The condition* (b) *is equivalent to the following condition:*

$$\begin{aligned}\nabla_X\omega_1 &= c(X)\omega_2 - b(X)\omega_3,\\ \nabla_X\omega_2 &= -c(X)\omega_1 + a(X)\omega_3,\\ \nabla_X\omega_3 &= b(X)\omega_1 - a(X)\omega_2,\end{aligned}$$

where a, b, c are local 1*-forms.*

Proof. It follows from the identities

$$\begin{aligned}(\nabla_X\omega_1)(Y,Z) &= \langle Y, (\nabla_X I)Z\rangle,\\ (\nabla_X\omega_2)(Y,Z) &= \langle Y, (\nabla_X J)Z\rangle,\\ (\nabla_X\omega_3)(Y,Z) &= \langle Y, (\nabla_X K)Z\rangle.\end{aligned}$$

q.e.d.

Using this lemma, we have that

Theorem 1.1. *The condition* (b) *is equivalent to that Ω is parallel, that is*

$$\nabla_X\Omega = 0$$

for any $X \in TM$.

In the following, we shall study the curvature of quaternionic Kähler manifold. First we have the following lemma.

Lemma 1.3. *If (M^{4n}, g) is a quaternionic Kähler manifold, then*

$$\begin{aligned}[R(X,Y), I] &= \gamma(X,Y)J - \beta(X,Y)K,\\ [R(X,Y), J] &= -\gamma(X,Y)I + \alpha(X,Y)K,\\ [R(X,Y), K] &= \beta(X,Y)I - \alpha(X,Y)J,\end{aligned}$$

where α, β and γ are local 2*-forms given by*

$$\begin{aligned}\alpha &= da + b\wedge c,\\ \beta &= db + c\wedge a,\\ \gamma &= dc + a\wedge b.\end{aligned}$$

Corollary 1.1. *If (M^{4n}, g) is a quarternionic Kähler manifold, then*

$$\begin{aligned}\langle R(X,Y)Z, IZ\rangle + \langle R(X,Y)JZ, KZ\rangle &= \alpha(X,Y)\,|Z|^2,\\ \langle R(X,Y)Z, JZ\rangle + \langle R(X,Y)KZ, IZ\rangle &= \beta(X,Y)\,|Z|^2,\\ \langle R(X,Y)Z, KZ\rangle + \langle R(X,Y)IZ, JZ\rangle &= \gamma(X,Y)\,|Z|^2.\end{aligned}$$

The following lemma is the key for quaternionic Kähler manifolds.

Lemma 1.4. *If (M^{4n}, g) is a quaternionic Kähler manifold and $n \geq 2$, then*

$$\alpha(X, IY) = \beta(X, JY) = \gamma(X, KY) = -\frac{1}{n+2}\mathrm{Ric}_M(X,Y). \tag{1.1}$$

As applications of the above lemma, one can show the following two main theorems on curvature of quaternionic Kähler manifolds.

Theorem 1.2. *If (M^{4n}, g) is a quaternionic Kähler manifold and $n \geq 2$, then (M^{4n}, g) is Einstein, that is, there is a constant δ such that*

$$\mathrm{Ric}_M(g) = 4(n+2)\delta g.$$

Theorem 1.3. *If (M^{4n}, g) is a quaternionic Kähler manifold and $n \geq 2$, then*

(1) *For any tangent vector X, the sectional curvature satisfies*

$$\begin{aligned}\langle R(X, IX)IX, X\rangle &+ \langle R(X, JX)JX, X\rangle \\ &+ \langle R(X, KX)KX, X\rangle = 12\delta\, |X|^4.\end{aligned}$$

(2) *For any tangent vector Y satisfying*

$$\langle Y, X\rangle = \langle Y, IX\rangle = \langle Y, JX\rangle = \langle Y, KX\rangle = 0,$$

the sectional curvature satisfies

$$\begin{aligned}&\langle R(X, Y)Y, X\rangle + \langle R(X, IY)IY, X\rangle \\ &\quad + \langle R(X, JY)JY, X\rangle + \langle R(X, KY)KY, X\rangle = 4\delta\, |X|^2\, |Y|^2,\end{aligned}$$

where $4(n+2)\delta$ is the Einstein constant.

Finally, we end this section with the following lemma.

Lemma 1.5. *Let $\gamma : [a, b] \to M$ be a geodesic with unit speed. If $S = 16n(n+2)\delta$, and $X_I(t), X_J(t), X_K(t)$ are parallel vector fields along γ such that $X_I(a) = I\gamma'(a), X_J(a) = J\gamma'(a), X_K(a) = K\gamma'(a)$, then*

$$\mathcal{K}(\gamma'(t), X_I(t)) + \mathcal{K}(\gamma'(t), X_J(t)) + \mathcal{K}(\gamma'(t), X_K(t)) = 12\delta,$$

for all t and γ.

Let Y be a tangent vector at $\gamma(a)$ satisfying $\langle\gamma'(a), Y\rangle = 0$, $\langle I\gamma'(a), Y\rangle = 0$, $\langle J\gamma'(a), Y\rangle = 0$, and $\langle K\gamma'(a), Y\rangle = 0$. If we denote the parallel vector fields $Y(t)$, $Y_I(t)$, $Y_J(t)$, and $Y_K(t)$ along γ with initial data $Y(a) = Y$, $Y_I(a) = IY$, $Y_J(a) = JY$, and $Y_K(a) = KY$, respectively, then

$$\mathcal{K}(\gamma'(t), Y(t)) + \mathcal{K}(\gamma'(t), Y_I(t)) + \mathcal{K}(\gamma'(t), Y_J(t)) + \mathcal{K}(\gamma'(t), Y_K(t)) = 4\delta,$$

for all t and γ.

Proof. By the discussion above, we know the 3-dimensional vector space $E(t)$ spanned by $X(t), Y(t), Z(t)$ does not depend on the choice of I, J, K. Hence it is parallel under the Levi-Civita connection. We consider $\langle R(\cdot, \gamma'(t))\gamma'(t), \cdot\rangle$ as a symmetric bilinear form on $E(t)$. Then $\mathcal{K}(\gamma'(t), X(t)) + \mathcal{K}(\gamma'(t), Y(t)) + \mathcal{K}(\gamma'(t), Z(t))$ is its trace on $E(t)$ which is independent of the choice of orthonormal basis. By the computation above it is equal to 12δ. The same argument also applies to the second part of the lemma. q.e.d.

2. Laplacian Comparison theorem

For a complete Riemannian manifold M and $p \in M$, let us denote the cut locus with respect to p by $\mathrm{Cut}\,(p)$.

Theorem 2.1. *Let (M^{4n}, g) be a complete quaternionic Kähler manifold with scalar curvature $\mathrm{S}_M \geq 16n(n+2)\delta$ and let $r(x)$ be the distance function to a fixed point $p \in M$. Then, for $x \notin Cut(p)$,*

$$\Delta r(x) \leq \begin{cases} 6\coth 2r(x) + 4(n-1)\coth r(x) & \text{when } \delta = -1 \\ (4n-3)r^{-1}(x) & \text{when } \delta = 0 \\ 6\cot 2r(x) + 4(n-1)\cot r(x) & \text{when } \delta = 1. \end{cases} \tag{2.1}$$

Proof. Let γ be the minimizing geodesic joining p to x. At x, we choose $\{e_1, e_2, \dots, e_n\}$, and two local almost complex structures I, J and $K = IJ$ such that $e_1 = \nabla r$ and

$$\{e_1, Ie_1, Je_1, Ke_1, e_2, Ie_2, Je_2, Ke_2, \dots, e_n, Ie_n, Je_n, Ke_n\}$$

is an orthonormal frame. By parallel translating along γ we obtain an orthonormal frame with $e_1 = \nabla r$. For the sake of convenience, we denote this frame by $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{4n}\}$. Since $|\nabla r|^2 = 1$ on $M \backslash Cut(p)$, by taking covariant derivative of this equation, we have

$$\begin{aligned} 0 &= |\nabla r|^2_{kl} \\ &= 2\sum_{i=1}^{4n} r_{ik} r_{il} + 2\sum_{i=1}^{4n} r_i r_{ikl}, \end{aligned} \tag{2.2}$$

for each $k, l = 2, \dots, 4n$. Since

$$r_{ikl} = r_{kli} + \sum_{j=1}^{n} R_{jkil} r_j,$$

with $R_{ijkl} = \langle R(\varepsilon_i, \varepsilon_j)\varepsilon_l, \varepsilon_k\rangle$, and $r_1 = 1$, $r_j = 0$, $j = 2, \dots, 4n$, we have

$$\sum_{i=1}^{4n} r_{ik} r_{il} + r_{kl1} + R_{1k1l} = 0. \tag{2.3}$$

In particular, if $k = l$, we have

$$\sum_{i=1}^{4n} r_{ik}^2 + r_{kk1} + \mathcal{K}(\varepsilon_1, \varepsilon_k) = 0, \tag{2.4}$$

where $\mathcal{K}(\varepsilon_1, \varepsilon_k) = R_{1k1k}$ is the sectional curvature of the 2-plane section spanned by $\varepsilon_1, \varepsilon_k$. Using the inequality

$$\sum_{k=2}^{4} r_{ik}^2 \geq \frac{1}{3}\left(\sum_{k=2}^{4} r_{kk}\right)^2,$$

and setting $f(t) = \sum_{k=2}^{4} r_{kk}$, (2.4) implies that

$$f'(t) + \frac{1}{3} f^2(t) + \sum_{k=2}^{4} \mathcal{K}(\varepsilon_1, \varepsilon_k) \leq 0. \tag{2.5}$$

By Lemma 1.5, we have

$$f'(t) + \frac{1}{3} f^2(t) + 12\delta \leq 0. \tag{2.6}$$

Since a smooth Riemannian metric is locally Euclidean, then $\lim_{t \to 0} tf(t) = 3$. By a standard comparison argument for ordinary differential equations, we conclude that

$$f(t) \leq \begin{cases} 6 \cot 2t & \text{when } \delta = 1 \\ 3t^{-1} & \text{when } \delta = 0 \\ 6 \coth 2t & \text{when } \delta = -1. \end{cases} \tag{2.7}$$

Similarly, using the inequality

$$\sum_{k=4i+1}^{4i+4} r_{ik}^2 \geq \frac{1}{4} \left(\sum_{k=4i+1}^{4i+4} r_{kk} \right)^2$$

for $1 \leq i \leq n-1$, and setting $h_i(t) = \sum_{k=4i+1}^{4i+4} r_{kk}$, (2.4) implies that

$$h_i'(t) + \frac{1}{4} h_i^2(t) + \sum_{k=4i+1}^{4i+4} \mathcal{K}(\varepsilon_1, \varepsilon_k) \leq 0. \tag{2.8}$$

Together with Lemma 1.5 asserting that

$$\sum_{k=4i+1}^{4i+4} \mathcal{K}(\varepsilon_1, \varepsilon_k) = 4\delta,$$

we have

$$h_i'(t) + \frac{1}{4} h_i^2(t) + 4\delta \leq 0. \tag{2.9}$$

Hence, as before, we conclude that

$$h_i(t) \leq \begin{cases} 4 \cot t & \text{when } \delta = 1 \\ 4t^{-1} & \text{when } \delta = 0 \\ 4 \coth t & \text{when } \delta = -1. \end{cases} \tag{2.10}$$

The result follows from the equation $\Delta r(x) = f(r(x)) + \sum_{i=1}^{n-1} h_i(r(x))$.

q.e.d.

Remark 2.1. The estimate in Theorem 2.1 is sharp since the right hand sides are exactly the Laplacian of the distance functions of quaternionic hyperbolic space $\mathbb{QH}^n$, quaternionic Euclidean space $\mathbb{Q}^n$, and quaternionic projective space $\mathbb{QP}^n$ respectively.

