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REFINED ANALYTIC TORSION

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Abstract

Given an acyclic representation α of the fundamental group of a compact oriented odd-dimensional manifold, which is close enough to an acyclic unitary representation, we define a refinement T_α of the Ray-Singer torsion associated to α, which can be viewed as the analytic counterpart of the refined combinatorial torsion introduced by Turaev. T_α is equal to the graded determinant of the odd signature operator up to a correction term, the *metric anomaly*, needed to make it independent of the choice of the Riemannian metric.

T_α is a holomorphic function on the space of such representations of the fundamental group. When α is a unitary representation, the absolute value of T_α is equal to the Ray-Singer torsion and the phase of T_α is proportional to the η-invariant of the odd signature operator. The fact that the Ray-Singer torsion and the η-invariant can be combined into one holomorphic function allows one to use methods of complex analysis to study both invariants. In particular, using these methods we compute the quotient of the refined analytic torsion and Turaev's refinement of the combinatorial torsion generalizing in this way the classical Cheeger-Müller theorem. As an application, we extend and improve a result of Farber about the relationship between the Farber-Turaev absolute torsion and the η-invariant.

As part of our construction of T_α we prove several new results about determinants and η-invariants of non self-adjoint elliptic operators.

1. Introduction

In this paper we refine the analytic torsion which has been introduced by Ray and Singer [**35**]. In our set-up we are given a complex flat vector bundle $E \to M$ over a closed oriented odd-dimensional manifold M and we denote by ∇ the flat connection on M. Whereas the Ray-Singer

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torsion $T^{\mathrm{RS}}(\nabla)$ is a positive real number, the proposed *refined analytic torsion* $T = T(\nabla)$ will be, in general, a complex number, and hence will have a nontrivial phase. The refined analytic torsion can be viewed as an analytic analogue of the refined combinatorial torsion, introduced by Turaev [**42**, **43**] and further developed by Farber and Turaev [**19**, **20**]. Though T is *not* equal to the Turaev torsion in general, the two torsions are very closely related, as described in Section 14.

Definition. In this paper the refined analytic torsion is defined as a non-zero complex number which is canonically associated to any acyclic flat connection lying in an open set of acyclic connections, which contains all acyclic Hermitian connections, see [**7**, **8**], where we extend this definition to arbitrary flat connections.

Relation to the η-invariant and the Ray-Singer torsion. If the connection ∇ is Hermitian, i.e., if there exists a Hermitian metric on E which is preserved by ∇, then the refined analytic torsion T is a complex number whose absolute value is equal to the Ray-Singer torsion and whose phase is determined by the η-invariant of the odd signature operator. When ∇ is not Hermitian, the relationship between the refined analytic torsion, the Ray-Singer torsion, and the η-invariant is slightly more complicated, cf. Section 12.

Analytic property. One of the most important properties of the refined analytic torsion is that it depends, in an appropriate sense, *holomorphically* on the connection ∇. The fact that the Ray-Singer torsion and the η-invariant can be combined into one holomorphic function allows us to use methods of complex analysis to study both invariants. In particular, using these methods we establish a relationship between the refined analytic torsion and Turaev's refinement of the combinatorial torsion which generalizes the classical Cheeger-Müller theorem about the equality between the Ray-Singer and the combinatorial torsion [**15**, **33**]. As an application, we generalize and improve a result of Farber about the comparison between the sign of the Farber-Turaev absolute torsion and the η-invariant, [**17**]. In fact, we compare the phase of the Turaev torsion and the η-invariant in a more general set-up.

Regularized determinant. Our construction of the refined analytic torsion uses determinants of non self-adjoint elliptic differential operators. In Section 4 and Appendix A we prove several new results about these determinants which generalize well known facts about determinants of self-adjoint differential operators. In particular, we express the determinant of a (not necessarily self-adjoint) operator D in terms of the determinant of D^2, the value at 0 of the ζ-function of D^2, and the η-invariant of D. Note that the η-invariant of a non-self-adjoint operator was defined and studied by Gilkey [**21**]. In this paper we use a *sign refined* version of Gilkey's construction, cf. Definition 4.2.

Related works. In [**42**, **43**], Turaev constructed a refined version of the combinatorial torsion for an arbitrary acyclic connection. This notion was later extended by Farber and Turaev [**19**, **20**]. In [**43**] Turaev posed the problem of constructing an analytic analogue of his torsion. In [**20**, §10.3], Farber and Turaev suggested that such an analogue should be related to the η-invariant. More precisely, one can ask if it can be defined in terms of regularized determinants of elliptic differential operators and, if so, whether the phase is related to the η-invariant of these differential operators. In the present paper we show that on the open neighborhood of the set of acyclic Hermitian connections, where the proposed refined analytic torsion $T(\nabla)$ is defined, $T(\nabla)$ solves this problem.

In [**7**] we extend the notion of refined analytic torsion to the set of all flat connections and in [**8**] we discuss properties and applications of the refined analytic torsion.

In addition to the works of Turaev [**42**, **43**] and Farber-Turaev [**19**, **20**] on their refined combinatorial torsion and the relation of its absolute value to the Ray-Singer torsion [**43**, **20**] as well as the study of its phase [**17**], we would like to mention a recent paper of Burghelea and Haller, [**13**]. In that paper, among many other topics, the authors address the question of whether the Ray-Singer torsion $T^{\mathrm{RS}}(\nabla)$ can be viewed as the absolute value of a (in an appropriate sense) holomorphic function $f(\nabla)$ on the space of acyclic connection ∇. Burghelea and Haller gave an affirmative answer to this question and showed that

$$T^{\mathrm{RS}}(\nabla) = |f_1(\nabla) \cdot f_2(\nabla)|, \tag{1.1}$$

where $f_1(\nabla)$ is Turaev's refinement of the combinatorial torsion and $f_2(\nabla)$ is an explicitly calculated holomorphic function.[1] The result of Burghelea and Haller is valid for manifolds of arbitrary dimension. If the dimension of the manifold is odd, the refined analytic torsion proposed in this paper allows to obtain an identity of the type (1.1). In contrast to [**13**], the holomorphic function on the right hand side of equality (1.1) is constructed in this paper in purely analytic terms, cf. Theorem 12.8. The quotient between the Ray-Singer torsion and the absolute value of Turaev's refinement of the combinatorial torsion is discussed in Section 14.

In response to a first version of our paper, Dan Burghelea kindly brought to our attention his ongoing project with Stefan Haller [**14**] where they consider, among other things, Laplace-type operators acting on forms obtained by replacing a Hermitian scalar product on a given complex vector bundle by a non-degenerate symmetric bilinear form. These operators are non self-adjoint and have complex-valued zeta-regularized determinants. Burghelea and Haller then express the

[1] Note that, in the case when the dimension of the manifold is odd, (1.1) is similar to our Theorem 14.3.

square of the Turaev torsion in terms of these determinants and some additional ingredients. [2]

The results of this paper were announced in [**6**].

2. Summary of the Main Results

Throughout this section M is a closed oriented manifold of odd dimension $\dim M = d = 2r-1$ and E is a complex vector bundle over M endowed with a flat connection ∇.

2.1. The odd signature operator. The refined analytic torsion is defined in terms of the odd signature operator, hence, let us begin by recalling the definition of this operator.

Let $\Omega^\bullet(M,E)$ denote the space of smooth differential forms on M with values in E and set

$$\Omega^{\text{even}}(M,E) = \bigoplus_{p=0}^{r-1} \Omega^{2p}(M,E),$$

where $r = \frac{\dim M+1}{2}$. Fix a Riemannian metric g^M on M and let $* : \Omega^\bullet(M,E) \to \Omega^{d-\bullet}(M,E)$ denote the Hodge $*$-operator. The *chirality* operator

$$\Gamma : \Omega^\bullet(M,E) \to \Omega^{d-\bullet}(M,E)$$

is then given by the formula, cf. [**4**, §3.2],

$$\Gamma\,\omega := i^r\,(-1)^{k(k+1)/2} * \omega, \qquad \omega \in \Omega^k(M,E).$$

The *odd signature operator* $B = B(\nabla, g^M) : \Omega^\bullet(M,E) \to \Omega^\bullet(M,E)$ is defined by

$$B := \Gamma\,\nabla + \nabla\,\Gamma.$$

It leaves $\Omega^{\text{even}}(M,E)$ invariant. Denote its restriction to $\Omega^{\text{even}}(M,E)$ by B_{even}. Then, for $\omega \in \Omega^{2p}(M,E)$, one has

$$B\,\omega = i^r\,(-1)^{p+1}\,\big(\,*\,\nabla - \nabla\,*\,\big)\,\omega \;\in\; \Omega^{d-2p-1}(M,E) \oplus \Omega^{d-2p+1}(M,E).$$

The odd signature operator was introduced by Atiyah, Patodi, and Singer, [**2**, p. 44], [**3**, p. 405], and, in the more general setting used here, by Gilkey, [**21**, p. 64–65].

The operator B_{even} is an elliptic differential operator of order one, whose leading symbol is symmetric with respect to any Hermitian metric h^E on E.

In this paper we define the refined analytic torsion in the case when the pair (∇, g^M) satisfies the following simplifying assumptions. The general case will be addressed in [**7**, **8**].

[2] Added in proof: D. Burghelea & S. Haller, *Complex valued Ray-singer torsion*, math.DG/0604484; M. Braverman & T. Kappeler, *Comparison of the refined analytic and the Burghelea-Haller torsions*, math.DG/0606398.

Assumption I. The connection ∇ is acyclic, i.e.,

$$\operatorname{Im}\big(\nabla|_{\Omega^{k-1}(M,E)}\big) = \operatorname{Ker}\big(\nabla|_{\Omega^{k}(M,E)}\big), \qquad \text{for every} \quad k = 0, \dots, d.$$

Assumption II. $B_{\text{even}} = B_{\text{even}}(\nabla, g^M)$ is bijective.

Note that if ∇ is a Hermitian connection then Assumption I implies Assumption II, cf. Subsection 6.6. Hence, all acyclic Hermitian connections satisfy Assumptions I and II. By a simple continuity argument, cf. Proposition 6.8, these two assumptions are then satisfied for all flat connections in an open neighborhood (in C^0-topology, cf. Subsection 6.7) of the set of acyclic Hermitian connections.

2.2. Graded determinant. Set

$$\begin{aligned} \Omega^k_+(M,E) &:= \operatorname{Ker}(\nabla\,\Gamma) \cap \Omega^k(M,E), \\ \Omega^k_-(M,E) &:= \operatorname{Ker}(\Gamma\,\nabla) \cap \Omega^k(M,E). \end{aligned} \tag{2.2}$$

Assumption II implies that $\Omega^k(M,E) = \Omega^k_+(M,E) \oplus \Omega^k_-(M,E)$, cf. Subsection 6.9. Hence, (2.2) defines a *grading* on $\Omega^k(M,E)$.

Define $\Omega^{\text{even}}_\pm(M,E) = \bigoplus_{p=0}^{r-1} \Omega^{2p}_\pm(M,E)$ and let $B^\pm_{\text{even}}$ denote the restriction of B_{even} to $\Omega^{\text{even}}_\pm(M,E)$. It is easy to see that B_{even} leaves the subspaces $\Omega^{\text{even}}_\pm(M,E)$ invariant and it follows from Assumption II that the operators $B^\pm_{\text{even}} : \Omega^{\text{even}}_\pm(M,E) \to \Omega^{\text{even}}_\pm(M,E)$ are bijective.

One of the central objects of this paper is the *graded determinant* of the operator B_{even}. To construct it we need to choose a *spectral cut* along a ray $R_\theta = \{\rho e^{i\theta} : 0 \le \rho < \infty\}$, where $\theta \in [-\pi, \pi)$ is an Agmon angle for B_{even}, cf. Definition 3.4. Since the leading symbol of B_{even} is symmetric, B_{even} admits an Agmon angle $\theta \in (-\pi, 0)$. Given such an angle θ, observe that it is an Agmon angle for $B^\pm_{\text{even}}$ as well. The graded determinant of B_{even} is the non-zero complex number defined by the formula

$$\operatorname{Det}_{\text{gr},\theta}(B_{\text{even}}) := \frac{\operatorname{Det}_\theta(B^+_{\text{even}})}{\operatorname{Det}_\theta(-B^-_{\text{even}})}. \tag{2.3}$$

By standard arguments, cf. Subsection 3.10, $\operatorname{Det}_{\text{gr},\theta}(B_{\text{even}})$ is independent of the choice of the Agmon angle $\theta \in (-\pi, 0)$.

2.3. A convenient choice of the Agmon angle. For $\mathcal{I} \subset \mathbb{R}$ we denote by $L_{\mathcal{I}}$ the solid angle

$$L_{\mathcal{I}} = \{\rho e^{i\theta} : 0 < \rho < \infty,\ \theta \in \mathcal{I}\}.$$

Though many of our results are valid for any Agmon angle $\theta \in (-\pi, 0)$, some of them are more easily formulated if the following conditions are satisfied:

(AG1) $\theta \in (-\pi/2, 0)$, and

(AG2) there are no eigenvalues of the operator B_{even} in the solid angles $L_{(-\pi/2,\theta]}$ and $L_{(\pi/2,\theta+\pi]}$.

For the sake of simplicity of exposition, we will assume that θ is chosen so that these conditions are satisfied throughout the Introduction. Since the leading symbol of B_{even} is symmetric (with respect to an arbitrary Hermitian metric on E), such a choice of θ is always possible.

2.4. Relationship with the Ray-Singer torsion and the η-invariant. For a pair (∇, g^M) satisfying Assumptions I and II set

$$\xi = \xi(\nabla, g^M, \theta) := \frac{1}{2} \sum_{k=0}^{d-1} (-1)^k \, \zeta'_{2\theta} \left(0, \, (\Gamma \nabla)^2\big|_{\Omega^k_+(M,E)}\right), \tag{2.4}$$

where $\zeta'_{2\theta}\big(s, \, (\Gamma \nabla)^2\big|_{\Omega^k_+(M,E)}\big)$ is the derivative with respect to s of the ζ-function of the operator $(\Gamma \nabla)^2\big|_{\Omega^k_+(M,E)}$ corresponding to the spectral cut along the ray $R_{2\theta}$, cf. Subsection 3.5, and θ is an Agmon angle satisfying (AG1)-(AG2).

Let $\eta = \eta(\nabla, g^M)$ denote the (sign refined) η-invariant of the operator $B_{\text{even}}(\nabla, g^M)$, cf. Definition 4.2. Theorem 7.2 implies that,

$$\operatorname{Det}_{\text{gr},\theta}(B_{\text{even}}) = e^{\xi(\nabla, g^M, \theta)} \cdot e^{-i\pi\eta(\nabla, g^M)}. \tag{2.5}$$

This representation of the graded determinant turns out to be very useful, e.g., in computing the metric anomaly of $\operatorname{Det}_{\text{gr},\theta}(B_{\text{even}})$.

If the connection ∇ is Hermitian, then (2.4) coincides with the well known expression for the logarithm of the Ray-Singer torsion $T^{\text{RS}} = T^{\text{RS}}(\nabla)$. Hence, for a Hermitian connection ∇ we have

$$\xi(\nabla, g^M, \theta) = \log \, T^{\text{RS}}(\nabla).$$

If ∇ is not Hermitian but is sufficiently close (in C^0-topology) to an acyclic Hermitian connection, then Theorem 8.2 states that

$$\log T^{\text{RS}}(\nabla) = \operatorname{Re} \, \xi(\nabla, g^M, \theta). \tag{2.6}$$

Combining (2.6) and (2.5), we get

$$\big| \operatorname{Det}_{\text{gr},\theta}(B_{\text{even}}) \big| = T^{\text{RS}}(\nabla) \cdot e^{\pi \operatorname{Im} \eta(\nabla, g^M)}. \tag{2.7}$$

If ∇ is Hermitian, then the operator B_{even} is self-adjoint (cf. Subsection 6.6) and $\eta = \eta(\nabla, g^M)$ is real. Hence, cf. Corollary 8.3, for the case of an acyclic Hermitian connection we obtain from (2.7)

$$\big| \operatorname{Det}_{\text{gr},\theta}(B_{\text{even}}) \big| = T^{\text{RS}}(\nabla).$$

2.5. Metric anomaly of the graded determinant. The graded determinant of the odd signature operator is not a differential invariant of the connection ∇ since, in general, it depends on the choice of the Riemannian metric g^M. Hence, we first investigate the *metric anomaly* of the graded determinant and then use it to "correct" the graded determinant and construct a differential invariant – the refined analytic torsion.

Suppose an acyclic connection ∇ is given. We call a Riemannian metric g^M on M *admissible for* ∇ if the operator $B_{\text{even}} = B_{\text{even}}(\nabla, g^M)$ satisfies Assumption II of Subsection 2.1. We denote the set of admissible metrics by $\mathcal{M}(\nabla)$. The set $\mathcal{M}(\nabla)$ might be empty. However, Proposition 6.8 implies that admissible metrics exist for all flat connections in an open neighborhood (in C^0-topology) of the set of acyclic Hermitian connections.

For each admissible metric $g^M \in \mathcal{M}(\nabla)$ choose an Agmon angle θ satisfying (AG1)-(AG2). Then the reduction of $\xi(\nabla, g^M, \theta)$ modulo $\pi\mathbb{Z}$ depends neither on the choice of θ nor on the choice of $g^M \in \mathcal{M}(\nabla)$, cf. Proposition 9.7.

The dependence of $\eta = \eta(\nabla, g^M)$ on g^M has been analyzed in [**3**] and [**21**]. In particular, it follows from the results in these papers that (cf. Proposition 9.5)

- *If* $\dim M \equiv 1 \pmod 4$ *then the reduction of* $\eta(\nabla, g^M)$ *modulo* $\mathbb{Z}$ *is independent of the choice of the admissible metric* g^M*;*
- *Suppose* $\dim M \equiv 3 \pmod 4$ *and let* $\eta_{\text{trivial}}(g^M)$ *denote the* η*-invariant of the odd signature operator associated to the trivial connection on the trivial line bundle over* M*. Then, modulo* $\mathbb{Z}$*,*

$$\eta(\nabla, g^M) - \eta_{\text{trivial}}(g^M) \cdot \operatorname{rank} E$$

is independent of the choice of the metric g^M*.*

2.6. Definition of the refined analytic torsion. The *refined analytic torsion* $T(\nabla)$ corresponding to an acyclic connection ∇, satisfying $\mathcal{M}(\nabla) \neq \emptyset$, is defined as follows: fix an admissible Riemannian metric $g^M \in \mathcal{M}(\nabla)$ and let $\theta \in (-\pi, 0)$ be an Agmon angle for $B_{\text{even}}(\nabla, g^M)$. Then

$$\begin{aligned} T(\nabla) &= T(M, E, \nabla) \\ &:= \operatorname{Det}_{\text{gr},\theta}(B_{\text{even}}) \cdot \exp\Big(i\pi\, \eta_{\text{trivial}}(g^M) \cdot \operatorname{rank} E \Big) \in \mathbb{C}\backslash 0. \end{aligned}$$

Note that if $\dim M \equiv 1 \pmod 4$ then $\eta_{\text{trivial}}(g^M) = 0$ and, hence, $T(\nabla) = \operatorname{Det}_{\text{gr},\theta}\big(B_{\text{even}}(\nabla, g^M)\big)$.

If ∇ is close enough to an acyclic Hermitian connection, then $\mathcal{M}(\nabla) \neq \emptyset$ and, it follows from the discussion of the metric anomaly of the graded determinant of B_{even} in Subsection 2.5, that $T(\nabla)$ *is independent of the choice of the admissible metric* g^M. Moreover, as $\operatorname{Det}_{\text{gr},\theta}(B_{\text{even}})$ is independent of the choice of the Agmon angle $\theta \in (-\pi, 0)$, so is $T(\nabla)$.

A simple example in Subsection 10.2 shows that *even when the connection* ∇ *is Hermitian, the refined analytic torsion can have an arbitrary phase.*

In Section 11 we also suggest an alternative definition of the refined analytic torsion, which is more convenient for some applications.

2.7. Comparison with the Ray-Singer torsion. The equality (2.7) implies that, if ∇ is C^0-close to an acyclic Hermitian connection, then

$$\log \frac{|T(\nabla)|}{T^{\mathrm{RS}}(\nabla)} = \pi \ \mathrm{Im}\ \eta(\nabla, g^M). \tag{2.8}$$

In particular, if ∇ is an acyclic Hermitian connection, then

$$\big|\, T(\nabla) \,\big| = T^{\mathrm{RS}}(\nabla).$$

Theorem 12.8 provides a local expression for the right hand side of (2.8). Following Farber, [**17**], we denote by $\mathbf{Arg}_\nabla$ the unique cohomology class $\mathbf{Arg}_\nabla \in H^1(M, \mathbb{C}/\mathbb{Z})$ such that for every closed curve $\gamma \in M$ we have

$$\det\big(\ \mathrm{Mon}_\nabla(\gamma)\ \big) = \exp\big(\, 2\pi i \langle \mathbf{Arg}_\nabla, [\gamma] \rangle\, \big),$$

where $\mathrm{Mon}_\nabla(\gamma)$ denotes the monodromy of the flat connection ∇ along the curve γ and $\langle \cdot, \cdot \rangle$ denotes the natural pairing

$$H^1(M, \mathbb{C}/\mathbb{Z}) \ \times \ H_1(M, \mathbb{Z}) \to \mathbb{C}/\mathbb{Z}.$$

Theorem 12.8 states that, if ∇ is C^0-close to an acyclic Hermitian connection, then

$$\log \frac{|T(\nabla)|}{T^{\mathrm{RS}}(\nabla)} = \pi \left\langle\, [L(p)] \cup \mathrm{Im}\, \mathbf{Arg}_\nabla, [M] \,\right\rangle, \tag{2.9}$$

where $L(p) = L_M(p)$ denotes the Hirzebruch L-polynomial in the Pontrjagin forms of the Riemannian metric on M. If $\dim M \equiv 3 \pmod 4$, then $L(p)$ has no component of degree $\dim M - 1$ and, hence, $|T(\nabla)| = T^{\mathrm{RS}}(\nabla)$.

2.8. The refined analytic torsion as a holomorphic function on the space of representations. One of the main properties of the refined analytic torsion $T(\nabla)$ is that, in an appropriate sense, it depends holomorphically on the connection. Note, however, that the space of connections is infinite dimensional and one needs to choose an appropriate notion of a holomorphic function on such a space. A possible choice is explained in Subsection 13.1. As an alternative one can view the refined analytic torsion as a holomorphic function on a finite dimensional space, which we shall now explain.