Remark 2.2. We actually proved the estimate for the Hessian of the distance function. In particular,

$$\sum_{k=2}^{4} r_{kk} \leq \begin{cases} 6\cot 2t & \text{when } \delta = 1 \\ 3t^{-1} & \text{when } \delta = 0 \\ 6\coth 2t & \text{when } \delta = -1. \end{cases} \tag{2.11}$$

Also for $1 \leq i \leq n-1$, we have

$$\sum_{k=4i+1}^{4i+4} r_{kk} \leq \begin{cases} 4\cot 2t & \text{when } \delta = 1 \\ 4t^{-1} & \text{when } \delta = 0 \\ 4\coth 2t & \text{when } \delta = -1. \end{cases} \tag{2.12}$$

Corollary 2.1. *Let (M^{4n}, g) be a complete quaternionic Kähler manifold with scalar curvature $\mathrm{S}_M \geq -16n(n+2)$. Then for any point $x \in M$ and $r > 0$, the area $A(r)$ of the geodesic spheres centered at x satisfies*

$$\frac{A'(r)}{A(r)} \leq 6\coth 2r + 4(n-1)\coth r. \tag{2.13}$$

In particular, $A(r) \leq C(\sinh 2r)^3(\sinh r)^{4(n-1)} \leq Ce^{(4n+2)r}$.

Corollary 2.2. *Let (M^{4n}, g) be a complete quaternionic Kähler manifold with scalar curvature $\mathrm{S}_M \geq -16n(n+2)$. Then for any point $x \in M$ and $0 < r_1 \leq r_2$, the volume of the geodesic balls centered at x satisfies*

$$\frac{V_x(r_2)}{V_x(r_1)} \leq \frac{V_{\mathbb{QH}^n}(r_2)}{V_{\mathbb{QH}^n}(r_1)}, \tag{2.14}$$

where $V_{\mathbb{QH}^n}(r)$ denotes the volume of the geodesic ball of radius r in $\mathbb{QH}^n$. In particular, $\lambda_1(M) \leq (2n+1)^2$.

Corollary 2.3. *Let (M^{4n}, g) be a complete quaternionic Kähler manifold with scalar curvature $\mathrm{S}_M \geq 16n(n+2)$. Then it is compact, and the diameter $d(M) \leq \frac{\pi}{2}$, which is the diameter of the model space $\mathbb{QP}^n$. Moreover, the volume of M is bounded by*

$$V(M) \leq V(\mathbb{QP}^n), \tag{2.15}$$

where $V_{\mathbb{QP}^n}$ is the volume of $\mathbb{QP}^n$.

3. Quaternionic harmonicity

In this section we will derive an over-determined system of harmonic functions with finite Dirichlet integral on a manifold with a parallel form. This result was first proved by Siu [**S**] for harmonic maps in his proof of the rigidity theorem for Kähler manifolds. Corlette [**C1**] gave a more systematic approach for harmonic map with finite energy from a finite-volume quaternionic hyperbolic space or Cayley hyperbolic plane to a manifold with nonpositive curvature. In [**L**], the second author generalized Siu's argument to harmonic functions with finite Dirichlet integral on a Kähler manifold. We will provide an argument that generalizes Corlette's argument to harmonic functions with finite Dirichlet integral on a complete manifold with a parallel form. We believe that it should be of independent interest.

Theorem 3.1. *Let M be a complete Riemannian manifold with a parallel p-form Ω. Assume that f is a harmonic function with its Dirichlet integral over geodesic balls centered at o of radius R satisfying the growth condition*

$$\int_{B_o(R)} |\nabla f|^2 dv = o(R^2)$$

as $R \to \infty$; then f satisfies

$$d * (df \wedge \Omega) = 0. \tag{3.1}$$

Before we prove the theorem, let us first recall the following operators and some of the basic properties. For an oriented real vector space V with an inner product, we have the Hodge star operator

$$* : \wedge^p V \to \wedge^{n-p} V.$$

For any $\theta \in \wedge^1 V$ and $v \in V$, we also have exterior multiplication and interior product operators

$$\varepsilon(\theta) : \wedge^p V \to \wedge^{p+1} V,$$

$$\ell(v) : \wedge^p V \to \wedge^{p-1} V.$$

For $\theta \in \wedge^1 V$ and $v \in V$ is the dual of θ by the inner product, if $\xi \in \wedge^p V$ we list the following identities among the operators:

1) $* * \xi = (-1)^{p(n-p)} \xi$,
2) $*\varepsilon(\theta)\xi = (-1)^p \ell(v) * \xi$,
3) $\varepsilon(\theta) * \xi = (-1)^{p-1} * \ell(v)\xi$,
4) $*\varepsilon(\theta) * \xi = (-1)^{(p-1)(n-p)} \ell(v)\xi$,
5) $\ell(v)\varepsilon(\theta')\xi + \varepsilon(\theta)\ell(v')\xi = 0$, where $v \perp v'$,
6) $\ell(v)\varepsilon(\theta)\xi + \varepsilon(\theta)\ell(v)\xi = \xi$.

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let $\eta : [0, +\infty) \to \mathbb{R}$ be a smooth function satisfying $\eta'(t) \leq 0$, and

$$\eta(t) = \begin{cases} 1 & \text{when } t \in [0, 1] \\ 0 & \text{when } t \in [2, +\infty]. \end{cases}$$

For $R \geq 1$, we define the cut-off function $\phi_R(x) = \eta(r(x)/R)$, where $r(x)$ is the distance function from a fixed point $o \in M$; then there is a positive constant C_1 depending on η and C such that

$$|\nabla \phi_R(x)| \leq C_1 \, R^{-1}.$$

Since $d^2 = 0$, then

$$\begin{aligned} 0 &= \int_M d\left\{\phi_R^2 * (df \wedge \Omega) \wedge d * (df \wedge *\Omega)\right\} \\ &= \int_M d(\phi_R^2) \wedge *(df \wedge \Omega) \wedge d * (df \wedge *\Omega) \\ &\quad + \int_M \phi_R^2 \, d * (df \wedge \Omega) \wedge d * (df \wedge *\Omega). \end{aligned} \tag{3.2}$$

We claim that

$$*d * (df \wedge \Omega) = (-1)^{n-1} d * (df \wedge *\Omega). \tag{3.3}$$

In fact, for any point $x \in M$, we can choose an orthonormal tangent basis $\{e_i\}_{i=1}^m$ in a neighborhood of x such that $\nabla_{e_i} e_j(x) = 0$. Denote by $\{\theta^i\}_{i=1}^m$ the dual basis of $\{e_i\}_{i=1}^m$. Then for $\omega \in \wedge^p(T^*M)$ we have

$$d\omega = \varepsilon(\theta^i)\nabla_{e_i}\omega.$$

Hence,

$$\begin{aligned} d * (df \wedge *\Omega) &= d * \varepsilon(df) * \Omega \\ &= (-1)^{(p-1)(m-p)} d[\ell(\nabla f)\Omega] \\ &= (-1)^{(p-1)(m-p)} \sum_{i=1}^m \varepsilon(\theta_i)\nabla_{e_i}(\ell(\nabla f)\Omega) \\ &\quad (-1)^{(p-1)(m-p)} \sum_{i,j=1}^m \varepsilon(\theta_i)(\nabla_{e_i}\nabla_{e_j} f)(\ell(e_j)\Omega) \\ &\quad (-1)^{(p-1)(m-p)} \sum_{i,j=1}^m f_{ij}\varepsilon(\theta_i)(\ell(e_j)\Omega), \end{aligned}$$

where $f_{ij} = \nabla_{e_i}\nabla_{e_j} f$ and the facts Ω is parallel and $\nabla_{e_i} e_j(x) = 0$ have been used. On the other hand,

$$
\begin{aligned}
(3.4)\quad & * d * (df \wedge \Omega) \\
& = *d * \varepsilon(df)\Omega \\
& = * \sum_{i=1}^{m} \varepsilon(\theta_i)\nabla_{e_i}(*\varepsilon(\sum_{j=1}^{m} f_j\theta_j) d[\ell(\nabla f)\Omega] \\
& \quad * \sum_{i,j=1}^{m} f_{ij}\varepsilon(\theta_i) * \varepsilon(\theta_j)(\Omega) \\
& \quad (-1)^{p(m-p-1)} \sum_{i,j=1}^{m} f_{ij}\ell(e_i)\varepsilon(\theta_j)\Omega \\
& \quad (-1)^{p(m-p-1)} \left(\sum_{i=1}^{m} f_{ii}\ell(e_i)\varepsilon(\theta_i)\Omega + \sum_{i\neq j}^{m} f_{ij}\ell(e_i)\varepsilon(\theta_j)\Omega \right) \\
& \quad (-1)^{p(m-p-1)} \left(\sum_{i=1}^{m} f_{ii}[\Omega - \varepsilon(\theta_i)\ell(e_i)\Omega] - \sum_{i\neq j}^{m} f_{ij}\varepsilon(\theta_j)\ell(e_i)\Omega \right) \\
& \quad (-1)^{p(m-p-1)} \sum_{i,j=1}^{m} f_{ij}\varepsilon(\theta_i)(\ell(e_j)\Omega),
\end{aligned}
$$

where we used $f_{ij} = f_{ji}$ and $\sum_{i=1}^m f_{ii} = 0$. So the claim is proved. By (3.2), we have

$$
\begin{aligned}
(3.5)\quad & \int_M \phi_R^2 \, |d * (df \wedge \Omega)|^2 dv \\
& = (-1)^m \int_M d(\phi_R^2) \wedge *(df \wedge \Omega) \wedge d * (df \wedge *\Omega) \\
& \leq 2 \left(\int_M \phi_R^2 \, |d * (df \wedge \Omega)|^2 dv \right)^{\frac{1}{2}} \left(\int_M |d\phi_R|^2 | * (df \wedge \Omega)|^2 dv \right)^{\frac{1}{2}}.
\end{aligned}
$$

On the other hand, (3.3) and the fact that ω is bounded implies that there exists a constant $C_2 > 0$, such that

$$
\begin{gathered}
| * (df \wedge \Omega)| \leq C_2 \, |df| \\
|d * (df \wedge *\Omega)| = |d * (df \wedge \Omega)|.
\end{gathered}
$$

Hence, combining with (3.5) and using the definition of ϕ_R, we conclude that

$$
\int_{B_o(R)} |d * (df \wedge \Omega)|^2 dv \leq C_1 R^{-2} \int_{B_o(2R)} |df|^2 dv.
$$

The assumption on the growth of the Dirichlet integral of f implies that the right hand side tends to zero as $R \to \infty$. Therefore $d * (df \wedge \Omega) = 0$, and the proof is complete. q.e.d.

Lemma 3.1. *Let (M^{4n}, g) be a quarternionic Kähler manifold and $n \geq 2$. If f is a function on M satisfying*

$$d * (df \wedge \Omega) = 0 \tag{3.6}$$

for the 4-form Ω determined by the quaternionic Kähler structure, then f is quaternionic harmonic; namely, for any nonzero tangent vector X,

$$f_{X,X} + f_{IX,IX} + f_{JX,JX} + f_{KX,KX} = 0,$$

where $f_{X,X} = \nabla df(X, X)$.