The set $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$ of all n-dimensional complex representations of $\pi_1(M)$ has a natural structure of a complex algebraic variety, cf. Subsection 13.6. Each representation $\alpha \in \mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$ gives rise to a vector bundle E_α with a flat connection ∇_α, whose monodromy is isomorphic to α, cf. Subsection 13.6. Let

$$\mathrm{Rep}_0(\pi_1(M), \mathbb{C}^n) \ \subset \ \mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$$

denote the set of all representations $\alpha \in \mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$ such that the connection ∇_α is acyclic. We also denote by

$$\mathrm{Rep}^u(\pi_1(M), \mathbb{C}^n) \ \subset \ \mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$$

the set of all unitary representations and set

$$\mathrm{Rep}_0^u(\pi_1(M),\mathbb{C}^n)=\mathrm{Rep}^u(\pi_1(M),\mathbb{C}^n)\cap\mathrm{Rep}_0(\pi_1(M),\mathbb{C}^n).$$

Denote by $V\subset\mathrm{Rep}_0(\pi_1(M),\mathbb{C}^n)$ the set of representations α for which there exists a metric g^M so that the odd signature operator $B_{\mathrm{even}}(\nabla,g^M)$ is bijective (i.e., Assumption II of Subsection 2.1 is satisfied). It is not difficult to show, cf. Subsection 13.7, that V is an open neighborhood of the set $\mathrm{Rep}_0^u(\pi_1(M),\mathbb{C}^n)$ of acyclic unitary representations.

For every $\alpha\in V$ one defines the refined analytic torsion $T_\alpha:=T(\nabla_\alpha)$. Corollary 13.11 states that the function $\alpha\mapsto T_\alpha$ is holomorphic on the open set of all non-singular points of V.

2.9. Comparison with Turaev's torsion. In [**42**, **43**], Turaev introduced a refinement $T_\alpha^{\mathrm{comb}}(\varepsilon,\mathfrak{o})$ of the combinatorial torsion associated to an acyclic representation α of $\pi_1(M)$. This refinement depends on an additional combinatorial data, denoted by ε and called the *Euler structure* as well as on the *cohomological orientation* of M, i.e., on the orientation $\mathfrak{o}$ of the determinant line of the cohomology $H^\bullet(M,\mathbb{R})$ of M. There are two versions of the Turaev torsion – the homological and the cohomological one. In this paper it turns out to be more convenient to use the cohomological Turaev torsion as it is defined by Farber and Turaev in Section 9.2 of [**20**]. For $\alpha\in\mathrm{Rep}_0(\pi_1(M),\mathbb{C}^n)$, the cohomological Turaev torsion $T_\alpha^{\mathrm{comb}}(\varepsilon,\mathfrak{o})$ is a non-vanishing complex number.

Theorem 10.2 of [**20**] computes the quotient of the Turaev and the Ray-Singer torsions. Combined with (2.9) this result leads to the following equality (cf. Subsection 14.7):

Let $c(\varepsilon)\in H_1(M,\mathbb{Z})$ be the characteristic class of the Euler structure ε, cf. [**43**] or Section 5.2 of [**20**]. Let $L_{d-1}(p)$ denote the component of $L(p)$ in dimension $d-1$. It represents a class in $H^{d-1}(M,\mathbb{Q})$, which can be lifted to a class in the integer cohomology $H^{d-1}(M,\mathbb{Z})$, cf. Lemma 14.5. We fix such a lift and we denote by $\widehat{L}_1\in H_1(M,\mathbb{Z})$ its Poincaré dual homology class. By Corollary 14.6 there exists a homology class $\beta_\varepsilon\in H_1(M,\mathbb{Z})$ such that $-2\beta_\varepsilon=\widehat{L}_1(p)+c(\varepsilon)$. Then there exists an open neighborhood $V'\subset V$ of $\mathrm{Rep}_0^u(\pi_1(M),\mathbb{C}^n)$ such that for every $\alpha\in V'$

$$\left|\frac{T_\alpha}{T_\alpha^{\mathrm{comb}}(\varepsilon,\mathfrak{o})}\right|=\left|e^{2\pi i\langle\mathbf{Arg}_\alpha,\beta_\varepsilon\rangle}\right|,\tag{2.10}$$

where $\mathbf{Arg}_\alpha:=\mathbf{Arg}_{\nabla_\alpha}\in H^1(M,\mathbb{C}/\mathbb{Z})$ is as in Subsection 2.7 and $\langle\cdot,\cdot\rangle$ denotes the natural pairing $H^1(M,\mathbb{C}/\mathbb{Z})\times H_1(M,\mathbb{Z})\to\mathbb{C}/\mathbb{Z}$.

Let Σ denote the set of singular points of the complex analytic set $\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$. By Corollary 13.11, the refined analytic torsion T_α is a non-vanishing holomorphic function of $\alpha\in V\backslash\Sigma$. By the very construction [**42**, **43**, **20**] the Turaev torsion is a non-vanishing holomorphic

function of $\alpha \in \mathrm{Rep}_0(\pi_1(M), \mathbb{C}^n)$. Hence, $T_\alpha / T_\alpha^{\mathrm{comb}}$ is a holomorphic function on $V' \backslash \Sigma$. By construction of the cohomology class $\mathbf{Arg}_\alpha$, for every homology class $z \in H_1(M, \mathbb{Z})$, the expression $e^{2\pi i \langle \mathbf{Arg}_\alpha, z \rangle}$ is a holomorphic function on $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$.

If the absolute values of two non-vanishing holomorphic functions are equal on a connected open set then the functions must be equal up to a factor $\mu \in \mathbb{C}$ with $|\mu| = 1$. This observation and (2.10) lead to the following generalization of the Cheeger-Müller theorem, cf. Theorem 14.8:

Theorem 2.10. *For each connected component C of V', there exists a constant $\phi_C = \phi_C(\varepsilon, \mathfrak{o}) \in \mathbb{R}$, depending on ε and $\mathfrak{o}$, such that*

$$\frac{T_\alpha}{T_\alpha^{\mathrm{comb}}(\varepsilon, \mathfrak{o})} = e^{i\phi_C}\, e^{2\pi i \langle \mathbf{Arg}_\alpha, \beta_\varepsilon \rangle}. \tag{2.11}$$

In the case when $\dim M \equiv 3 (\mathrm{mod}\, 4)$ and $c(\varepsilon) = 0$ formula (2.11) simplifies, cf. Corollary 14.6.

2.11. Application: Phase of the Turaev torsion of a unitary representation. We denote the phase of a complex number z by

$$\mathbf{Ph}(z) \ \in \ [0, 2\pi)$$

so that $z = |z| e^{i\, \mathbf{Ph}(z)}$. Set $\eta_\alpha := \eta(\nabla_\alpha, g^M)$.

Suppose $\alpha_1,\ \alpha_2 \in \mathrm{Rep}_0^u(\pi_1(M), \mathbb{C}^n)$ are unitary representations which lie in the same connected component of V', where $V' \subset V$ is the open neighborhood of $\mathrm{Rep}_0^u(\pi_1(M), \mathbb{C}^n)$ defined in Subsection 2.9. As an application of (2.11) one obtains, cf. Theorem 14.13, that, modulo $2\pi\,\mathbb{Z}$,

$$\begin{aligned} \mathbf{Ph}(T_{\alpha_1}^{\mathrm{comb}}(\varepsilon, \mathfrak{o})) + \pi\, \eta_{\alpha_1} + 2\pi \left\langle \mathbf{Arg}_{\alpha_1}, \beta_\varepsilon \right\rangle \\ \equiv\ \mathbf{Ph}(T_{\alpha_2}^{\mathrm{comb}}(\varepsilon, \mathfrak{o})) + \pi\, \eta_{\alpha_2} + 2\pi \left\langle \mathbf{Arg}_{\alpha_2}, \beta_\varepsilon \right\rangle. \end{aligned} \tag{2.12}$$

2.12. Sign of the absolute torsion and a theorem of Farber. Suppose that the Stiefel-Whitney class $w_{d-1}(M) \in H^{d-1}(M, \mathbb{Z}_2)$ vanishes (which is always the case when $\dim M \equiv 3 \pmod 4$, cf. [**32**]). Then one can choose an Euler structure ε such that $c(\varepsilon) = 0$, cf. [**19**, §3.2]. Assume, in addition, that the first Stiefel-Whitney class $w_1(E_\alpha)$, viewed as a homomorphism $H_1(M, \mathbb{Z}) \to \mathbb{Z}_2$, vanishes on the 2-torsion subgroup of $H_1(M, \mathbb{Z})$. In this case there is also a canonical choice of the cohomological orientation $\mathfrak{o}$, cf. [**19**, §3.3]. Then the Turaev torsions $T_\alpha^{\mathrm{comb}}(\varepsilon, \mathfrak{o})$ corresponding to different choices of ε with $c(\varepsilon) = 0$ and the canonically chosen $\mathfrak{o}$ will be the same.

If the above assumptions on $w_{d-1}(M)$ and $w_1(E_\alpha)$ are satisfied, then the number

$$T_\alpha^{\mathrm{abs}} := T_\alpha^{\mathrm{comb}}(\varepsilon, \mathfrak{o}) \ \in \ \mathbb{C} \backslash \{0\}, \qquad (c(\varepsilon) \,=\, 0),$$

is canonically defined, i.e., is independent of any choices. It was introduced by Farber and Turaev, [**19**], who called it the *absolute torsion*[3] . If $\alpha \in \mathrm{Rep}_0^u(\pi_1(M), \mathbb{C}^n)$, then T_α^{abs} is a real number, cf. Theorem 3.8 of [**19**]. In Subsection 14.14 we show that, under the above assumptions, (2.12) implies that *if* α_1, $\alpha_2 \in \mathrm{Rep}_0^u(\pi_1(M), \mathbb{C}^n)$ *are unitary representations which lie in the same connected component of* V' *then the following statements hold:*

1) *in the case* $\dim M \equiv 3 \pmod 4$

$$\mathrm{sign}\,\big(T_{\alpha_1}^{\mathrm{abs}}\big) \cdot e^{i\pi\eta_{\alpha_1}} = \mathrm{sign}\,\big(T_{\alpha_2}^{\mathrm{abs}}\big) \cdot e^{i\pi\eta_{\alpha_2}},$$

2) *in the case* $\dim M \equiv 1 \pmod 4$

$$\mathrm{sign}\,\big(T_{\alpha_1}^{\mathrm{abs}}\big) \cdot e^{i\pi\left(\eta_{\alpha_1} - \langle [L(p)] \cup \mathbf{Arg}_{\alpha_1}, [M]\rangle\right)} = \mathrm{sign}\,\big(T_{\alpha_2}^{\mathrm{abs}}\big) \cdot e^{i\pi\left(\eta_{\alpha_2} - \langle [L(p)] \cup \mathbf{Arg}_{\alpha_2}, [M]\rangle\right)}.$$

For the special case when there is a real analytic path α_t of unitary representations connecting α_1 and α_2 such that the twisted deRham complex (6.63) is acyclic for all but finitely many values of t, Theorem 14.15 was established by Farber, using a completely different method,[4] see [**17**], Theorems 2.1 and 3.1.

3. Preliminaries on Determinants of Elliptic Operators

In this section we briefly review the main facts about the ζ-regularized determinants of elliptic operators. At the end of the section (cf. Subsection 3.11) we define a sign-refined version of the graded determinant – a notion, which plays a central role in this paper.

3.1. Setting. Throughout this paper let E be a complex vector bundle over a smooth compact manifold M and let $D : C^\infty(M, E) \to C^\infty(M, E)$ be an elliptic differential operator of order $m \geq 1$. Denote by $\sigma(D)$ the leading symbol of D.

3.2. Choice of an angle. Our aim is to define the ζ-function and the determinant of D. For this we will need to define the complex powers of D. As usual, to define complex powers we need to choose a *spectral cut* in the complex plane. We will restrict ourselves to the simplest spectral cuts given by a ray

$$R_\theta = \big\{\rho e^{i\theta} : 0 \leq \rho < \infty\big\}, \qquad 0 \leq \theta < 2\pi. \tag{3.13}$$

Consequently, we have to choose an angle $\theta \in [0, 2\pi)$.

[3] In [**19**], Farber and Turaev defined the absolute torsion also in the case when the representation α is not necessarily acyclic.

[4] Note that Farber's definition of the η-invariant differs from ours by a factor of 2 and also that the sign in front of $\langle [L(p)] \cup \mathbf{Arg}_{\alpha_1}, [M]\rangle$ in [**17**] has to be replaced by the opposite one.

Definition 3.3. The angle θ is a *principal angle* for an elliptic operator D if

$$\operatorname{spec}\big(\sigma(D)(x,\xi)\big) \cap R_\theta = \emptyset, \qquad \text{for all} \quad x \in M,\ \xi \in T^*_x M \backslash \{0\}.$$

If $\mathcal{I} \subset \mathbb{R}$ we denote by $L_{\mathcal{I}}$ the solid angle

$$L_{\mathcal{I}} = \big\{\rho e^{i\theta} : 0 < \rho < \infty,\, \theta \in \mathcal{I}\big\}.$$

Definition 3.4. The angle θ is an *Agmon angle* for an elliptic operator D if it is a principal angle for D and there exists $\varepsilon > 0$ such that

$$\operatorname{spec}(D) \cap L_{[\theta-\varepsilon,\theta+\varepsilon]} = \emptyset.$$

3.5. ζ-function and determinant. Let θ be an Agmon angle[5] for D. Assume, in addition, that D is injective. In this case, the ζ-function $\zeta_\theta(s, D)$ of D is defined as follows.

Since D is invertible, there exists a small number $\rho_0 > 0$ such that

$$\operatorname{spec}(D) \cap \big\{z \in \mathbb{C};\, |z| < 2\rho_0\big\} = \emptyset.$$

Define the contour $\Gamma = \Gamma_{\theta,\rho_0} \subset \mathbb{C}$ consisting of three curves

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3,$$

where

$$\Gamma_1 = \big\{\rho e^{i\theta} : \infty > \rho \geq \rho_0\big\}, \quad \Gamma_2 = \big\{\rho_0 e^{i\alpha} : \theta < \alpha < \theta + 2\pi\big\}, \tag{3.14}$$
$$\Gamma_3 = \big\{\rho e^{i(\theta+2\pi)} : \rho_0 \leq \rho < \infty\big\}.$$

For $\operatorname{Re} s > \frac{\dim M}{m}$, the operator

$$D_\theta^{-s} = \frac{i}{2\pi} \int_{\Gamma_{\theta,\rho_0}} \lambda^{-s} (D - \lambda)^{-1}\, d\lambda \tag{3.15}$$

is a pseudo-differential operator with continuous kernel $D_\theta^{-s}(x, y)$, cf. [**39**, **40**]. When the angle θ is fixed we will often write D^{-s} for D_θ^{-s}.

We define

$$\zeta_\theta(s, D) = \operatorname{Tr} D_\theta^{-s} = \int_M \operatorname{tr} D_\theta^{-s}(x, x)\, dx, \qquad \operatorname{Re} s > \frac{\dim M}{m}. \tag{3.16}$$

It was shown by Seeley [**39**] (see also [**40**]) that $\zeta_\theta(s, D)$ has a meromorphic extension to the whole complex plane and that 0 is a regular value of $\zeta_\theta(s, D)$.

More generally, let Q be a pseudo-differential operator of order q. We set

$$\zeta_\theta(s, Q, D) = \operatorname{Tr} Q\, D_\theta^{-s}, \qquad \operatorname{Re} s > (q + \dim M)/m. \tag{3.17}$$

[5]The existence of an Agmon angle is an additional assumption on D, though a very mild one. In particular, if D possesses a principal angle it also possesses an Agmon angle, cf. the discussion in Subsection 3.10.

This function also has a meromorphic extension to the whole complex plane, cf. [**47**, §3.22], [**24**, Th. 2.7], and [**25**]. Moreover, if Q is a 0-th order pseudo-differential projection, i.e., a 0-th order pseudo-differential operator satisfying $Q^2 = Q$, then by [**46**, §7], [**47**] (see also [**9**, **34**]), $\zeta_\theta(s, Q, D)$ is regular at 0.

Definition 3.6. The ζ-regularized determinant of D is defined by the formula

$$\mathrm{Det}_\theta(D) := \exp\left(-\frac{d}{ds}\big|_{s=0}\zeta_\theta(s, D)\right). \tag{3.18}$$

Roughly speaking, (3.18) says that the logarithm $\log \mathrm{Det}_\theta(D)$ of the determinant of D is equal to $-\zeta'_\theta(0, D)$. However, the logarithm is a multivalued function. Hence, $\log \mathrm{Det}_\theta(D)$ is defined only up to a multiple of $2\pi i$, while $-\zeta'_\theta(0, D)$ is a well defined complex number. We denote by $\mathrm{LDet}_\theta(D)$ the particular value of the logarithm of the determinant such that

$$\mathrm{LDet}_\theta(D) = -\zeta'_\theta(0, D). \tag{3.19}$$

Let us emphasize that the equality (3.19) is the definition of the number $\mathrm{LDet}_\theta(D)$.

We will need the following generalization of Definition 3.6.

Definition 3.7. Suppose Q is a 0-th order pseudo-differential projection commuting with D. Then $V := \mathrm{Im}\, Q$ is a D invariant subspace of $C^\infty(M, E)$. The ζ-regularized determinant of the restriction $D|_V$ of D to V is defined by the formula

$$\mathrm{Det}_\theta(D|_V) := e^{\mathrm{LDet}_\theta(D|_V)}, \tag{3.20}$$

where

$$\mathrm{LDet}_\theta(D|_V) = -\frac{d}{ds}\big|_{s=0}\zeta_\theta(s, Q, D). \tag{3.21}$$

Remark 3.8. From the representation of $\zeta_\theta(s, Q, D)$ for $\mathrm{Re}\, s > \frac{\dim M}{m}$ by the eigenvalues of $D|_V$, cf. (3.26) below, it follows that the right hand side of (3.21) is independent of Q except through $\mathrm{Im}(Q)$. This justifies the notation $\mathrm{LDet}_\theta(D|_V)$. However, we need to know that V is the image of a 0-th order pseudo-differential projection Q to ensure that $\zeta_\theta(s, D)$ has a meromorphic extension to the whole s-plane with $s = 0$ being a regular point.

3.9. Case of a self-adjoint leading symbol. Let us assume now that

$$\sigma(D)^*(x, \xi) = \sigma(D)(x, \xi), \qquad (x, \xi) \in T^*M, \tag{3.22}$$

where $\sigma(D)^*(x, \xi)$ denotes the adjoint of the linear operator $\sigma(D)(x, \xi)$ with respect to some fixed scalar product on the fibers on E. This assumption implies that D can be written as a sum $D = D' + A$ where

D' is a self-adjoint differential operator of order m and A is a differential operator of order smaller than m.

Though the operator D is not self-adjoint in general, the assumption (3.22) guarantees that it has nice spectral properties. More precisely, cf. [**31**, §I.6], the space $L^2(M,E)$ of square integrable sections of E is the closure of the algebraic direct sum of finite dimensional D-invariant subspaces

$$L^2(M,E) = \overline{\bigoplus_{k\geq 1} \Lambda_k} \tag{3.23}$$

such that the restriction of D to Λ_k has a unique eigenvalue λ_k and $\lim_{k\to\infty} |\lambda_k| = \infty$. In general, the sum (3.23) is not a sum of mutually orthogonal subspaces.

The space Λ_k are called the *space of root vectors of D with eigenvalue λ_k*. We call the dimension of the space Λ_k the (*algebraic*) *multiplicity* of the eigenvalue λ_k and we denote it by m_k.

Assume now that θ is an Agmon angle for D. As, for $\operatorname{Re} s > \frac{\dim M}{m}$, the operator D_θ^{-s} is of trace class, we conclude by Lidskii's theorem, [**30**], [**36**, Ch. XI], that the ζ-function (3.16) is equal to the sum (including the algebraic multiplicities) of the eigenvalues of D_θ^{-s}. Hence,

$$\zeta_\theta(s,D) = \sum_{k=1}^{\infty} m_k\ (\lambda_k)_\theta^{-s} = \sum_{k=1}^{\infty} m_k\ e^{-s\log_\theta \lambda_k}, \tag{3.24}$$

where $\log_\theta(\lambda)$ denotes the branch of the logarithm in $\mathbb{C}\backslash R_\theta$ with

$$\theta < \operatorname{Im}\log_\theta(\lambda) < \theta + 2\pi.$$

3.10. Dependence of the determinant on the angle. Assume now that θ is only a principal angle for D. Then, cf. [**39**, **40**], there exists $\varepsilon > 0$ such that $\operatorname{spec}(D)\cap L_{[\theta-\varepsilon,\theta+\varepsilon]}$ is finite and $\operatorname{spec}(\sigma(D))\cap L_{[\theta-\varepsilon,\theta+\varepsilon]} = \emptyset$. Thus we can choose an Agmon angle $\theta' \in (\theta-\varepsilon, \theta+\varepsilon)$ for D. In this subsection we show that $\operatorname{Det}_{\theta'}(D)$ is independent of the choice of this angle θ'. For simplicity, we will restrict ourselves to the case when D has a self-adjoint leading symbol.

Let $\theta'' > \theta'$ be another Agmon angle for D in $(\theta-\varepsilon, \theta+\varepsilon)$. Then there are only finitely many eigenvalues $\lambda_{r_1}, \dots, \lambda_{r_l}$ of D in the solid angle $L_{[\theta',\theta'']}$. We have

$$\log_{\theta''}\lambda_k = \begin{cases} \log_{\theta'}\lambda_k, & \text{if } k \notin \{r_1,\dots,r_l\}; \\ \log_{\theta'}\lambda_k + 2\pi i, & \text{if } k \in \{r_1,\dots,r_l\}. \end{cases} \tag{3.25}$$

Hence

$$\zeta'_{\theta'}(0,D)-\zeta'_{\theta''}(0,D) = \frac{d}{ds}\big|_{s=0}\sum_{i=1}^{l} m_{r_i}\, e^{-s\log_{\theta'}(\lambda_{r_i})}(1-e^{-2\pi i s}) = 2\pi i\sum_{i=1}^{l} m_{r_i} \tag{3.26}$$

and, by Definition 3.6,

$$\mathrm{Det}_{\theta''}(D)=\mathrm{Det}_{\theta'}(D). \tag{3.27}$$

Note that the equality (3.27) holds only because both angles θ' and θ'' are close to a given principal angle θ so that the intersection $\mathrm{spec}\,(D)\cap L_{[\theta',\theta'']}$ is finite. If there are infinitely many eigenvalues of D in the solid angle $L_{[\theta',\theta'']}$ then $\mathrm{Det}_{\theta''}(D)$ and $\mathrm{Det}_{\theta'}(D)$ might be different.