Proof. Let

$$\begin{aligned}\{e_A\}_{A=1}^{4n} = \{&e_1, e_2, \ldots, e_n, Ie_1, Ie_2, \ldots, Ie_n,\\ &Je_1, Je_2, \ldots, Je_n, Ke_1, Ke_2, \ldots, Ke_n\}\end{aligned}$$

be an orthonormal basis of TM and $\{\omega_A\}$ the dual basis with $e_1 = \frac{X}{\|X\|}$. Since Ω is parallel, by (3.4) and (3.6), we have

$$\begin{aligned}0 &= \sum_{A=1}^{4n} (\nabla_{e_A} df) \wedge \ell(e_A)\Omega\\ &= \sum_{A,B=1}^{4n} f_{e_A,e_B}\, \omega_B \wedge \ell(e_A)\Omega,\end{aligned}$$

where we have used the fact that f is a harmonic function. Hence, equation (3.6) implies

$$\sum_{A,B=1}^{4n} f_{e_A,e_B}\, \omega_B \wedge \ell(e_A)\Omega = 0.$$

Comparing the coefficient of $\omega_i \wedge I\omega_i \wedge J\omega_i \wedge K\omega_i$ on both sides by the explicit formula for Ω given before, we obtain that

$$6\,(f_{e_i,e_i} + f_{Ie_i,Ie_i} + f_{Je_i,Je_i} + f_{Ke_i,Ke_i}) = 0$$

for all e_i, $(1 \leq i \leq n)$. So the proof is complete. q.e.d.

The following corollary is an immediate consequence of the lemma.

Corollary 3.1. *Let M^{4n} be a complete quaternionic Kähler manifold. Assume that f is a harmonic function with its Dirichlet integral satisfying the growth condition*

$$\int_{B_o(R)} |\nabla f|^2 dv = o(R^2)$$

as $R \to \infty$; then f must satisfy

$$d * (df \wedge \Omega) = 0, \tag{3.7}$$

where Ω is the parallel 4-form determined by the quaternionic Kähler structure. Moreover, f is quaternionic harmonic.

4. Uniqueness of infinite volume end

Recall that for any complete manifold, if $\lambda_1(M) > 0$ then M must be nonparabolic. In particular, M must have at least one nonparabolic ends. It was also proved in [**LW1**] that under the assumption that $\lambda_1(M) > 0$, an end is nonparabolic if and only if it has infinite volume.

Let us assume that M has at least two nonparabolic ends, E_1 and E_2. A construction of Li-Tam [**LT**] asserts that one can construct a nonconstant bounded harmonic function with finite Dirichlet integral. The harmonic function f can be obtained by taking a convergent subsequence of the harmonic functions f_R, as $R \to +\infty$, satisfying

$$\Delta f_R = 0 \qquad \text{on } B(R),$$

with boundary conditions

$$f_R = 1 \quad \text{on } \partial B(R) \cap E_1$$

and

$$f_R = 0 \quad \text{on } \partial B(R) \setminus E_1.$$

It follows from the maximum principle that $0 \leq f_R \leq 1$, hence $0 \leq f \leq 1$. We need the following estimates from [**LW1**](Lemma 1.1 and 1.2 in [**LW1**]), and [**LW3**](Lemma 5.1 in [**LW3**]).

Lemma 4.1. *Let M be a complete Riemannian manifold with $\lambda_1(M) > 0$. Suppose M has at least two nonparabolic ends and E be an end of M. Then for the harmonic function f constructed above, it must satisfy the following growth conditions:*

1) *There exists a constant a such that $f - a \in L^2(E)$. Moreover, the function $f - a$ must satisfy the decay estimate*

$$\int_{E(R+1)\setminus E(R)} (f-a)^2 \leq C \exp(-2\sqrt{\lambda_1(E)}R)$$

for some constant $C > 0$ depending on f, $\lambda_1(E)$ and the dimension of M.

2) *The Dirichlet integral of the function f must satisfy the decay estimate*

$$\int_{E(R+1)\setminus E(R)} |\nabla f|^2 \leq C \exp(-2\sqrt{\lambda_1(E)}R),$$

and

$$\int_{E(R)} \exp(-2\sqrt{\lambda_1(E)}r(x))|\nabla f|^2 \leq CR$$

for R sufficiently large.

Lemma 4.2. *Let M be a complete Riemannian manifold with at least two nonparabolic ends and $\lambda_1(M) > 0$. Then for the harmonic function f constructed above, for any $t \in (\inf f, \sup f)$ and $(a, b) \subset (\inf f, \sup f)$,*

$$\int_{\mathcal{L}(a,b)} |\nabla f|^2 = (b-a) \int_{l(b)} |\nabla f|,$$

where

$$l(t) = \{x \in M | f(x) = t\},$$

and

$$\mathcal{L}(a, b) = \{x \in M | a < f(x) < b\}.$$

Moreover,

$$\int_{l(t)} |\nabla f| = \int_{l(b)} |\nabla f|.$$

We are now ready to prove Theorem 0.5.

Proof of Theorem 0.5. Suppose to the contrary that there exist two ends E_1 and E_2 with infinite volume. The assumption that $\lambda_1(M) > 0$ implies that they are nonparabolic. By the construction above, there exists a harmonic function f with finite energy such that

$$\liminf_{x\to\infty,\, x\in E_1} f(x) = 1$$

and

$$\liminf_{x\to\infty,\, x\in E_2} f(x) = 0.$$

The Bochner formula implies that

$$\frac{1}{2}\Delta|\nabla f|^2 = \mathrm{Ric}_M(\nabla f, \nabla f) + |\nabla^2 f|^2. \tag{4.1}$$

We now choose an orthonormal basis $\{e_A\}_{A=1}^{4n}$ satisfying

$$\{e_1, e_2, \dots, e_n, Ie_1, Ie_2, \dots, Ie_n, Je_1, Je_2, \dots, Je_n, Ke_1, Ke_2, \dots, Ke_n\}$$

with $e_1 = \frac{\nabla f}{|\nabla f|}$. Corollary 3.1 implies that

$$\sum_{i=0}^{3} f_{(in+1)(in+1)} = 0.$$

Therefore, applying the arithmetic-geometric means, we have

$$
\begin{aligned}
(4.2) \qquad |\nabla^2 f|^2 &= \sum_{A,B=1}^{4n} f_{AB}^2 \\
&\geq f_{11}^2 + \sum_{i=1}^{3} f_{(in+1)(in+1)}^2 + 2\sum_{A=2}^{4n} f_{1A}^2 \\
&\geq f_{11}^2 + \frac{1}{3}\left(\sum_{i=1}^{3} f_{(in+1)(in+1)}\right)^2 + 2\sum_{A=2}^{4n} f_{1A}^2 \\
&\geq \frac{4}{3}|\nabla|\nabla f||^2,
\end{aligned}
$$

hence combining with (4.1) we obtain

$$
(4.3) \qquad \frac{1}{2}\Delta|\nabla f|^2 \geq -4(n+2)|\nabla f|^2 + \frac{4}{3}|\nabla|\nabla f||^2.
$$

If we write $u = |\nabla f|^{\frac{2}{3}}$, then

$$
(4.4) \qquad \Delta u \geq -\frac{8(n+2)}{3}u.
$$

We want to prove that the above inequality is actually an equality. The argument follows from that in [**LW4**] after making suitable modification to fit our situation. For any compactly supported smooth function ϕ on M, we have

$$
\begin{aligned}
(4.5) \qquad 0 &\leq \int_M \phi^2 u\left(\Delta u + \frac{8(n+2)}{3}u\right) \\
&\leq -2\int_M \phi u\langle\nabla u, \nabla\phi\rangle - \int_M \phi^2|\nabla u|^2 + \lambda_1(M)\int_M (\phi u)^2 \\
&\leq -2\int_M \phi u\langle\nabla u, \nabla\phi\rangle - \int_M \phi^2|\nabla u|^2 + \int_M |\nabla(\phi u)|^2 \\
&= \int_M |\nabla\phi|^2 u^2.
\end{aligned}
$$

Let us choose $\phi = \psi\chi$ to be the product of two compactly supported functions. For any $\varepsilon \in (0, \frac{1}{2})$, we define

$$
\chi(x) = \begin{cases} 0 & \text{on } \mathcal{L}(0, \sigma\varepsilon) \cup \mathcal{L}(1-\frac{\varepsilon}{2}, 1) \\ (\log 2)^{-1}(\log f - \log(\frac{\varepsilon}{2})) & \text{on } \mathcal{L}(\frac{\varepsilon}{2}, \varepsilon) \cap (M \setminus E_1) \\ (\log 2)^{-1}(\log(1-f) - \log(\frac{\varepsilon}{2})) & \text{on } \mathcal{L}(1-\varepsilon, 1-\frac{\varepsilon}{2}) \cap E_1 \\ 1 & \text{otherwise.} \end{cases}
$$

For $R > 1$ we define

$$\psi = \begin{cases} 1 & \text{on } B(R-1) \\ R-r & \text{on } B(R)\setminus B(R-1) \\ 0 & \text{on } M\setminus B(R). \end{cases}$$

Applying to the right hand side of (4.5), we obtain

$$\int_M |\nabla\phi|^2 u^2 \le 2\int_M |\nabla\psi|^2\chi^2|\nabla f|^{\frac{4}{3}} + 2\int_M |\nabla\chi|^2\psi^2|\nabla f|^{\frac{4}{3}}. \tag{4.6}$$

Since $\mathrm{Ric}_M \ge -4(n+2)$, the local estimate of Cheng-Yau [**CY**] (see also [**LW2**]) implies that there exists a constant depending on n such that

$$|\nabla f|(x) \le C|1-f(x)|.$$

On E_1, the first term of (4.6) satisfies

$$\int_M |\nabla\psi|^2\chi^2|\nabla f|^{\frac{4}{3}} \le \left(\int_\Omega |\nabla f|^2\right)^{\frac{2}{3}}\left(\int_\Omega 1\right)^{\frac{1}{3}}, \tag{4.7}$$

where $\Omega = E_1 \cap (B(R)\setminus B(R-1)) \cap (\mathcal{L}(1-\varepsilon, 1-\frac{\varepsilon}{2}) \cup \mathcal{L}(\frac{\varepsilon}{2}, \varepsilon)$. Since

$$\begin{aligned}\int_\Omega 1 &\le 4\int_\Omega \frac{(1-f)^2}{\varepsilon^2} \\ &\le \frac{4}{\varepsilon^2}\int_\Omega (1-f)^2 \\ &\le 4C\varepsilon^{-2}\exp(-2\sqrt{\lambda_1}R),\end{aligned}$$

where in the last inequality we have used Lemma 4.1. Again by Lemma 4.1, from (4.7) we have

$$\int_M |\nabla\psi|^2\chi^2|\nabla f|^{\frac{4}{3}} \le C\varepsilon^{-\frac{2}{3}}\exp(-2\sqrt{\lambda_1}R). \tag{4.8}$$

For the second term of (4.6) we have

$$\begin{aligned}&\int_{E_1} |\nabla\chi|^2\psi^2|\nabla f|^{\frac{4}{3}} \\ &\le (\log 2)^{-2}\int_{\mathcal{L}(1-\varepsilon,1-\frac{\varepsilon}{2})\cap E_1\cap B(R)} |\nabla f|^{\frac{4}{3}+2}(1-f)^{-2} \\ &\le C(\log 2)^{-2}\int_{\mathcal{L}(1-\varepsilon,1-\frac{\varepsilon}{2})\cap E_1\cap B(R)} |\nabla f|^2(1-f)^{-\frac{2}{3}}.\end{aligned}$$