3.11. Graded determinant. Let $D: C^\infty(M,E)\to C^\infty(M,E)$ be a differential operator with a self-adjoint leading symbol. Suppose that $Q_j: C^\infty(M,E)\to C^\infty(M,E)$ $(j=0,\dots,d)$ are 0-th order pseudo-differential projections commuting with D. Set $V_j:=\mathrm{Im}\,Q_j$ and assume that

$$C^\infty(M,E)=\bigoplus_{j=0}^{d} V_j.$$

Definition 3.12. Assume that D is injective and that $\theta\in[0,2\pi)$ is an Agmon angle for the operator $(-1)^j D|_{V_j}$, for every $j=0,\dots,d$. The *graded determinant* $\mathrm{Det}_{\mathrm{gr},\theta}(D)$ of D (with respect to the grading defined by the pseudo-differential projections Q_j) is defined by the formula

$$\mathrm{Det}_{\mathrm{gr},\theta}(D):=e^{\mathrm{LDet}_{\mathrm{gr},\theta}(D)}, \tag{3.28}$$

where

$$\mathrm{LDet}_{\mathrm{gr},\theta}(D):=\sum_{j=0}^{d}(-1)^j\ \mathrm{LDet}_\theta\ \big((-1)^j D|_{V_j}\big). \tag{3.29}$$

The following is an important example of the above situation: Let $E=\bigoplus_{j=0}^{d}E_j$ be a *graded* vector bundle over M. Suppose that for each $j=0,\dots,d$, there is an injective elliptic differential operator

$$D_j:\ C^\infty(M,E_j)\longrightarrow C^\infty(M,E_j),$$

such that $\theta\in[0,2\pi)$ is an Agmon angle for $(-1)^j D_j$ for all $j=0,\dots,d$. We denote by

$$D=\bigoplus_{j=0}^{d} D_j:\ C^\infty(M,E)\longrightarrow C^\infty(M,E) \tag{3.30}$$

the direct sum of the operators D_j. Then (3.29) reduces to

$$\mathrm{LDet}_{\mathrm{gr},\theta}(D) := \sum_{j=0}^{d} (-1)^j \ \mathrm{LDet}_\theta \ \big((-1)^j D_j\big). \tag{3.31}$$

4. The η-invariant of a non Self-Adjoint Operator and the Determinant

It is well known, cf. [**41**, **48**], that the phase of the determinant of a self-adjoint elliptic differential operator D can be expressed in terms of the η-invariant of D and the ζ-function of D^2. In this section we extend this result to non self-adjoint operators.

Throughout this section we use the notation introduced in Section 3 and assume that $D : C^\infty(M, E) \to C^\infty(M, E)$ is an elliptic differential operator of order m with self-adjoint leading symbol, cf. Subsection 3.9. We also assume that 0 is not in the spectrum of D.

4.1. η-invariant. First, we recall the definition of the η-function of D for a non-self-adjoint operator, cf. [**21**].

Definition 4.2. Let θ be an Agmon angle for D, cf. Definition 3.4. Using the spectral decomposition of D defined in Subsection 3.9, we define the η-function of D by the formula

$$\eta_\theta(s, D) = \sum_{\mathrm{Re}\,\lambda_k > 0} m_k \ (\lambda_k)_\theta^{-s} - \sum_{\mathrm{Re}\,\lambda_k < 0} m_k \ (-\lambda_k)_\theta^{-s}. \tag{4.32}$$

Note that, by definition, the purely imaginary eigenvalues of D do not contribute to $\eta_\theta(s, D)$.

It was shown by Gilkey, [**21**], that $\eta_\theta(s, D)$ has a meromorphic extension to the whole complex plane $\mathbb{C}$ with isolated simple poles, and that it is regular at 0. Moreover, the number $\eta_\theta(0, D)$ is independent of the Agmon angle θ.

Since the leading symbol of D is self-adjoint, the angles $\pm\pi/2$ are principal angles for D, cf. Definition 3.3. In particular, there are at most finitely many eigenvalues of D on the imaginary axis.

Let m_+ (respectively, m_-) denote the number of eigenvalues (counted with their algebraic multiplicities, cf. Subsection 3.9) of D on the positive (respectively, negative) part of the imaginary axis.

Definition 4.3. The η-invariant $\eta(D)$ of D is defined by the formula

$$\eta(D) = \frac{\eta_\theta(0, D) + m_+ - m_-}{2}. \tag{4.33}$$

In view of (3.25), $\eta(D)$ is independent of the angle θ.

Let $D(t)$ be a smooth 1-parameter family of operators. Then $\eta(D(t))$ is not smooth but may have integer jumps when eigenvalues cross the

imaginary axis. Because of this, the η-invariant is usually considered modulo integers. However, in this paper we will be interested in the number $e^{i\pi\eta(D)}$, which changes its sign when $\eta(D)$ is changed by an odd integer. Hence, we will consider the η-invariant as a complex number.

Remark 4.4. Note that our definition of $\eta(D)$ is slightly different from the one suggested by Gilkey in [**21**]. In fact, in our notation, Gilkey's definition is $\eta(D) + m_-$. Hence, reduced modulo integers the two definitions coincide. However, the number $e^{i\pi\eta(D)}$ will be multiplied by $(-1)^{m_-}$ if we replace one definition by the other. In this sense, Definition 4.3 can be viewed as a *sign refinement* of the definition given in [**21**].

4.5. Relationship between the η-invariant and the determinant. Since the leading symbol of D is self-adjoint, the angles $\pm\pi/2$ are principal for D. Hence, cf. Subsection 3.10, there exists an Agmon angle $\theta \in (-\pi/2, 0)$ such that there are no eigenvalues of D in the solid angles $L_{(-\pi/2,\theta]}$ and $L_{(\pi/2,\theta+\pi]}$. Then 2θ is an Agmon angle for the operator D^2.

Theorem 4.6. *Let $D : C^\infty(M, E) \to C^\infty(M, E)$ be a bijective elliptic differential operator of order m with self-adjoint leading symbol. Let $\theta \in (-\pi/2, 0)$ be an Agmon angle for D such that there are no eigenvalues of D in the solid angles $L_{(-\pi/2,\theta]}$ and $L_{(\pi/2,\theta+\pi]}$ (hence, there are no eigenvalues of D^2 in the solid angle $L_{(-\pi,2\theta]}$). Then* [6]

$$\mathrm{LDet}_\theta(D) = \frac{1}{2}\,\mathrm{LDet}_{2\theta}(D^2) - i\pi\left(\eta(D) - \frac{1}{2}\zeta_{2\theta}(0, D^2)\right). \tag{4.34}$$

In particular,

$$\mathrm{Det}_\theta(D) = e^{-\frac{1}{2}\,\zeta'_{2\theta}(0,D^2)} \cdot e^{-i\pi\,(\eta(D)-\frac{1}{2}\zeta_{2\theta}(0,D^2))}. \tag{4.35}$$

Remark 4.7.

a. Let θ be as in Theorem 4.6 and suppose that $\theta' \in (-\pi, 0)$ is another angle such that both θ' and $\theta' + \pi$ are Agmon angles for D. Then, by (3.26) and (3.27),

$$\begin{aligned} &\mathrm{Det}_{\theta'}(D) = \mathrm{Det}_\theta(D), \\ &\zeta'_{2\theta}(0, D^2) \equiv \zeta'_{2\theta'}(0, D^2) \qquad \mod 2\pi i. \end{aligned} \tag{4.36}$$

In particular,

$$e^{-\frac{1}{2}\,\zeta'_{2\theta'}(0,D^2)} = \pm\, e^{-\frac{1}{2}\,\zeta'_{2\theta}(0,D^2)}. \tag{4.37}$$

[6] Recall that we denote by $\mathrm{LDet}_\theta(D)$ the particular branch of the logarithm of the determinant of D defined by formula (3.19).

Clearly, $\zeta_{\theta_1}(0,D^2)=\zeta_{\theta_2}(0,D^2)$ if there are finitely many eigenvalues of D^2 in the solid angle $L_{[\theta_1,\theta_2]}$. Hence, $\zeta_{2\,\theta}\,(0,\,D^2)=\zeta_{2\theta'}(0,D^2)$. We then conclude from (4.35), (4.36), and (4.37) that

$$\mathrm{Det}_{\theta'}(D)=\pm\, e^{-\frac{1}{2}\,\zeta'_{2\theta'}(0,D^2)}\cdot e^{-i\pi\,(\eta(D)-\frac{\zeta_{2\theta'}(0,D^2)}{2})}. \tag{4.38}$$

In other words, for (4.35) to hold we need the precise assumptions on θ which are specified in Theorem 4.6. But "up to a sign" it holds for every spectral cut in the lower half plane.

b. If instead of the spectral cut R_θ in the lower half-plane we use the spectral cut $R_{\theta+\pi}$ in the upper half-plane we will get a similar formula

$$\mathrm{LDet}_{\theta+\pi}(D)=\frac{1}{2}\,\mathrm{LDet}_{2\theta}(D^2)+i\pi\,\left(\eta(D)-\frac{1}{2}\zeta_{2\theta}(0,D^2)\right), \tag{4.39}$$

whose proof is a verbatim repetition of the proof of (4.34), cf. below.

c. *If the dimension of M is odd*, then the ζ-function of an elliptic differential operator of even order vanishes at 0, cf. [**39**]. In particular, $\zeta_{2\theta}(0,D^2)=0$. Hence, (4.34) simplifies to

$$\mathrm{LDet}_{\theta}(D)=\frac{1}{2}\,\mathrm{LDet}_{2\theta}(D^2)-i\pi\,\eta(D). \tag{4.40}$$

Proof. Let Π_+ and Π_- denote the spectral projections of D corresponding to the solid angles $L_{(-\pi/2,\pi/2)}$ and $L_{(\pi/2,3\pi/2)}$ respectively. Let P_+ and P_- denote the spectral projections of D corresponding to the rays $R_{\pi/2}$ and $R_{-\pi/2}$ respectively (here we use the notation introduced in (3.13)). Set $\tilde\Pi_\pm=\Pi_\pm+P_\pm$. Since D is injective $\mathrm{Id}=\tilde\Pi_++\tilde\Pi_-$. Clearly

$$\begin{aligned}\zeta_\theta(s,D)&=\mathrm{Tr}\left[\tilde\Pi_+\,D_\theta^{-s}\right]+e^{-i\pi s}\,\mathrm{Tr}\left[\tilde\Pi_-\,(-D)_\theta^{-s}\right];\\ \zeta_{2\theta}(s/2,D^2)&=\mathrm{Tr}\left[\tilde\Pi_+\,D_\theta^{-s}\right]+\mathrm{Tr}\left[\tilde\Pi_-\,(-D)_\theta^{-s}\right].\end{aligned}$$

Hence, using the notation introduced in (3.17), we obtain

$$\begin{aligned}\zeta_\theta(s,D)&=\zeta_\theta(s,\tilde\Pi_+,D)+e^{-i\pi s}\,\zeta_\theta(s,\tilde\Pi_-,-D);\\ \zeta_{2\theta}(s/2,D^2)&=\zeta_\theta(s,\tilde\Pi_+,D)+\zeta_\theta(s,\tilde\Pi_-,-D).\end{aligned} \tag{4.41}$$

As, by assumption, the solid angles $L_{(-\pi/2,\theta]}$ and $L_{(\pi/2,\theta+\pi]}$ do not contain eigenvalues of D, it follows that

$$\begin{aligned}\eta(s,D)&=\mathrm{Tr}\left[\Pi_+D_\theta^{-s}\right]-\mathrm{Tr}\left[\Pi_-(-D)_\theta^{-s}\right]\\ &=\zeta_\theta(s,\Pi_+,D)-\zeta_\theta(s,\Pi_-,-D).\end{aligned} \tag{4.42}$$

Recall that the projectors $P_\pm$ have finite rank, which we denoted by $m_\pm$, cf. Subsection 4.1. Hence,

$$\zeta_\theta(0,P_\pm,\pm D)=\mathrm{rank}\,P_\pm=m_\pm.$$

Combining this equality with (4.42), and using (4.33), we obtain

$$\eta(D) = \frac{\zeta_\theta(0,\tilde{\Pi}_+,D) - \zeta_\theta(0,\tilde{\Pi}_-,-D)}{2}. \tag{4.43}$$

From (4.41) and (4.43), we get

$$\begin{aligned}\zeta'_\theta(0,D) &= \zeta'_\theta(0,\tilde{\Pi}_+,D) \\ &\quad + \zeta'_\theta(0,\tilde{\Pi}_-,-D) - i\pi\,\zeta_\theta(0,\tilde{\Pi}_-,-D) \\ &= \frac{1}{2}\,\zeta'_{2\theta}(0,D^2) \;-\; i\pi\,\Big(\frac{\zeta_\theta(0,\tilde{\Pi}_+,D)+\zeta_\theta(0,\tilde{\Pi}_-,-D)}{2} \\ &\quad - \frac{\zeta_\theta(0,\tilde{\Pi}_+,D)-\zeta_\theta(0,\tilde{\Pi}_-,-D)}{2}\Big) \\ &= \frac{1}{2}\,\zeta'_{2\theta}(0,D^2) - i\pi\,\Big(\frac{1}{2}\zeta_{2\theta}(0,D^2) - \eta(D)\Big).\end{aligned} \tag{4.44}$$

Since, by definition (3.19) of the logarithm of the determinant

$$\mathrm{LDet}_\theta(D) = -\,\zeta'_\theta(0,D),$$

equality (4.34) follows from (4.44). q.e.d.