Using the co-area formula and Lemma 4.2, we have

$$\begin{aligned}
&\int_{\mathcal{L}(1-\varepsilon,1-\frac{\varepsilon}{2})\cap E_1\cap B(R)} |\nabla f|^2(1-f)^{-\frac{2}{3}} \\
&\leq \int_{1-\varepsilon}^{1-\frac{\varepsilon}{2}} (1-t)^{-\frac{2}{3}} \int_{l(t)\cap E_1\cap B(R)} |\nabla f| dA dt \\
&\leq C \int_{l(b)} |\nabla f| dA \int_{1-\varepsilon}^{1-\frac{\varepsilon}{2}} (1-t)^{-\frac{2}{3}} dt \\
&= -3C[(1-t)^{\frac{1}{3}}]_{1-\varepsilon}^{1-\frac{\varepsilon}{2}} \int_{l(b)} |\nabla f| dA \\
&= 3C\varepsilon^{\frac{1}{3}} \int_{l(b)} |\nabla f| dA.
\end{aligned}$$

Combining the above inequality with (4.8), we have

$$\int_{E_1} |\nabla \phi|^2 u^2 \leq C(\varepsilon^{\frac{2}{3}} \exp(-2\sqrt{\lambda_1}R) + \varepsilon^{\frac{1}{3}}). \tag{4.9}$$

A similar argument using f instead of $1-f$ on the other end yields the estimate

$$\int_{M\setminus E_1} |\nabla \phi|^2 u^2 \leq C(\varepsilon^{\frac{2}{3}} \exp(-2\sqrt{\lambda_1}R) + \varepsilon^{\frac{1}{3}}).$$

Letting $R \to \infty$ and $\varepsilon \to 0$, we have

$$\Delta u = -\frac{8(n+2)}{3} u \tag{4.10}$$

with $\lambda_1(M) = \frac{8(n+2)}{3}$, since f is nonconstant and u cannot be identically zero. Therefore, all the inequalities used to prove (4.4) are equalities. Thus there exists a function μ, such that,

$$(f_{AB}) = \begin{pmatrix} D_1 & & & \\ & D_2 & & \\ & & D_2 & \\ & & & D_2 \end{pmatrix}, \tag{4.11}$$

where D_1 and D_2 are $n \times n$ matrices defined by

$$D_1 = \begin{pmatrix} -3\mu & & & \\ & 0 & & \\ & & \dots & \\ & & & 0 \end{pmatrix}$$

and

$$D_2 = \begin{pmatrix} \mu & & & \\ & 0 & & \\ & & \dots & \\ & & & 0 \end{pmatrix}.$$

Since $f_{1\alpha} = 0$ for $\alpha \neq 1$ implies that $|\nabla f|$ is constant along the level set of f. Moreover, regularity of the equation (4.10) implies that $|\nabla f|$ can never be zero. Hence M must be diffeomorphic to $\mathbb{R} \times N$, where N is given by the level set of f. Also, N must be compact since we assume that M has at least 2 ends.

Fix a level set N_0 of f, consider $(-\varepsilon, \varepsilon) \times N_0 \subset M$. Note that $\{e_A\}$ is an orthonormal basis of TM such that e_1 is the normal vector to N_0 and $\{e_\alpha\}$ are the tangent vectors of N_0. We shall compute the sectional curvature

$$\mathcal{K}(e_1, e_\alpha) = \langle R(e_1, e_\alpha)e_\alpha, e_1\rangle.$$

We claim that

$$\nabla_{e_1} e_1 = 0.$$

Indeed, it suffices to prove all integral curves $\eta(t)$ of the vector field $e_1 = \frac{\nabla f}{|\nabla f|}$ emanating from N_0 are geodesics. For any point $\eta(t_0)$, let γ be the geodesic realizing the distance between $\eta(t_0)$ and N_0. Then γ is perpendicular to every level set N_t. So γ' is parallel to e_1 along γ. This implies γ coincides with the integral curve of e_1.

Let $(h_{\alpha\beta})$ with $2 \leq \alpha, \beta \leq 4n$ be the second fundamental form of the level set of f. Then

$$h_{\alpha\beta}\, f_1 = -f_{\alpha\beta}, \tag{4.12}$$

and

$$\nabla_{e_\alpha} e_1 = -\sum_{\beta=2}^{4n} h_{\alpha\beta} e_\beta.$$

By the definition of curvature tensor, we have

$$\begin{aligned}
&\langle R(e_1, e_\alpha)e_1, e_\alpha\rangle \langle \nabla_{e_1}\nabla_{e_\alpha} e_1 - \nabla_{e_\alpha}\nabla_{e_1} e_1 - \nabla_{[e_1,e_\alpha]} e_1, e_\alpha\rangle \\
&= \langle \nabla_{e_1}\nabla_{e_\alpha} e_1, e_\alpha\rangle - \langle \nabla_{[e_1,e_\alpha]} e_1, e_\alpha\rangle \\
&= \langle \nabla_{e_1}\nabla_{e_\alpha} e_1, e_\alpha\rangle - \langle \nabla_{\nabla_{e_1} e_\alpha - \nabla_{e_\alpha} e_1} e_1, e_\alpha\rangle \\
&= \langle \nabla_{e_1}\nabla_{e_\alpha} e_1, e_\alpha\rangle - \sum_{\beta=2}^{4n} \langle \nabla_{e_1} e_\alpha, e_\beta\rangle \langle \nabla_{e_\beta} e_1, e_\alpha\rangle \\
&\quad + \sum_{\beta=2}^{4n} \langle \nabla_{e_\alpha} e_1, e_\beta\rangle \langle \nabla_{e_\beta} e_1, e_\alpha\rangle \\
&= -\sum_{\beta=2}^{4n} \langle \nabla_{e_1}(h_{\alpha\beta} e_\beta), e_\alpha\rangle + \sum_{\beta=2}^{4n} h_{\alpha\beta} \langle \nabla_{e_1} e_\alpha, e_\beta\rangle + \sum_{\beta=2}^{4n} h_{\alpha\beta}^2 \\
&= -\sum_{\beta=2}^{4n} \langle (e_1 h_{\alpha\beta}) e_\beta, e_\alpha\rangle - \sum_{\beta=2}^{4n} h_{\alpha\beta} \langle \nabla_{e_1} e_\beta, e_\alpha\rangle
\end{aligned}$$

$$+\sum_{\beta=2}^{4n} h_{\alpha\beta}\langle\nabla_{e_1}e_\alpha, e_\beta\rangle + h_{\alpha\beta}^2$$
$$= -e_1 h_{\alpha\alpha} + 2\sum_{\beta=2}^{4n} h_{\alpha\beta}\langle\nabla_{e_1}e_\alpha, e_\beta\rangle + h_{\alpha\beta}^2.$$

Therefore,

$$\mathcal{K}(e_1, e_\alpha) = e_1 h_{\alpha\alpha} - 2\sum_{\beta=2}^{4n} h_{\alpha\beta}\langle\nabla_{e_1}e_\alpha, e_\beta\rangle - \sum_{\beta=2}^{4n} h_{\alpha\beta}^2. \tag{4.13}$$

Since $h_{\alpha\beta}$ is diagonal, this implies that

$$\mathcal{K}(e_1, e_\alpha) = e_1 h_{\alpha\alpha} - h_{\alpha\alpha}^2.$$

Combining with (4.11) and (4.12), we conclude that

$$\mathcal{K}(e_1, e_2) = \mathcal{K}(e_1, Ie_2) = \mathcal{K}(e_1, Je_2) = \mathcal{K}(e_1, Ke_2) = 0,$$

which implies M is Ricci flat by Theorem 1.3. This contradicts to the assumption that $\lambda_1 > \frac{8(n+2)}{3} > 0$. Therefore M must have only one end with infinite volume. q.e.d.

5. Maximal first eigenvalue

In this section, we will consider the case when $\lambda_1(M)$ is of maximal value.

Proof of Theorem 0.6. According to Theorem 0.5, we know that M has exactly one nonparabolic end. Suppose that M has more than one end. Then there must exist at least one end with finite volume. We divide the rest of the proof into several parts. The first part follows exactly as that in the proof of the corresponding theorem in the Kähler case (Theorem 3.1) in [**LW5**]. For the sake of completeness, we will give a quick outline of it.

Part 1. Assume that E_1 is such an end with finite volume given by $M \setminus B_p(1)$. Then we can choose a ray $\eta : [0, +\infty)$ such that $\eta(0) = p$ and $\eta[1, +\infty) \subset E_1$. The Busemann function corresponding to γ is defined by

$$\beta(x) = \lim_{t\to+\infty}[t - d(x, \eta(t))].$$

The Laplacian comparison theorem, Theorem 2.1, asserts that

$$\Delta\beta \geq -2(2n+1)$$

in the sense of distribution. We define the function $f = \exp((2n+1)\beta)$, and using the fact that $|\nabla\beta| = 1$ almost everywhere, we have

$$\Delta f = (2n+1)\exp((2n+1)\beta)\Delta\beta + (2n+1)^2$$
$$\geq -(2n+1)^2 f.$$

Similar to the proof of above theorem, we conclude that for any compactly supported function ϕ,

$$\begin{aligned} 0 &\le \int_M (\Delta f + (2n+1)^2 f) f \phi^2 \\ &\le \int_M f^2 |\nabla \phi|^2. \end{aligned}$$

By choosing the function ϕ to be

$$\phi = \begin{cases} 1, & \text{on } B_p(R); \\ \frac{2R - r(x)}{R}, & \text{on } B_p(2R) \setminus B_p(R); \\ 0, & \text{on } M \setminus B_p(2R); \end{cases}$$

we obtain

$$\begin{aligned} &\int_{M \cap E_1} f^2 |\nabla \phi|^2 \\ &\le \frac{1}{R^2} \int_{(B_p(2R) \setminus B_p(R)) \cap E_1} f^2 \\ &\le \frac{1}{R^2} \sum_{i=1}^{[R]} \int_{(B_p(R+i) \setminus B_p(R+i-1)) \cap E_1} f^2 \\ &\le \frac{C}{R^2} \sum_{i=1}^{[R]} (V_{E_1}(R+i) - V_{E_1}(R+i-1)) \exp(2(2n+1)(R+i)) \end{aligned}$$

where $V_{E_1}(R+i)$ denotes the volume of the set $E_1 \cap B_p(R+i)$. On the other hand, the volume estimate in Theorem 1.4 of [**LW1**] implies that

$$V_{E_1}(\infty) - V_{E_1}(R) \le C \exp(-2(2n+1)R),$$

hence

$$\begin{aligned} &V_{E_1}(R+i) - V_{E_1}(R+i-1) \\ &= V_{E_1}(\infty) - V_{E_1}(R+i-1) - (V_{E_1}(\infty) - V_{E_1}(R+i)) \\ &\le C \exp(-2(2n+1)(R+i)). \end{aligned}$$

Therefore, we conclude that

$$\int_{M \cap E_1} f^2 |\nabla \phi|^2 \le \frac{C}{R}.$$

Let us now denote $E_2 = M \setminus (B_p(1) \cup E_1)$ to be the other end of M. When $x \in E_2$, following the argument in Theorem 3.1 of [**LW4**], we have

$$\beta(x) \le -d(p,x) + 2.$$

Therefore,

$$\begin{aligned}\int_{E_2} f^2|\nabla\phi|^2 &\le \frac{1}{R^2}\int_{(B_p(2R)\setminus B_p(R))\cap E_2} f^2\\ &= \frac{C}{R^2}\int_{(B_p(2R)\setminus B_p(R))\cap E_2} \exp(-2(2n+1)(r-2))\\ &\le \frac{C}{R}.\end{aligned}$$

Letting $R \to +\infty$, we conclude that

$$\Delta f + (2n+1)^2 f = 0, \tag{5.1}$$

and all inequalities used are indeed equalities and f is smooth by regularity of the equation (5.1). Moreover, $|\nabla\beta| = 1$, and

$$\Delta\beta = -2(2n+1).$$

This implies that M must be diffeomorphic to $\mathbb{R}\times N$, where N is given by the level set of β. We choose an orthonormal basis $\{e_i\}_{i=1}^{4n}$ as follows

$$\{e_1, e_2, \dots, e_n, Ie_1, Ie_2, \dots, Ie_n, Je_1, Je_2, \dots, Je_n, Ke_1, Ke_2, \dots, Ke_n\}$$

with $e_1 = \nabla\beta$. Applying the Bochner formula to β, we get

$$\begin{aligned}0 &= \frac{1}{2}\Delta|\nabla\beta|^2\\ &= \sum_{i,j=1}^{4n}\beta_{ij}^2 + \mathrm{Ric}_M(\nabla\beta,\nabla\beta) + \sum_{i=1}^{4n}\beta_i(\Delta\beta)_i\\ &= \sum_{i,j=1}^{4n}\beta_{ij}^2 - 4(n+2).\end{aligned}$$

By the comparison theorem, we have

$$\sum_{i=0}^{3}\beta_{(in+1)(in+1)} = -6.$$

Hence

$$(\beta_{\alpha\beta}) = \begin{pmatrix} D_1 & & & \\ & D_2 & & \\ & & D_2 & \\ & & & D_2 \end{pmatrix},$$

where D_1 and D_2 are $n\times n$ matrices defined by

$$D_1 = \begin{pmatrix} 0 & & & \\ & -1 & & \\ & & \dots & \\ & & & -1 \end{pmatrix}$$

and

$$D_2 = \begin{pmatrix} -2 & & & \\ & -1 & & \\ & & \cdots & \\ & & & -1 \end{pmatrix}.$$

Part 2. For a fixed level set N_0 of β, we consider $(-\varepsilon, \varepsilon) \times N_0 \subset M$. Note that $\{e_i\}$ is an orthonormal basis of TM such that e_1 is the normal vector to N_0 and $\{e_\alpha\}$, for $2 \leq \alpha \leq 4n$, are the tangent vectors of N_0. We shall compute the sectional curvature

$$\mathcal{K}(e_1, e_\alpha) = \langle R(e_1, e_\alpha)e_\alpha, e_1 \rangle.$$

The fact that $\nabla_{e_1} e_1 = 0$ implies that the integral curves of e_1 are geodesics. Let $(h_{\alpha\gamma})$ be the second fundamental form of the level set of $\nabla\beta$. Then

$$\begin{aligned} h_{\alpha\gamma} &= \langle \nabla_{e_\alpha} e_\gamma, e_1 \rangle \\ &= \langle \nabla_{e_\alpha} e_\gamma, \nabla\beta \rangle \\ &= -\beta_{\alpha\gamma} \end{aligned}$$

and

$$\nabla_{e_\alpha} e_1 = -\sum_{\gamma=2}^{4n} h_{\alpha\gamma} e_\gamma. \tag{5.2}$$

By (4.13) in the proof of Theorem 0.5 we have

$$\langle R(e_1, e_\alpha)e_1, e_\alpha \rangle = -e_1 h_{\alpha\alpha} + 2\sum_{\gamma=2}^{4n} h_{\alpha\gamma} \langle \nabla_{e_1} e_\gamma, e_\beta \rangle + \sum_{\gamma=2}^{4n} h_{\alpha\gamma}^2.$$

Since $(h_{\alpha\gamma})$ are constant and diagonal, then

$$\mathcal{K}(e_1, e_\alpha) = -h_{\alpha\alpha}^2.$$

In particular, we have

$$\mathcal{K}(e_1, e_\alpha) = \begin{cases} -4 & \text{when } \alpha = in + 1,\ i = 1, 2, 3 \\ -1 & \text{otherwise.} \end{cases}$$

On the other hand, we also have

$$\begin{aligned} \mathcal{K}(e_{n+1}, e_{2n+1}) + \mathcal{K}(e_{n+1}, e_{3n+1}) &= -12 - \mathcal{K}(e_1, e_{n+1}) = -8 \\ \mathcal{K}(e_{n+1}, e_{2n+1}) + \mathcal{K}(e_{3n+1}, e_{2n+1}) &= -8 \\ \mathcal{K}(e_{3n+1}, e_{2n+1}) + \mathcal{K}(e_{n+1}, e_{3n+1}) &= -8, \end{aligned}$$

hence

$$\mathcal{K}(e_{n+1}, e_{2n+1}) = \mathcal{K}(e_{n+1}, e_{3n+1}) = \mathcal{K}(e_{2n+1}, e_{3n+1}) = -4.$$

Since for $\alpha = 2, 3, \ldots, n$,

$$\begin{aligned}\mathcal{K}(Ie_1, e_\alpha) &= -\langle R(Ie_1, e_\alpha)Ie_1, e_\alpha\rangle \\ &= -\langle IR(Ie_1, e_\alpha)Ie_1, Ie_\alpha\rangle \\ &= \langle R(Ie_1, e_\alpha)e_1, Ie_\alpha\rangle \\ &= \langle R(e_1, Ie_\alpha)Ie_1, e_\alpha\rangle \\ &= \mathcal{K}(e_1, Ie_\alpha) \\ &= -1,\end{aligned}$$

and $\mathcal{K}(Je_1, e_\alpha) = \mathcal{K}(Ke_1, e_\alpha) = -1$, we have

$$\mathcal{K}(e_{in+1}, e_\alpha) = -1,$$

for all $i = 0, 1, 2, 3$ and $\alpha \neq 1, n+1, 2n+1, 3n+1$.

Let $\mathcal{K}^N(e_\alpha, e_\gamma)$ denote the sectional curvature of the level set with induced metric. By Gaussian equation,

$$\mathcal{K}^N(e_\alpha, e_\gamma) - \mathcal{K}(e_\alpha, e_\gamma) = h_{\alpha\alpha}h_{\gamma\gamma},$$

it is straightforward to obtain

$$\mathcal{K}^N(e_{n+1}, e_{2n+1}) = \mathcal{K}^N(e_{n+1}, e_{3n+1}) = \mathcal{K}^N(e_{2n+1}, e_{3n+1}) = 0,$$

and

$$\mathcal{K}^N(e_{in+1}, e_\alpha) = 1, \tag{5.3}$$

for all $i = 1, 2, 3$ and $\alpha \neq 1, n+1, 2n+1, 3n+1$.

Part 3. There is a natural map φ_t between the level sets N_0 and N_t given by the gradient flow of β. Since the integral curves are geodesics, $d\varphi_t(X)$ are Jacobi fields along corresponding curves. Let $(N, g_0) = N_0$ with the induced metric. We can consider φ as a flow on N. We claim that

$$d\varphi_t|_{V_1} = e^{2t}\,\mathrm{id}$$

and

$$d\varphi_t|_{V_2} = e^{t}\,\mathrm{id},$$

where $TN = V_1 \oplus V_2$, $V_1 = \mathrm{span}\{Ie_1, Je_1, Ke_1\}$ and $V_2 = V_1^\perp$. Indeed, for any point $q \in N_0$, denote $e_1(t) = \nabla\beta(\varphi(t))$ and $\{\varepsilon_\alpha(t)\}_{\alpha=2}^{4n}$ to be the parallel transport of the orthonormal base $\{e_\alpha\}_{\alpha=2}^{4n}$ of N_0 at q along $\varphi_t(q)$. Since both V_1 and V_2 are φ-invariant, we have, in particular,

$$\langle \nabla_{e_1(t)}\varepsilon_\alpha, \varepsilon_\gamma\rangle = 0, \tag{5.4}$$

when $\alpha \in \{n+1, 2n+1, 3n+1\}$, and $\gamma \notin \{n+1, 2n+1, 3n+1\}$.

Now we can compute $R_{1\alpha 1\gamma}$. Then

$$
\begin{aligned}
(5.5)\qquad & \langle R(e_1,\varepsilon_\alpha)e_1,\varepsilon_\gamma\rangle \\
&= \langle \nabla_{e_1}\nabla_{\varepsilon_\alpha}e_1 - \nabla_{\varepsilon_\alpha}\nabla_{e_1}e_1 - \nabla_{[e_1,\varepsilon_\alpha]}e_1,\varepsilon_\gamma\rangle \\
&= \langle \nabla_{e_1}\nabla_{\varepsilon_\gamma}e_1,\varepsilon_\alpha\rangle - \langle \nabla_{[e_1,\varepsilon_\alpha]}e_1,\varepsilon_\gamma\rangle \\
&= \langle \nabla_{e_1}\nabla_{\varepsilon_\alpha}e_1,\varepsilon_\gamma\rangle - \langle \nabla_{\nabla_{e_1}\varepsilon_\alpha - \nabla_{\varepsilon_\alpha}e_1}e_1,\varepsilon_\gamma\rangle \\
&= \langle \nabla_{e_1}\nabla_{\varepsilon_\alpha}e_1,\varepsilon_\gamma\rangle - \sum_{\tau=2}^{4n}\langle \nabla_{e_1}\varepsilon_\alpha,\varepsilon_\tau\rangle\langle\nabla_{\varepsilon_\tau}e_1,\varepsilon_\gamma\rangle \\
&\quad + \sum_{\tau=2}^{4n}\langle \nabla_{\varepsilon_\alpha}e_1,\varepsilon_\tau\rangle\langle\nabla_{\varepsilon_\tau}e_1,\varepsilon_\gamma\rangle \\
&= -\sum_{\tau=2}^{4n}\langle \nabla_{e_1}(h_{\alpha\tau}\varepsilon_\tau),\varepsilon_\gamma\rangle + \sum_{\tau=2}^{4n} h_{\gamma\tau}\langle\nabla_{e_1}\varepsilon_\alpha,\varepsilon_\tau\rangle + \sum_{\tau=2}^{4n} h_{\alpha\tau}h_{\tau\gamma} \\
&= -e_1 h_{\alpha\gamma} - \sum_{\tau=2}^{4n} h_{\alpha\tau}\langle\nabla_{e_1}\varepsilon_\tau,\varepsilon_\gamma\rangle \\
&\quad + \sum_{\tau=2}^{4n} h_{\gamma\tau}\langle\nabla_{e_1}\varepsilon_\alpha,\varepsilon_\tau\rangle + \sum_{\tau=2}^{4n} h_{\alpha\tau}h_{\tau\gamma}.
\end{aligned}
$$

We see that $(h_{\alpha\gamma})$ is diagonal and

$$
h_{\alpha\alpha} = \begin{cases} 2, & \text{when } \alpha = n+1, 2n+1, 3n+1; \\ 1, & \text{otherwise.} \end{cases}
$$

Therefore, when $\alpha \neq \gamma$,

$$
\begin{aligned}
R_{1\alpha 1\gamma} &= -h_{\alpha\alpha}\langle\nabla_{e_1}\varepsilon_\alpha,\varepsilon_\gamma\rangle + h_{\gamma\gamma}\langle\nabla_{e_1}\varepsilon_\alpha,\varepsilon_\gamma\rangle \\
&= (h_{\gamma\gamma} - h_{\alpha\alpha})\langle\nabla_{e_1}\varepsilon_\alpha,\varepsilon_\gamma\rangle.
\end{aligned}
$$

Since $h_{\alpha\alpha} = h_{\gamma\gamma}$ when $\alpha,\gamma \in \{n+1, 2n+1, 3n+1\}$ and $\alpha,\gamma \notin \{n+1, 2n+1, 3n+1\}$, using (5.4), we have

$$
R_{1\alpha 1\gamma} = 0, \text{ for all } \alpha \neq \gamma.
$$

Define

$$
J_\alpha(t) = \begin{cases} e^{-2t}\varepsilon_\alpha, & \text{when } \alpha \in \{n+1, 2n+1, 3n+1\}; \\ e^{-t}\varepsilon_\alpha, & \text{when } \alpha \notin \{n+1, 2n+1, 3n+1\}. \end{cases}
$$

Since

$$
\nabla_{\frac{\partial}{\partial t}} d\varphi_t(e_\alpha)|_{t=0} = [e_1, e_\alpha] = -\nabla_{e_\alpha}e_1,
$$

we see that J_α satisfies the Jacobi equation and initial conditions $J_\alpha(0) = e_\alpha$ and $J'_\alpha(0) = e_\alpha = \nabla_{\frac{\partial}{\partial t}} d\varphi_t(e_\alpha)|_{t=0}$. By the uniqueness theorem for the Jacobi equations, we have $d\varphi_t(e_\alpha) = J_\alpha$. The claim is proved.