4.8. Determinant of a self-adjoint operator and the η-invariant. If the operator D is, in addition, self-adjoint, then $\eta(D)$ and $\zeta_{2\theta}(0,D^2)$ are real and the number $\mathrm{Det}_{2\theta}(D^2)$ is positive, cf. Corollary A.3 in Section A. Hence, formula (4.35) leads to

$$|\,\mathrm{Det}_\theta(D)| = \sqrt{\mathrm{Det}_{2\theta}(D^2)}, \tag{4.45}$$

$$\mathbf{Ph}\,\big(\,\mathrm{Det}_\theta(D)\big) \equiv -\pi\,\Big(\eta(D) - \frac{1}{2}\zeta_{2\theta}(0,D^2)\Big), \qquad \mathrm{mod}\ 2\pi, \tag{4.46}$$

where $\mathbf{Ph}\,\big(\,\mathrm{Det}_\theta(D)\big)$ denotes the phase of the complex number $\mathrm{Det}_\theta(D)$.

If D is not self-adjoint, (4.45) is not true in general, because the numbers $\mathrm{LDet}_{2\theta}(D^2)$, $\eta(D)$, and $\zeta_{2\theta}(0,D^2)$ need not be real. However, they are real and a version of (4.45) and (4.46) holds for a class of injective elliptic differential operators *whose spectrum is symmetric with respect to the real axis*. Though we will not use this result we present it in the Appendix A for the sake of completeness.

4.9. η-invariant and graded determinant. Suppose now that $D = \bigoplus_{j=0}^{d} D_j$ as in (3.30). Choose $\theta \in (-\pi/2, 0)$ such that there are no eigenvalues of D_j in the solid angles $L_{(-\pi/2,\theta]}$ and $L_{(\pi/2,\theta+\pi]}$ for every $0 \le j \le d$. From the definition of the η-invariant it follows that

$$\eta\big(\pm D_j\big) = \pm\,\eta(D_j).$$

Combining this equality with (3.29) and (4.34) we obtain

$$\mathrm{LDet}_{\mathrm{gr},\theta}(D) = \frac{1}{2}\sum_{j=0}^{d}(-1)^j\ \mathrm{LDet}_{2\theta}(D_j^2) - i\pi\ \left(\eta(D) - \frac{1}{2}\sum_{j=0}^{d}(-1)^j\zeta_{2\theta}(0,D_j^2)\right), \tag{4.47}$$

where

$$\eta(D) = \sum_{j=0}^{d}\eta(D_j)$$

is the η-invariant of the operator $D = \bigoplus_{j=0}^{d} D_j$.

Finally, note that, as in Remark 4.7.c, if the dimension of M is odd, then $\zeta_{2\theta}(0, D_j^2) = 0$, and (4.47) takes the form

$$\mathrm{LDet}_{\mathrm{gr},\theta}(D) = \frac{1}{2}\sum_{j=0}^{d}(-1)^j\ \mathrm{LDet}_{2\theta}(D^2) - i\pi\,\eta(D). \tag{4.48}$$

4.10. Generalization. All the constructions and theorems of this section easily generalize to operators acting on a subspace of the space $C^\infty(M,E)$ of sections of E.

Let $D : C^\infty(M,E) \to C^\infty(M,E)$ be an injective elliptic differential operator with a self-adjoint leading symbol. Let $Q : C^\infty(M,E) \to C^\infty(M,E)$ be a 0-th order pseudo-differential projection commuting with D. Then $V := \mathrm{Im}\,Q \subset C^\infty(M,E)$ is a D-invariant subspace. Hence, the decomposition (3.23) implies that

$$V = \overline{\bigoplus_{k\geq 1}\ (\,\Lambda_k \cap V\,)}$$

and the restriction $D|_V$ of D to V has the same eigenvalues $\lambda_1, \lambda_2, \ldots$ as D but with new multiplicities $m_1^V, m_2^V, \ldots$. Note, that now $m_i^V \geq 0$ might vanish for certain i's. Let m_+^V (respectively, m_-^V) denote the number of eigenvalues (counted with their algebraic multiplicities) of $D|_V$ on the positive (respectively, negative) part of the imaginary axis. Set

$$\eta_\theta(s, D|_V) = \sum_{\mathrm{Re}\,\lambda_k>0} m_k^V\ (\lambda_k)_\theta^{-s} - \sum_{\mathrm{Re}\,\lambda_k<0} m_k^V\ (-\lambda_k)_\theta^{-s}, \tag{4.49}$$

$$\eta(D|_V) = \frac{\eta_\theta(0, D|_V) + m_+^V - m_-^V}{2}.$$

A verbatim repetition of the proof of Theorem 4.6 implies

$$\mathrm{LDet}_\theta(D|_V) = \frac{1}{2}\,\mathrm{LDet}_{2\theta}(D^2|_V) - i\pi\left(\eta(D|_V) - \frac{1}{2}\zeta_{2\theta}(0, D^2|_V)\right), \tag{4.50}$$

where we used the notation

$$\zeta_{2\theta}(s, D^2|_V) = \zeta_{2\theta}(s, Q, D^2), \tag{4.51}$$

cf. (3.17).

Finally, suppose that $V = \bigoplus_{j=0}^d V_j$ is given as in Definition 3.12. Then

$$\mathrm{LDet}_{\mathrm{gr},\theta}(D) = \frac{1}{2}\sum_{j=0}^d (-1)^j\,\mathrm{LDet}_{2\theta}(D^2|_{V_j}) - i\pi\left(\eta(D) - \frac{1}{2}\sum_{j=0}^d (-1)^j \zeta_{2\theta}(0, D^2|_{V_j})\right). \tag{4.52}$$

Note, however, that an analogue of (4.48) does not necessarily hold in this case even if $\dim M$ is odd, because $\zeta_{2\theta}(s, D^2|_{V_j})$, defined by (4.51), is not a ζ-function of a *differential* operator and does not necessarily vanish at 0.

5. Determinant as a Holomorphic Function

In this section we explain that the determinant can be viewed as a holomorphic function on the space of elliptic differential operators. We also discuss some applications of this result, which will be used in Section 13 to show that the refined analytic torsion is a holomorphic function and in Section 9 for studying the dependence of the graded determinant on the Riemannian metric.

5.1. Holomorphic curves in a Fréchet space. Let $\mathcal{E}$ be a complex Fréchet space and let $\mathcal{O} \subset \mathbb{C}$ be an open set. Recall (cf., e.g., [**37**, Def. 3.30]) that a map $\gamma : \mathcal{O} \to \mathcal{E}$ is called *holomorphic* if for every $\lambda \in \mathcal{O}$ the following limit exists,

$$\lim_{\mu\to\lambda} \frac{\gamma(\mu) - \gamma(\lambda)}{\mu - \lambda}.$$

We will refer to a holomorphic map $\gamma : \mathcal{O} \to \mathcal{E}$ as a *holomorphic curve* in $\mathcal{E}$.

Let $\mathcal{Z} \subset \mathcal{E}$ be a subset of a complex Fréchet space. By a holomorphic curve in $\mathcal{Z}$ we understand a holomorphic map $\gamma : \mathcal{O} \to \mathcal{E}$ such that $\gamma(\lambda) \in \mathcal{Z}$ for all $\lambda \in \mathcal{O}$.

Suppose now that $V \subset \mathbb{C}^n$ is an open set. We call a map $f : V \to \mathcal{Z}$ holomorphic if for each holomorphic curve $\gamma : \mathcal{O} \to V$ the composition

$f \circ \gamma : \mathcal{O} \to \mathcal{Z}$ is a holomorphic curve in $\mathcal{Z}$. Note that if $\mathcal{Z} = \mathbb{C}$ then, by Hartogs' theorem (cf., e.g., [**26**, Th. 2.2.8]), the above definition is equivalent to the standard definition of a holomorphic map.

5.2. The space of smooth functions as a Fréchet space. The space $C_b^\infty(\mathbb{R}^d)$ of bounded smooth complex-valued functions on $\mathbb{R}^d$ with bounded derivatives has a natural structure of a Fréchet space (cf., e.g., [**49**, Ch. I]) with topology defined by the semi-norms

$$\|f\|_\alpha := \sup_{x\in\mathbb{R}^d} |\, \partial_x^\alpha f(x)\,|\,, \tag{5.53}$$

where $\alpha = (\alpha_1, \dots, \alpha_d) \in (\mathbb{Z}_{\geq 0})^d$ and $\partial_x^\alpha := \big(\frac{\partial}{\partial x_1}\big)^{\alpha_1} \cdots \big(\frac{\partial}{\partial x_d}\big)^{\alpha_d}$.

5.3. A Fréchet space structure on the space of differential operators. Let M be a closed d-dimensional manifold and let E be a complex vector bundle over M. Denote by $\mathrm{Diff}_m(M, E)$ the set of differential operators $D : C^\infty(M, E) \to C^\infty(M, E)$ of order $\leq m$ with smooth coefficients. It has a natural structure of a Fréchet space defined as follows. Consider a pair (ϕ, Φ) where $\phi : U \to \mathbb{R}^d$ is a diffeomorphism (with $U \subset M$ an open set), and $\Phi : E|_U \to \mathbb{C}^l \times U$ is a bundle map which identifies the restriction $E|_U$ of E to U with the trivial bundle $\mathbb{C}^l \times U \to U$. We refer to (ϕ, Φ) as a *coordinate pair*.

Using the maps ϕ and Φ we can identify the restriction of an operator $D \in \mathrm{Diff}_m(M, E)$ to U with the operator

$$D_{(\phi,\Phi)} := \sum_{|\beta|\leq m} a^\beta_{(\phi,\Phi)}(x)\ \partial_x^\beta \in \mathrm{Diff}_m(\mathbb{R}^d, \mathbb{C}^l \times \mathbb{R}^d), \tag{5.54}$$

where $|\beta| = \sum_{j=1}^d \beta_j$ and $a^\beta_{(\phi,\Phi)}(x) = \big\{\, a^\beta_{(\phi,\Phi);i,j}(x)\,\big\}_{i,j=1}^l$ are smooth bounded matrix-valued functions on $\mathbb{R}^d$, called the *coefficients* of D with respect to the coordinate pair (ϕ, Φ).

We now define a structure of a Fréchet space on $\mathrm{Diff}_m(M, E)$ using the semi-norms

$$\|D\|^\alpha_{(\phi,\Phi);\beta;i,j} := \big\|\, a^\beta_{(\phi,\Phi);i,j}\,\big\|_\alpha, \tag{5.55}$$

where (ϕ, Φ) runs over all coordinate pairs, $\alpha, \beta \in (\mathbb{Z}_{\geq 0})^d$ with $|\beta| \leq m$, $1 \leq i, j \leq l$, and the norm on the right hand side of (5.55) is defined by (5.53).