Part 4. We have now a family of metrics on N written as

$$ds_t^2 = e^{4t}\sum_{i=1}^{3}\omega_{in+1}^2 + e^{2t}\sum_{i=0}^{3}\sum_{\alpha=2}^{n}\omega_{in+\alpha}^2,$$

and the metric of M can be rewritten as

$$ds^2 = dt^2 + e^{4t}\sum_{p=2}^{4}\omega_p^2 + e^{2t}\sum_{\alpha=5}^{4n}\omega_\alpha^2, \tag{5.6}$$

where $\{\omega_2, \omega_3, \omega_4, \ldots, \omega_{4n}\}$ is the dual coframe to $\{e_2, e_3, e_4, \ldots, e_{4n}\}$ at N_0. We also choose that $Ie_{4s-3} = e_{4s-2}$, $Je_{4s-3} = e_{4s-1}$, and $Ke_{4s-3} = e_{4s}$ for $s = 1, \ldots, n$, with $e_1 = \frac{\partial}{\partial t}$. In particular, the second fundamental form on N_t must be a diagonal matrix when written in terms of the basis $\{e_i\}_{i=2}^{4n}$ with eigenvalues given by

$$(\langle \nabla_{e_i} e_j, e_1\rangle) \begin{pmatrix} 2I_3 & 0 \\ 0 & I_{4(n-1)} \end{pmatrix}, \tag{5.7}$$

where I_k denotes the $k\times k$ identity matrix. Also, the sectional curvatures of the sections containing e_1 are given by

$$\mathcal{K}(e_1, e_p) = -4 \qquad \text{for} \qquad 2 \le p \le 4$$

and

$$\mathcal{K}(e_1, e_\alpha) = -1 \qquad \text{for} \qquad 5 \le \alpha \le 4n.$$

The Guass curvature equation also asserts that

$$R_{ijkl} = \bar{R}_{ijkl} + h_{li}h_{kj} - h_{ki}h_{lj},$$

where $\bar{R}_{ijkl}$ is the curvature tensor on N_t. In particular,

$$R_{ijkl} = \begin{cases} \bar{R}_{ijkl} + \delta_{li}\delta_{kj} - \delta_{ki}\delta_{lj} & \text{if} \quad 5 \le i,j,k,l \le 4n \\ \bar{R}_{ijkl} + 4\delta_{li}\delta_{kj} - 4\delta_{ki}\delta_{lj} & \text{if} \quad 2 \le i,j,k,l \le 4 \\ \bar{R}_{ijkl} + 2 \quad \text{if} \quad 2 \le i = l \le 4 & \text{and} \quad 5 \le k = j \le 4n \\ \bar{R}_{ijkl} + 2 \quad \text{if} \quad 2 \le k = j \le 4 & \text{and} \quad 5 \le i = l \le 4n \\ \bar{R}_{ijkl} - 2 \quad \text{if} \quad 2 \le i = k \le 4 & \text{and} \quad 5 \le j = l \le 4n \\ \bar{R}_{ijkl} - 2 \quad \text{if} \quad 2 \le j = l \le 4 & \text{and} \quad 5 \le i = k \le 4n \\ \bar{R}_{ijkl} - 2 \quad \text{if} \quad 2 \le k = i \le 4 & \text{and} \quad 5 \le j = l \le 4n \\ \bar{R}_{ijkl} \qquad \text{otherwise.} \end{cases} \tag{5.8}$$

We will now use (5.6) to compute the curvature tensor of M and hence N_0. Using the orthonormal coframe

$$\eta_1 = \omega_1 = dt,$$

$$\eta_p = e^{2t}\omega_p$$

$$\eta_\alpha = e^t\omega_\alpha$$

for $2 \leq p \leq 4$ and $5 \leq \alpha \leq 4n$, we obtain the first structural equations

$$d\eta_1 = 0, \tag{5.9}$$

$$\begin{aligned} d\eta_p &= 2e^{2t}\,\omega_1 \wedge \omega_p + e^{2t}\sum_{q=2}^{4}\omega_{pq}\wedge\omega_q + e^{2t}\sum_{\alpha=5}^{4n}\omega_{p\alpha}\wedge\omega_\alpha \\ &= -2\eta_p\wedge\eta_1 + \sum_{q=2}^{4}\omega_{pq}\wedge\eta_q + e^{t}\sum_{\alpha=5}^{4n}\omega_{p\alpha}\wedge\eta_\alpha, \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} d\eta_\alpha &= e^{t}\,\omega_1\wedge\omega_\alpha + e^{t}\sum_{p=2}^{4}\omega_{\alpha p}\wedge\omega_p + e^{t}\sum_{\beta=5}^{4n}\omega_{\alpha\beta}\wedge\omega_\beta \\ &= -\eta_\alpha\wedge\eta_1 + e^{-t}\sum_{p=2}^{4}\omega_{\alpha p}\wedge\eta_p + \sum_{\beta=5}^{4n}\omega_{\alpha\beta}\wedge\eta_\beta, \end{aligned} \tag{5.11}$$

where ω_{ij} are the connection forms of N_0. In the above and all subsequent computations, we will adopt the convention that $5 \leq \alpha, \beta \leq 4n$, $2 \leq i, j \leq 4n$, $2 \leq o, p, q, r \leq 4$, $2 \leq s, t \leq m$, and $1 \leq A, B \leq 4n$.

Note that using the endomorphism I and the fact that $\nabla I = cJ - bK$, we have

$$\begin{aligned} \omega_{ij}(X) &= \langle \bar{\nabla}_X e_j, e_i\rangle \\ &= \langle I\nabla_X e_j, Ie_i\rangle \\ &= \langle \nabla_X Ie_j, Ie_i\rangle + c(X)\,\langle Je_j, Ie_i\rangle - b(X)\,\langle ke_j, Ie_i\rangle \\ &= \langle \nabla_X Ie_j, Ie_j\rangle + c(X)\,\langle e_j, Ke_i\rangle + b(X)\langle e_j, Je_i\rangle \end{aligned}$$

for any tangent vector X to N_0, where $\bar{\nabla}$ denotes the connection on N_0. Hence, we conclude that

$$\omega_{ij} = \omega_{I_iI_j} + c\,\langle e_j, Ke_i\rangle + b\,\langle e_j, Je_i\rangle, \tag{5.12}$$

where I_i denotes the index corresponding to $Ie_ie_{I_i}$. Similarly, we have

$$\omega_{ij} = \omega_{J_iJ_j} + c\,\langle e_j, Ke_i\rangle + a\,\langle e_j, Ie_i\rangle,$$

and

$$\omega_{ij} = \omega_{K_iK_j} + b\,\langle e_j, Je_i\rangle + a\,\langle e_j, Ie_i\rangle.$$

Together with (5.7), we conclude that

$$\omega_{2(4s-1)}(e_{4s}) = -1 = -\omega_{2(4s)}(e_{4s-1}),$$

$$\omega_{2(4s-3)}(e_{4s-2}) = -1 = -\omega_{2(4s-2)}(e_{4s-3}),$$

for all $2 \leq s \leq n$, and

$$\omega_{2\alpha}(e_\beta) = 0 \qquad \text{otherwise.}$$

Similarly,

$$\begin{aligned}\omega_{2\alpha}(e_p) &= \langle \nabla_{e_p} e_\alpha, e_2\rangle\\ &= -\langle \nabla_{e_p} I e_\alpha, e_1\rangle\\ &= 0.\end{aligned}$$

These identities imply that

$$\begin{aligned}\omega_{2(4s-3)} &= -\omega_{4s-2},\\ \omega_{2(4s-2)} &= \omega_{4s-3},\\ \omega_{2(4s-1)} &= -\omega_{4s},\\ \omega_{2(4s)} &= \omega_{4s-1}.\end{aligned}\tag{5.13}$$

A similar calculation using the endomorphisms J and K yields

$$\begin{aligned}\omega_{3(4s-3)} &= -\omega_{4s-1},\\ \omega_{3(4s-2)} &= \omega_{4s},\\ \omega_{3(4s-1)} &= \omega_{4s-3},\\ \omega_{3(4s)} &= -\omega_{4s-2},\end{aligned}\tag{5.14}$$

and

$$\begin{aligned}\omega_{4(4s-3)} &= -\omega_{4s},\\ \omega_{4(4s-2)} &= -\omega_{4s-1},\\ \omega_{4(4s-1)} &= \omega_{4s-2},\\ \omega_{4(4s)} &= \omega_{4s-3}.\end{aligned}\tag{5.15}$$

We claim that the connection forms are given by

$$\begin{aligned}\eta_{1p} &= -\eta_{p1}\\ &= 2\eta_p \qquad \text{for} \qquad 2 \le p \le 4,\end{aligned}\tag{5.16}$$

$$\begin{aligned}\eta_{1\alpha} &= -\eta_{\alpha 1}\\ &= \eta_\alpha \qquad \text{for} \qquad 5 \le \alpha \le 4n,\end{aligned}\tag{5.17}$$

$$\eta_{pq} = -\eta_{qp} = \omega_{pq},\tag{5.18}$$

$$\eta_{p\alpha} = -\eta_{\alpha p} = e^t\,\omega_{p\alpha},\tag{5.19}$$

$$\eta_{(4s)\beta} = -\eta_{\beta(4s)}$$
$$= \begin{cases} \omega_{(4s)\beta} - (1-e^{-2t})\,\eta_2 & \text{if} \quad \beta = 4s-1 \\ \omega_{(4s)\beta} + (1-e^{-2t})\,\eta_3 & \text{if} \quad \beta = 4s-2 \\ \omega_{(4s)\beta} - (1-e^{-2t})\,\eta_4 & \text{if} \quad \beta = 4s-3 \\ \omega_{(4s)\beta} & \text{if} \quad \beta \neq 4s-1, 4s-2, \text{or } 4s-3, \end{cases} \tag{5.20}$$

$$\eta_{(4s-1)\beta} = -\eta_{\beta(4s-1)}$$
$$= \begin{cases} \omega_{(4s-1)\beta} + (1-e^{-2t})\,\eta_2 & \text{if} \quad \beta = 4s \\ \omega_{(4s-1)\beta} - (1-e^{-2t})\,\eta_4 & \text{if} \quad \beta = 4s-2 \\ \omega_{(4s-1)\beta} - (1-e^{-2t})\,\eta_3 & \text{if} \quad \beta = 4s-3 \\ \omega_{(4s-1)\beta} & \text{if} \quad \beta \neq 4s, 4s-2, \text{or } 4s-3, \end{cases} \tag{5.21}$$

$$\eta_{(4s-2)\beta} = -\eta_{\beta(4s-2)}$$
$$= \begin{cases} \omega_{(4s-2)\beta} - (1-e^{-2t})\,\eta_3 & \text{if} \quad \beta = 4s \\ \omega_{(4s-2)\beta} + (1-e^{-2t})\,\eta_4 & \text{if} \quad \beta = 4s-1 \\ \omega_{(4s-2)\beta} - (1-e^{-2t})\,\eta_2 & \text{if} \quad \beta = 4s-3 \\ \omega_{(4s-2)\beta} & \text{if} \quad \beta \neq 4s, 4s-1, \text{or } 4s-3, \end{cases} \tag{5.22}$$

$$\eta_{(4s-3)\beta} = -\eta_{\beta(4s-3)}$$
$$= \begin{cases} \omega_{(4s-3)\beta} + (1-e^{-2t})\,\eta_4 & \text{if} \quad \beta = 4s \\ \omega_{(4s-3)\beta} + (1-e^{-2t})\,\eta_3 & \text{if} \quad \beta = 4s-1 \\ \omega_{(4s-3)\beta} + (1-e^{-2t})\,\eta_2 & \text{if} \quad \beta = 4s-2 \\ \omega_{(4s-3)\beta} & \text{if} \quad \beta \neq 4s, 4s-1, \text{or } 4s-2. \end{cases} \tag{5.23}$$

Indeed, if we substitute (5.16−5.23) into the first structural equations

$$d\eta_A = \eta_{A1} \wedge \eta_1 + \sum_{q=2}^{4} \eta_{Aq} \wedge \eta_q + \sum_{\beta=5}^{4n} \eta_{A\beta} \wedge \eta_\beta,$$

we obtain (5.9), (5.10), and (5.11).

To compute the curvature, we consider the second structural equations. In particular,

$$\begin{aligned} & d\eta_{1p} - \eta_{1q} \wedge \eta_{qp} - \eta_{1\alpha} \wedge \eta_{\alpha p} \\ &\quad = 2d\eta_p - 2\eta_q \wedge \eta_{qp} - \eta_\alpha \wedge \eta_{\alpha p} \\ &\quad = -4\eta_p \wedge \eta_1 + \eta_\alpha \wedge \eta_{\alpha p} \\ &\quad = -4\eta_p \wedge \eta_1 + e^t\,\omega_{p\alpha} \wedge \eta_\alpha. \end{aligned}$$

Hence, using $(5.13-5.15)$, we have

$$\begin{aligned} R_{1p1p} &= -4, \\ R_{12(4s-1)(4s)} &= -\,2 = -R_{12(4s)(4s-1)}, \\ R_{12(4s-3)(4s-2)} &= -\,2 = -R_{12(4s-2)(4s-3)}, \\ R_{13(4s)(4s-2)} &= -\,2 = -R_{13(4s-2)(4s)}, \\ R_{13(4s-1)(4s-3)} &= 2 = -R_{13(4s-3)(4s-1)}, \\ R_{14(4s)(4s-3)} &= 2 = -R_{14(4s-3)(4s)}, \\ R_{14(4s-1)(4s-2)} &= 2 = -R_{14(4s-2)(4s-1)}, \end{aligned}$$

and

$$R_{1pAB} = 0, \qquad \text{otherwise.}$$

Also,

$$\begin{aligned} d\eta_{1\alpha} &- \eta_{1q}\wedge\eta_{q\alpha} - \eta_{1\beta}\wedge\eta_{\beta\alpha} \\ &= d\eta_\alpha - 2\eta_q\wedge\eta_{q\alpha} - \eta_\beta\wedge\eta_{\beta\alpha} \\ &= -\eta_\alpha\wedge\eta_1 + e^t\,\omega_{q\alpha}\wedge\eta_q, \end{aligned}$$

hence

$$\begin{aligned} R_{1\alpha1\alpha} &= -1, \\ R_{1(4s)(4s-1)2} &= -\,1 = -R_{1(4s-1)(4s)2}, \\ R_{1(4s)(4s-2)3} &= 1 = -R_{1(4s-2)(4s)3}, \\ R_{1(4s)(4s-3)4} &= -\,1 = -R_{1(4s-3)(4s)4}, \\ R_{1(4s-1)(4s-3)3} &= -\,1 = -R_{1(4s-3)(4s-1)3}, \\ R_{1(4s-1)(4s-2)4} &= -\,1 = -R_{1(4s-2)(4s-1)4}, \\ R_{1(4s-2)(4s-3)2} &= -\,1 = -R_{1(4s-3)(4s-2)2}, \end{aligned}$$

and

$$R_{1\alpha AB} = 0 \qquad \text{otherwise.}$$

Similarly,

$$\begin{aligned} d\eta_{pq} &- \eta_{p1}\wedge\eta_{1q} - \eta_{pr}\wedge\eta_{rq} - \eta_{p\beta}\wedge\eta_{\beta q} \\ &= d\omega_{pq} + 4\eta_p\wedge\eta_q - \omega_{pr}\wedge\omega_{rq} - e^{2t}\omega_{p\beta}\wedge\omega_{\beta q} \\ &= \bar\Omega_{pq} + (1-e^{2t})\,\omega_{p\beta}\wedge\omega_{\beta q} + 4\eta_p\wedge\eta_q, \end{aligned}$$

where

$$\bar\Omega_{pq} = \frac{1}{2}\bar R_{pqij}\omega_j\wedge\omega_i$$

is the curvature form of N_0. In particular, this implies that

$$R_{pqro} = \begin{cases} -4+e^{-4t}\,\bar R_{pqpq} & \text{if} \qquad r=p \text{ and } o=q \\ 4+e^{-4t}\,\bar R_{pqqp} & \text{if} \qquad r=q \text{ and } o=p \\ e^{-4t}\,\bar R_{pqro} & \text{otherwise,} \end{cases} \tag{5.24}$$

$$R_{23(4s)(4s-3)} = e^{-2t}\,\bar{R}_{23(4s)(4s-3)} - 2(e^{-2t}-1), \tag{5.25}$$

$$R_{23(4s-1)(4s-2)} = e^{-2t}\,\bar{R}_{23(4s-1)(4s-2)} - 2(e^{-2t}-1), \tag{5.26}$$

$$R_{24(4s)(4s-2)} = e^{-2t}\,\bar{R}_{24(4s)(4s-2)} - 2(e^{-2t}-1), \tag{5.27}$$

$$R_{24(4s-1)(4s-3)} = e^{-2t}\,\bar{R}_{24(4s-1)(4s-3)} + 2(e^{-2t}-1), \tag{5.28}$$

$$R_{34(4s)(4s-1)} = e^{-2t}\,\bar{R}_{34(4s)(4s-1)} - 2(e^{-2t}-1), \tag{5.29}$$

$$R_{34(4s-2)(4s-3)} = e^{-2t}\,\bar{R}_{34(4s-2)(4s-3)} - 2(e^{-2t}-1), \tag{5.30}$$

and

$$R_{pq\alpha\beta} = e^{-2t}\,\bar{R}_{pq\alpha\beta}, \qquad \text{otherwise.} \tag{5.31}$$

We now continue with our curvature computation and consider

$$\begin{aligned}
&d\eta_{p\alpha} - \eta_{p1}\wedge\eta_{1\alpha} - \eta_{pq}\wedge\eta_{q\alpha} - \eta_{p\beta}\wedge\eta_{\beta\alpha}\\
&= d(e^t\,\omega_{p\alpha}) + 2\eta_p\wedge\eta_\alpha - \omega_{pq}\wedge e^t\,\omega_{q\alpha} - e^t\,\omega_{p\beta}\wedge\eta_{\beta\alpha}\\
&= e^t\,\eta_1\wedge\omega_{p\alpha} + \frac{1}{2}e^t\bar{R}_{p\alpha ij}\omega_j\wedge\omega_i + 2\eta_p\wedge\eta_\alpha + e^t\,\omega_{p\beta}\wedge(\omega_{\beta\alpha}-\eta_{\beta\alpha}),
\end{aligned}$$

where $\bar{R}_{p\alpha ij}$ is the curvature tensor of N_0. Using (5.13−5.15) and (5.20−5.23), we have

$$\begin{aligned}
&\frac{1}{2}R_{2(4s)AB}\eta_B\wedge\eta_A\\
&= \eta_1\wedge\eta_{(4s-1)} + \frac{1}{2}e^t\bar{R}_{2(4s)ij}\,\omega_j\wedge\omega_i - 2\eta_{(4s)}\wedge\eta_2 + (1-e^{-2t})\,\eta_{(4s)}\wedge\eta_2\\
&\qquad + (1-e^{-2t})\,\eta_{(4s-3)}\wedge\eta_3 + (1-e^{-2t})\,\eta_{(4s-2)}\wedge\eta_4\\
&= \eta_1\wedge\eta_{(4s-1)} + \frac{1}{2}e^t\bar{R}_{2(4s)ij}\,\omega_j\wedge\omega_i - (1+e^{-2t})\,\eta_{(4s)}\wedge\eta_2\\
&\qquad + (1-e^{-2t})\,\eta_{(4s-3)}\wedge\eta_3 + (1-e^{-2t})\,\eta_{(4s-2)}\wedge\eta_4.
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}R_{2(4s-1)AB}\eta_B\wedge\eta_A\\
&= -\eta_1\wedge\eta_{(4s)} + \frac{1}{2}e^t\bar{R}_{2(4s-1)ij}\,\omega_j\wedge\omega_i - (1+e^{-2t})\,\eta_{(4s-1)}\wedge\eta_2\\
&\qquad + (1-e^{-2t})\eta_{(4s-2)}\wedge\eta_3 - (1-e^{-2t})\,\eta_{(4s-3)}\wedge\eta_4.
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}R_{2(4s-2)AB}\eta_B\wedge\eta_A\\
&= \eta_1\wedge\eta_{4s-3} + \frac{1}{2}e^t\bar{R}_{2(4s-2)ij}\,\omega_j\wedge\omega_i - (1+e^{-2t})\,\eta_{(4s-2)}\wedge\eta_2\\
&\qquad + (1-e^{-2t})\,\eta_{(4s-1)}\wedge\eta_3 - (1-e^{-2t})\,\eta_{(4s)}\wedge\eta_4.
\end{aligned}$$

$$
\begin{aligned}
&\frac{1}{2}R_{2(4s-3)AB}\eta_B\wedge\eta_A\\
&=-\eta_1\wedge\eta_{4s-2}+\frac{1}{2}e^t\bar{R}_{2(4s-3)ij}\,\omega_j\wedge\omega_i-(1+e^{-2t})\,\eta_{(4s-3)}\wedge\eta_2\\
&\qquad-(1-e^{-2t})\,\eta_{(4s)}\wedge\eta_3+(1-e^{-2t})\,\eta_{(4s-1)}\wedge\eta_4.
\end{aligned}
$$

There are similar formulas for the curvature tensors of the form $R_{3\alpha AB}$ and $R_{4\alpha AB}$.