5.4. Holomorphic curves in the space of differential operators. Suppose $\mathcal{O} \subset \mathbb{C}$ is an open set and consider a map $\gamma : \mathcal{O} \to \mathrm{Diff}_m(M, E)$. For every coordinate pair (ϕ, Φ) we denote by $a^\beta_{(\phi,\Phi)}(x; \lambda)$ the coefficients of $\gamma(\lambda)$ with respect to the coordinate pair (ϕ, Φ).

Clearly, γ is a holomorphic curve in $\mathrm{Diff}_m(M, E)$ with respect to the Fréchet space structure introduced in Subsection 5.3 if and only if for every coordinate pair (ϕ, Φ), every $\beta \in (\mathbb{Z}_{\geq 0})^d$ with $|\beta| \leq m$, and every

$1 \leq i,j \leq l$, the map $\lambda \mapsto a^{\beta}_{(\phi,\Phi);i,j}(x;\lambda)$ is a holomorphic curve in $C^{\infty}_{b}(\mathbb{R}^{d})$.

The following lemma follows immediately from the definitions.

Lemma 5.5. *Let $\mathcal{O} \subset \mathbb{C}$ be an open set and, for $i = 1, 2$, let $\gamma_i : \mathcal{O} \to \mathrm{Diff}_{m_i}(M,E)$ be a holomorphic curve. Then $\lambda \mapsto \gamma(\lambda) := \gamma_1(\lambda) \circ \gamma_2(\lambda)$ is a holomorphic curve in $\mathrm{Diff}_{m_1+m_2}(M,E)$. Here $\gamma_1(\lambda) \circ \gamma_2(\lambda)$ is the composition of the differential operators $\gamma_1(\lambda)$ and $\gamma_2(\lambda)$.*

5.6. Determinant of a holomorphic curve of operators. Let

$$\mathrm{Ell}_m(M,E) \ \subset \ \mathrm{Diff}_m(M,E)$$

denote the open set of elliptic differential operators of order m and let $\mathrm{Ell}_{m,\theta}(M,E) \subset \mathrm{Ell}_m(M,E)$ be the open subset of operators which have θ as an Agmon angle. We denote by $\mathrm{Ell}'_{m,\theta}(M,E)$ the open subset of invertible operators in $\mathrm{Ell}_{m,\theta}(M,E)$. According to Subsection 3.5, the function

$$\mathrm{LDet}_{\theta} : \mathrm{Ell}'_{m,\theta}(M,E) \ \longrightarrow \ \mathbb{C}$$

is well defined. For $D \in \mathrm{Ell}_{m,\theta}(M,E)$ we set

$$\mathrm{Det}_{\theta}(D) = \begin{cases} \exp\left(\mathrm{LDet}_{\theta}(D)\right), & \text{if } D \text{ is invertible;} \\ 0, & \text{otherwise.} \end{cases} \tag{5.56}$$

Further, we denote by $\mathrm{Ell}_{m,(\theta_1,\theta_2)}(M,E) \subset \mathrm{Ell}_m(M,E)$ the open subset of operators for which all the angles $\theta \in (\theta_1,\theta_2)$ are principal, cf. Subsection 3.2. Any operator $D \in \mathrm{Ell}_{m,(\theta_1,\theta_2)}(M,E)$ has an Agmon angle $\theta \in (\theta_1,\theta_2)$ and, by (3.27), the determinant $\mathrm{Det}_{\theta}(D)$ is independent of the choice of θ in the interval (θ_1,θ_2). The following theorem is well known, cf., for example, [**29**, Corollary 4.2],

Theorem 5.7. *Let E be a complex vector bundle over a closed manifold M and let $\mathcal{O} \subset \mathbb{C}$ be an open set.*

a. *Suppose $\gamma : \mathcal{O} \to \mathrm{Ell}'_{m,\theta}(M,E)$ is a holomorphic curve in*

$$\mathrm{Ell}'_{m,\theta}(M,E) \ \subset \ \mathrm{Diff}_m(M,E).$$

Then the function $\mathcal{O} \to \mathbb{C}$, $\lambda \mapsto \mathrm{LDet}_{\theta}\left(\gamma(\lambda)\right) \in \mathbb{C}$ is holomorphic.

b. *Given angles $\theta_1 < \theta_2$ and an operator $D \in \mathrm{Ell}_{m,(\theta_1,\theta_2)}(M,E)$, denote by $\mathrm{Det}_{(\theta_1,\theta_2)}(D)$ the determinant $\mathrm{Det}_{\theta}(D)$ defined using any Agmon angle $\theta \in (\theta_1,\theta_2)$. Let $\gamma : \mathcal{O} \to \mathrm{Ell}_{m,(\theta_1,\theta_2)}(M,E)$ be a holomorphic curve in $\mathrm{Ell}_{m,(\theta_1,\theta_2)} \subset \mathrm{Diff}_m(M,E)$. Then*

$$\mathcal{O} \ \longrightarrow \ \mathbb{C}, \qquad \lambda \ \mapsto \ \mathrm{Det}_{(\theta_1,\theta_2)}\left(\gamma(\lambda)\right) \tag{5.57}$$

is a holomorphic function.

Remark 5.8. The theorem implies that the function $\mathrm{Det}_{(\theta_1,\theta_2)}(D)$ is *Gâteaux holomorphic* on $\mathrm{Ell}_{m,(\theta_1,\theta_2)}(M,E)$, cf. [**16**, Def. 3.1]. Moreover, since $\mathrm{Det}_{(\theta_1,\theta_2)}$ is continuous on $\mathrm{Ell}_{m,(\theta_1,\theta_2)}(M,E)$ it follows that this function is holomorphic in the sense of Definition 3.6 of [**16**]. However, since there seems to be no standard notion of a holomorphic function on a Fréchet space, we prefer to avoid this terminology.

Corollary 5.9. *Suppose $E \to M$ is a complex Hermitian vector bundle over a closed manifold M. Let $\mathrm{Ell}'_{m,\mathrm{sa}}(M,E)$ denote the set of invertible elliptic operators of order m with self-adjoint leading symbol and let $\gamma : \mathcal{O} \to \mathrm{Ell}'_{m,\mathrm{sa}}(M,E)$ be a holomorphic curve in $\mathrm{Ell}'_{m,\mathrm{sa}}(M,E)$. Then the function*

$$\mathcal{O} \;\longrightarrow\; \mathbb{C}, \qquad \lambda \;\mapsto\; e^{2\pi i\eta(\gamma(\lambda))} \tag{5.58}$$

is holomorphic.

Proof. By formula (4.35) of Theorem 4.6

$$e^{2\pi i\eta(\gamma(\lambda))} \;=\; \frac{\mathrm{Det}_{(-\pi,0)}\big(\gamma(\lambda)^2\big)}{\Big(\,\mathrm{Det}_{(-\pi/2,0)}\big(\gamma(\lambda)\big)\,\Big)^2}\cdot e^{i\pi\zeta_{2\theta}\left(0,\gamma(\lambda)^2\right)}. \tag{5.59}$$

By Lemma 5.5, $\lambda \mapsto \gamma(\lambda)^2$ is a holomorphic curve in $\mathrm{Ell}'_{2m,\mathrm{sa}}(M,E)$. Hence, by Theorem 5.7.b the quotient on the right hand side of (5.59) is a holomorphic function in λ. It remains to show that $\zeta_{2\theta}\big(0,\gamma(\lambda)^2\big)$ depends holomorphically on λ.

First, note that by (3.25), $\zeta_{2\theta}\big(0,\gamma(\lambda)^2\big)$ is independent of θ. By a result of Seeley [**39**] (see also [**40**]), the value $\zeta_{2\theta}\big(0,\gamma(\lambda)^2\big)$ of the zeta-function of $\gamma(\lambda)^2$ is given by a local formula, i.e., by an integral over M of a $\mathbb{C}$-valued differential form ϕ whose value at a point $x \in M$ is a rational function of the symbol of $\gamma(\lambda)$ and a finite number of its derivatives. It follows that the function $\mathcal{O} \to \mathbb{C}$, $\lambda \mapsto \zeta_{2\theta}(0,\gamma(\lambda)^2)$ is holomorphic. q.e.d.

Another important consequence of Theorem 5.7 is the following

Corollary 5.10. *Let $V \subset \mathbb{C}^n$ be an open set and let*

$$f : V \;\to\; \mathrm{Ell}_{m,(\theta_1,\theta_2)}(M,E)$$

be a holomorphic map in the sense of Subsection 5.1. *Then the set*

$$\Sigma := \big\{\lambda \in V :\, f(\lambda) \text{ is not invertible}\big\}$$

is a complex analytic subset of V. In particular, if V is connected then so is $V\backslash\Sigma$.

Proof. In view of Hartogs' theorem ([**26**, Th. 2.2.8]), Theorem 5.7.b implies that the function $V \to \mathbb{C}$, $\lambda \mapsto \mathrm{Det}_{(\theta_1,\theta_2)}(f(\lambda))$ is holomorphic on V. By (5.56), $\Sigma = \big\{\lambda \in V :\, \mathrm{Det}_{(\theta_1,\theta_2)}(f(\lambda)) = 0\big\}$. q.e.d.

6. Graded Determinant of the Odd Signature Operator

In this section we define the graded determinant of the Atiyah-Patodi-Singer odd signature operator, [**3**, **21**], of a flat vector bundle E over a closed Riemannian manifold M. In Section 8 we show that, if E admits an invariant Hermitian metric, then the absolute value of this determinant is equal to the Ray-Singer analytic torsion [**35**]. There is a similar, though slightly more complicated, relationship between the graded determinant and the Ray-Singer torsion in the general case, cf. Theorem 8.2. Thus, the graded determinant of the odd signature operator can be viewed as a *refinement of the Ray-Singer torsion*.

6.1. Setting. Let M be a smooth closed oriented manifold of odd dimension $d = 2r - 1$ and let $E \to M$ be a complex vector bundle over M endowed with a flat connection ∇. We denote by ∇ also the induced differential

$$\nabla : \Omega^\bullet(M, E) \longrightarrow \Omega^{\bullet+1}(M, E),$$

where $\Omega^k(M, E)$ denotes the space of smooth differential forms of M with values in E of degree k.

6.2. Odd signature operator. Fix a Riemannian metric g^M on M and let $* : \Omega^\bullet(M, E) \to \Omega^{d-\bullet}(M, E)$ denote the Hodge $*$-operator. Define the *chirality operator* $\Gamma : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E)$ by the formula

$$\Gamma\,\omega := i^r\,(-1)^{\frac{k(k+1)}{2}}\,*\,\omega, \qquad \omega \in \Omega^k(M, E), \tag{6.60}$$

with r given as above by $r = \frac{d+1}{2}$. This operator is equal to the operator defined in §3.2 of [**4**] as follows from applying Proposition 3.58 of [**4**] in the case $\dim M$ is odd. Note that $\Gamma^2 = 1$ and that Γ is self-adjoint with respect to the scalar product on $\Omega^\bullet(M, E)$ induced by the Riemannian metric g^M and by an arbitrary Hermitian metric on E.

Definition 6.3. The *odd signature operator* is the operator

$$B = B(\nabla, g^M) : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E)$$

defined by

$$B = \Gamma\,\nabla + \nabla\,\Gamma. \tag{6.61}$$

We denote by B_k the restriction of B to the space $\Omega^k(M, E)$.

Explicitly, for $\omega \in \Omega^k(M, E)$ one has

$$B_k\,\omega := i^r(-1)^{\frac{k(k+1)}{2}+1}\left((-1)^k * \nabla - \nabla *\right)\omega \in \Omega^{d-k-1}(M, E) \oplus \Omega^{d-k+1}(M, E). \tag{6.62}$$

The odd signature operator was introduced by Atiyah, Patodi, and Singer, [**2**, p. 44], [**3**, p. 405], in the case when E is endowed with a Hermitian metric *invariant with respect to* ∇ (i.e., invariant under parallel transport by ∇). The general case was studied by Gilkey, [**21**, p. 64–65].