Continuing with our computation of the second structural equations using $(5.13-5.15)$, we have

(5.32)

$$
\begin{aligned}
&d\eta_{(4s-1)(4s)}-\eta_{(4s-1)1}\wedge\eta_{1(4s)}-\eta_{(4s-1)q}\wedge\eta_{q(4s)}-\eta_{(4s-1)\beta}\wedge\eta_{\beta(4s)}\\
&=d\omega_{(4s-1)(4s)}+2e^{-2t}\,\eta_1\wedge\eta_2+(1-e^{-2t})\,d\eta_2\\
&\quad+\eta_{(4s-1)}\wedge\eta_{(4s)}-e^{2t}\,\omega_{(4s-1)q}\wedge\omega_{q(4s)}\\
&\quad-(\omega_{(4s-1)(4s-2)}-(1-e^{-2t})\,\eta_4)\wedge(\omega_{(4s-2)(4s)}-(1-e^{-2t})\,\eta_3)\\
&\quad-(\omega_{(4s-1)(4s-3)}-(1-e^{-2t})\,\eta_3)\wedge(\omega_{(4s-3)(4s)}+(1-e^{-2t})\,\eta_4)\\
&=\frac{1}{2}\bar{R}_{(4s-1)(4s)ij}\,\omega_j\wedge\omega_i+(1-e^{2t})\,\omega_{(4s-1)q}\wedge\omega_{q(4s)}+2\eta_1\wedge\eta_2\\
&\quad+(1-e^{-2t})\,\omega_{2q}\wedge\eta_q+e^t(1-e^{-2t})\,\omega_{2\beta}\wedge\eta_\beta-\eta_{(4s)}\wedge\eta_{(4s-1)}\\
&\quad+(1-e^{-2t})\,\omega_{(4s-1)(4s-2)}\wedge\eta_3+(1-e^{-2t})\,\eta_4\wedge\omega_{(4s-2)(4s)}\\
&\quad+(1-e^{-2t})\,\eta_3\wedge\omega_{(4s-3)(4s)}-(1-e^{-2t})\,\omega_{(4s-1)(4s-3)}\wedge\eta_4\\
&\quad+2(1-e^{-2t})^2\,\eta_3\wedge\eta_4\\
&=\frac{1}{2}\bar{R}_{(4s-1)(4s)ij}\,\omega_j\wedge\omega_i+(2-e^{-2t})\,\eta_{(4s-1)}\wedge\eta_{(4s)}\\
&\quad-2(1-e^{-2t})\,\eta_{(4s-3)}\wedge\eta_{(4s-2)}+2\eta_1\wedge\eta_2+(1-e^{-2t})\,\omega_{2q}\wedge\eta_q\\
&\quad+2(1-e^{-2t})\,\eta_{(4r-3)}\wedge\eta_{(4r-2)}+2(1-e^{-2t})\,\eta_{(4r-1)}\wedge\eta_{(4r)}\\
&\quad+(1-e^{-2t})\,(\omega_{(4s-1)(4s-2)}-\omega_{(4s-3)(4s)})\wedge\eta_3\\
&\quad-(1-e^{-2t})\,(\omega_{(4s-2)(4s)}+\omega_{(4s-1)(4s-3)})\wedge\eta_4+2(1-e^{-2t})^2\,\eta_3\wedge\eta_4.
\end{aligned}
$$

Note that (5.22) asserts that

$$
\begin{aligned}
(1-e^{-2t})\,\omega_{2q}\wedge\eta_q&=(1-e^{-2t})\,(-\omega_{14}\wedge\eta_3+c\wedge\eta_3+\omega_{13}\wedge\eta_4-b\wedge\eta_4)\\
&=(1-e^{-2t})\,(4e^{-2t}\,\eta_3\wedge\eta_4+c\wedge\eta_3-b\eta_4),
\end{aligned}
$$

$$
\begin{aligned}
&(1-e^{-2t})\,(\omega_{(4s-1)(4s-2)}-\omega_{(4s-3)(4s)})\wedge\eta_3\\
&=-(1-e^{-2t})\,c\wedge\eta_3,
\end{aligned}
$$

and

$$\begin{aligned}
&- (1 - e^{-2t})\,(\omega_{(4s-2)(4s)} + \omega_{(4s-1)(4s-3)}) \wedge \eta_4 \\
&= (1 - e^{-2t})\, b \wedge \eta_4.
\end{aligned}$$

Hence, substituting into (5.32), we obtain

$$\begin{aligned}
&\frac{1}{2} R_{(4s-1)(4s)AB}\, \eta_B \wedge \eta_A \\
&= \frac{1}{2}\bar{R}_{(4s-1)(4s)ij}\, \omega_j \wedge \omega_i + (2 - e^{-2t})\, \eta_{(4s-1)} \wedge \eta_{(4s)} \\
&\quad - 2(1 - e^{-2t})\, \eta_{(4s-3)} \wedge \eta_{(4s-2)} + 2\eta_1 \wedge \eta_2 \\
&\quad + 2(1 - e^{-2t})\, \eta_{(4r-3)} \wedge \eta_{(4r-2)} \\
&\quad + 2(1 - e^{-2t})\, \eta_{(4r-1)} \wedge \eta_{(4r)} + 2(1 - e^{-4t})\, \eta_3 \wedge \eta_4 \\
&= 2\eta_1 \wedge \eta_2 + \frac{1}{2}\bar{R}_{(4s-1)(4s)pq}\, e^{-4t}\, \eta_q \wedge \eta_p + 2(1 - e^{-4t})\, \eta_3 \wedge \eta_4 \\
&\quad + \bar{R}_{(4s-1)(4s)p\alpha}\, e^{-3t}\, \eta_\alpha \wedge \eta_p + \frac{1}{2}\bar{R}_{(4s-1)(4s)\alpha\beta}\, e^{-2t}\, \eta_\beta \wedge \eta_\alpha \\
&\quad + (2 - e^{-2t})\, \eta_{(4s-1)} \wedge \eta_{(4s)} - 2(1 - e^{-2t})\, \eta_{(4s-3)} \wedge \eta_{(4s-2)} \\
&\quad + 2(1 - e^{-2t})\, \eta_{(4r-3)} \wedge \eta_{(4r-2)} + 2(1 - e^{-2t})\, \eta_{(4r-1)} \wedge \eta_{(4r)}.
\end{aligned}$$

A similar computation yields the curvature tensor of the form $R_{(4s-1)\,(4s-2)AB}$, $R_{(4s-1)\,(4s-3)AB}$, $R_{(4s-2)\,(4s-3)AB}$, $R_{(4s-2)\,(4s)AB}$, and $R_{(4s-3)(4s)AB}$. It remains to compute

(5.33)

$$\begin{aligned}
&\frac{1}{2} R_{(4s-3)(4r)AB}\, \eta_B \wedge \eta_A \\
&= d\eta_{(4s-3)(4r)} - \eta_{(4s-3)1} \wedge \eta_{1(4r)} - \eta_{(4s-3)q} \wedge \eta_{q(4r)} - \eta_{(4s-3)\beta} \wedge \eta_{\beta(4r)} \\
&= d\omega_{(4s-3)(4r)} + \eta_{(4s-3)} \wedge \eta_{(4r)} - e^{2t}\, \omega_{(4s-3)q} \wedge \omega_{q(4r)} - \eta_{(4s-3)\beta} \wedge \eta_{\beta(4r)} \\
&= \frac{1}{2}\bar{R}_{(4s-3)(4r)ij}\, \omega_j \wedge \omega_i + (1 - e^{2t})\, \omega_{(4s-3)q} \wedge \omega_{q(4r)} \\
&\quad - (1 - e^{-2t})\, (\eta_4 \wedge \omega_{(4s)(4r)} + \eta_3 \wedge \omega_{(4s-1)(4r)}) \\
&\quad - (1 - e^{-2t})\, (\eta_2 \wedge \omega_{4s-2)(4r)} + \omega_{(4s-3)(4r-1)} \wedge \eta_2 \\
&\quad - \omega_{(4s-3)(4r-2)} \wedge \eta_3 + \omega_{(4s-3)(4r-3)} \wedge \eta_4) + \eta_{(4s-3)} \wedge \eta_{(4r)}.
\end{aligned}$$

Using (5.12 – 5.14), we can write

$$\omega_{(4s-3)q} \wedge \omega_{q(4r)} = -\eta_(4s-2) \wedge \eta_{(4r-1)} + \eta_{(4s-1)} \wedge \eta_{(4r-2)} - \eta_{(4s)} \wedge \eta_{(4r-3)}).$$

Also using (5.12) asserts that

$$\begin{aligned}
\omega_{(4s-3)(4r-1)} &= \omega_{(4s-2)(4r)}, \\
\omega_{(4s-3)(4r-2)} &= -\omega_{((4s-1)(4r)} \\
\omega_{((4s-3)(4r-3)} &= \omega_{((4s)(4r)}.
\end{aligned}$$

Hence (5.33) becomes

$$
\begin{aligned}
&\frac{1}{2} R_{(4s-3)(4r)AB}\, \eta_B \wedge \eta_A \\
&= \frac{1}{2} \bar{R}_{(4s-3)(4r)pq}\, e^{-4t}\, \eta_q \wedge \eta_p + \bar{R}_{(4s-3)(4r)p\beta}\, e^{-3t}\, \eta_\beta \wedge \eta_p \\
&\quad + \frac{1}{2} \bar{R}_{(4s-3)(4r)\alpha\beta}\, e^{-2t}\, \eta_\beta \wedge \eta_\alpha \\
&\quad - (1 - e^{2t})\, \eta_{(4s-2)} \wedge \eta_{(4r-1)} + (1 - e^{-2t})\, \eta_{(4s-1)} \wedge \eta_{(4r-2)} \\
&\quad - (1 - e^{-2t})\, \eta_{(4s)} \wedge \eta_{(4r-3)} + \eta_{(4s-3)} \wedge \eta_{4r}.
\end{aligned}
$$

So we have determined all curvature tensores of M. Note that the quaternionic curvatures satisfy

$$
\begin{aligned}
&\mathcal{K}(e_1, e_2) + \mathcal{K}(e_1, e_3) + \mathcal{K}(e_1, e_4) = -12 \\
&\mathcal{K}(e_2, e_1) + \mathcal{K}(e_2, e_3) + \mathcal{K}(e_2, e_4) = -12 + e^{-2t}\, (\mathcal{K}^N(e_2, e_3) + \mathcal{K}^N(e_2, e_4)) \\
&\mathcal{K}(e_3, e_1) + \mathcal{K}(e_3, e_2) + \mathcal{K}(e_3, e_4) = -12 + e^{-2t}\, (\mathcal{K}^N(e_3, e_2) + \mathcal{K}^N(e_3, e_4)) \\
&\mathcal{K}(e_4, e_1) + \mathcal{K}(e_4, e_2) + \mathcal{K}(e_4, e_3) = -12 + e^{-2t}\, (\bar{\mathcal{K}}(e_4, e_2) + \mathcal{K}^N(e_4, e_3)).
\end{aligned}
$$

In particular, this implies that

$$
\mathcal{K}^N(e_2, e_3) = \mathcal{K}^N(e_2, e_4) = \mathcal{K}^N(e_3, e_4) = 0.
$$

Also, for $2 \le p \le 4$, we have

$$
\begin{aligned}
&\sum_{i=0}^{3} \mathcal{K}(e_1, e_{(4s-i)}) = -4 \\
&\sum_{i=0}^{3} \mathcal{K}(e_p, e_{(4s-i)} = -4 + e^{-2t} \left(\sum_{i=0}^{3} \bar{\mathcal{K}}(e_p, e_{(4s-i)}) - 4 \right),
\end{aligned}
$$

implying

$$
\sum_{i=0}^{3} \mathcal{K}^N(e_p, e_{(4s-i)}) = 4.
$$

We also have

$$
\sum_{i=1}^{3} \mathcal{K}(e_{(4s)}, e_{(4s-1)}) = -12 + e^{-2t} \left(\sum_{i=1}^{3} \mathcal{K}^N(e_{(4s)}, e_{(4s-i)}) + 9 \right),
$$

implying

$$
\sum_{i=1}^{3} \mathcal{K}^N(e_{(4s)}, e_{(4s-i)}) = -9.
$$

Lastly,

$$
\sum_{i=0}^{3} \mathcal{K}(e_{(4s)}, e_{(4r-i)}) = -4 + e^{-2t} \sum_{i=0}^{3} \mathcal{K}^N(e_{(4s)}, e_{(4r-i)}),
$$

implying

$$\sum_{i=0}^{3} \mathcal{K}^N(e_{(4s)}, e_{(4r-i)}) = 0.$$

The above computation determined the whole curvature tensor for M and N_0. In particular, if M has bounded curvature, then from the formulas about the components of curvature tensors of M, all curvature components are determined as those of $\mathbb{QH}^n$. So it must be covered by $\mathbb{QH}^n$. q.e.d.

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