Lemma 6.4. *Suppose that E is endowed with a Hermitian metric h^E. Denote by $\langle\cdot,\cdot\rangle$ the scalar product on $\Omega^\bullet(M,E)$ induced by h^E and the Riemannian metric g^M on M.*

1. *B is elliptic and its leading symbol is symmetric with respect to the Hermitian metric h^E.*
2. *If, in addition, the metric h^E is invariant with respect to the connection ∇, then B is symmetric with respect to the scalar product $\langle\cdot,\cdot\rangle$,*
$$B^* = B.$$
If the metric h^E is not invariant, then, in general, B is not symmetric.

The proof of the lemma is a simple calculation. The first part is already stated in [**3**, p. 405]. The second part is proven in the Remark on page 65 of [**21**].

6.5. Assumptions. In this paper we study the odd signature operator B and the analytic torsion under the following simplifying assumptions. The general case is addressed in [**7**].

Assumption I. The connection ∇ is *acyclic*, i.e., the *twisted deRham complex*

$$0 \to \Omega^0(M,E) \xrightarrow{\nabla} \Omega^1(M,E) \xrightarrow{\nabla} \cdots \cdots \xrightarrow{\nabla} \Omega^d(M,E) \to 0 \tag{6.63}$$

is acyclic,

$$\operatorname{Im}\big(\nabla|_{\Omega^{k-1}(M,E)}\big) = \operatorname{Ker}\big(\nabla|_{\Omega^k(M,E)}\big) \qquad \text{for every} \quad k = 1,\dots,d.$$

Assumption II. The odd signature operator

$$B : \Omega^\bullet(M,E) \to \Omega^\bullet(M,E)$$

is bijective.

6.6. Hermitian connection. Suppose that there exists a Hermitian metric h^E on E invariant with respect to ∇ (in this case we say that the connection ∇ is *Hermitian*). Then Assumption II follows from Assumption I. Indeed, in this case the operator B is symmetric with respect to the scalar product $\langle\cdot,\cdot\rangle$, defined by the metrics g^M and h^E, cf. Lemma 6.4. Hence, we only need to show that $\operatorname{Ker} B = \{0\}$. Let ∇^* denote the formal adjoint of the operator $\nabla : \Omega^\bullet(M,E) \to \Omega^{\bullet+1}(M,E)$ with respect to the scalar product $\langle\cdot,\cdot\rangle$. Since the metric h^E is flat, we obtain, cf. [**45**, §6.1],

$$\nabla^* = \Gamma\,\nabla\,\Gamma. \tag{6.64}$$

Using this identity and the definition (6.61) of B we see that

$$B^2 = \nabla^*\nabla + \nabla\nabla^* \tag{6.65}$$

is the Laplacian. Thus $\operatorname{Ker} B = \operatorname{Ker} B^2$ is isomorphic to the cohomology of the complex (6.63), and, hence, is trivial by Assumption I. Conversely, these arguments show at the same time that, in the case considered, Assumption II implies Assumption I.

6.7. Connections which are close to a Hermitian connection. In this paper we are interested in the study of connections which are *close to a Hermitian connection* in the following sense:

Let $\Omega^1(M, \operatorname{End}(E))$ denote the space of differential 1-forms on M with values in the bundle $\operatorname{End}(E)$ of endomorphisms of E. A Hermitian metric on E and a Riemannian metric on M define a natural norm $|\cdot|$ on the bundle $\Lambda^1(T^*M) \otimes \operatorname{End}(E) \to M$. Using this norm we define the sup-norm

$$\|\omega\|_{\sup} := \max_{x \in M} |\omega(x)|, \qquad \omega \in \Omega^1(M, \operatorname{End} E)$$

on $\Omega^1(M, \operatorname{End} E)$. The topology defined by this norm is independent of the metrics and is called the C^0-topology on $\Omega^1(M, \operatorname{End} E)$.

Let $\mathcal{C}(E)$ denote the space of connections on the bundle E. By choosing a connection ∇_0 we can identify this space with $\Omega^1(M, \operatorname{End} E)$ associating to a connection $\nabla \in \mathcal{C}(E)$ the 1-form $\nabla - \nabla_0 \in \Omega^1(M, \operatorname{End} E)$. By this identification the C^0-topology on $\Omega^1(M, \operatorname{End} E)$ provides a topology on $\mathcal{C}(E)$ which is independent of the choice of ∇_0 and is called the C^0-*topology* on the space of connections.

Finally, we denote by $\operatorname{Flat}(E) \subset \mathcal{C}(E)$ the set of flat connections on E and by $\operatorname{Flat}'(E, g^M) \subset \operatorname{Flat}(E)$ the set of flat connections satisfying Assumptions I and II of Subsection 6.5. The topology induced on these sets by the C^0-topology on $\mathcal{C}(E)$ is also called the C^0-topology. The discussion of the previous subsection implies that $\operatorname{Flat}'(E, g^M)$ *contains all the acyclic Hermitian connections.*

Proposition 6.8. $\operatorname{Flat}'(E, g^M)$ *is a* C^0*-open subset of* $\operatorname{Flat}(E)$*, which contains all acyclic Hermitian connections on* E*.*

Proof. We already know that $\operatorname{Flat}'(E, g^M)$ contains all acyclic Hermitian connections on E. Hence it is enough to show that $\operatorname{Flat}'(E, g^M)$ is open in C^0-topology.

Let $\nabla \in \operatorname{Flat}'(E, g^M)$ and suppose that $\nabla' \in \operatorname{Flat}(E)$ is sufficiently close to ∇ in C^0-topology. Let $B = B(\nabla, g^M)$, $B' = B(\nabla', g^M)$ denote the odd signature operators associated to the connections ∇ and ∇', respectively. Then $B - B'$ is a 0'th order differential operator on $\Omega^\bullet(M, E)$ and, hence, is bounded. Moreover, if ∇ is close to ∇' in the C^0-topology, then $B' - B : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E)$ is small in the operator norm, when $\Omega^\bullet(M, E)$ is endowed with the L^2-norm induced by the Riemannian metric on M and the Hermitian metric on E. We refer to this operator norm as the *standard* operator norm and denote it by $\|\cdot\|$.

Since the operator B satisfies Assumption II, its inverse B^{-1} can be viewed as a bounded operator on the L^2-completion $L^2\Omega^\bullet(M,E)$ of $\Omega^\bullet(M,E)$. If $B-B'$ is sufficiently small so that $\|(B'-B)B^{-1}\|<1$, then B', viewed as an unbounded operator on $L^2\Omega^\bullet(M,E)$, has a bounded inverse given by the formula

$$(B')^{-1} = B^{-1}\left(\operatorname{Id}+(B'-B)B^{-1}\right)^{-1}.$$

By elliptic theory, $(B')^{-1}$ maps the space of smooth forms $\Omega^\bullet(M,E)$ to itself. Hence, B' satisfies Assumption II. q.e.d.

6.9. Decomposition of the odd signature operator. Set

$$\Omega^{\text{even}}(M,E) := \bigoplus_{p=0}^{r-1} \Omega^{2p}(M,E), \quad \Omega^{\text{odd}}(M,E) := \bigoplus_{p=1}^{r} \Omega^{2p-1}(M,E),$$

$$B_{\text{even}} := \bigoplus_{p=0}^{r-1} B_{2p} : \Omega^{\text{even}}(M,E) \longrightarrow \Omega^{\text{even}}(M,E),$$

$$B_{\text{odd}} := \bigoplus_{p=1}^{r} B_{2p-1} : \Omega^{\text{odd}}(M,E) \longrightarrow \Omega^{\text{odd}}(M,E).$$

Using that $\Gamma^2=1$, we obtain

$$B_{\text{odd}} = \Gamma\circ B_{\text{even}}\circ\Gamma\,\big|_{\Omega^{\text{odd}}(M,E)}. \tag{6.66}$$

Hence, the whole information about the odd signature operator is encoded in its *even part* B_{even}. The operator B_{even} can be expressed by the following formula, which is slightly simpler than (6.62):

$$B_{\text{even}}\,\omega := i^r(-1)^{p+1}\left(*\,\nabla-\nabla\,*\right)\omega, \qquad \text{for} \quad \omega\in\Omega^{2p}(M,E). \tag{6.67}$$

Assume now that $\nabla\in\operatorname{Flat}'(E,g^M)$, i.e., that Assumptions I and II of Subsection 6.5 are satisfied. From Assumption I we conclude that the kernel and the image of the operator $\nabla:\Omega^\bullet(M,E)\to\Omega^\bullet(M,E)$ coincide. Hence,

$$\begin{aligned}&\operatorname{Ker}(\Gamma\nabla)=\operatorname{Ker}\nabla=\operatorname{Im}\nabla=\operatorname{Im}(\nabla\Gamma)\\ &\operatorname{Ker}(\nabla\Gamma)=\Gamma\left(\operatorname{Ker}\nabla\right)=\Gamma\left(\operatorname{Im}\nabla\right)=\operatorname{Im}(\Gamma\nabla).\end{aligned} \tag{6.68}$$

We set

$$\begin{aligned}&\Omega^k_+(M,E) := \operatorname{Ker}(\nabla\Gamma)\cap\Omega^k(M,E)=\left(\Gamma\ \operatorname{Ker}\nabla\right)\cap\Omega^k(M,E);\\ &\Omega^k_-(M,E) := \operatorname{Ker}(\Gamma\nabla)\cap\Omega^k(M,E)=\operatorname{Ker}\nabla\cap\Omega^k(M,E),\end{aligned} \tag{6.69}$$

and refer to $\Omega^k_+(M,E)$ and $\Omega^k_-(M,E)$ as the positive and negative subspaces of $\Omega^k(M,E)$.[7]

[7]Note, that our grading is opposite to the one considered in [**12**, §2].

Assumption II of Subsection 6.5 then implies that, for all $k = 0, \dots, d$,

$$\operatorname{Ker}\left(\nabla\,\Gamma\big|_{\Omega^k(M,E)}\right) \cap \operatorname{Ker}\left(\Gamma\,\nabla\big|_{\Omega^k(M,E)}\right) = \{0\} \tag{6.70}$$

(as B is one-to-one) and

$$\operatorname{Im}\left(\nabla\,\Gamma\big|_{\Omega^{d-k+1}(M,E)}\right) + \operatorname{Im}\left(\Gamma\,\nabla\big|_{\Omega^{d-k-1}(M,E)}\right) = \Omega^k(M,E) \tag{6.71}$$

(as B is onto). Combining (6.70) and (6.71) with (6.68) and (6.69) we conclude that

$$\Omega^k(M,E) = \Omega^k_+(M,E) \,\oplus\, \Omega^k_-(M,E). \tag{6.72}$$

Clearly,

$$\Gamma : \Omega^k_\pm(M,E) \;\longrightarrow\; \Omega^{d-k}_\mp(M,E), \qquad\qquad k = 0, \dots, d. \tag{6.73}$$

Remark 6.10. Suppose that h^E is a flat Hermitian metric on E. Then, using (6.64) and (6.68), we obtain

$$\Omega^k_+(M,E) = \operatorname{Ker}\,\nabla^* \,\cap\, \Omega^k(M,E), \tag{6.74}$$

where ∇^* is the adjoint of ∇ with respect to the scalar product induced by the metrics g^M and h^E.

Let $B^\pm_k$ denote the restriction of B to $\Omega^k_\pm(M,E)$. By (6.61) and (6.68),

$$\begin{aligned} B^+_k &= \Gamma\,\nabla : \Omega^k_+(M,E) \;\longrightarrow\; \Omega^{d-k-1}_+(M,E), \\ B^-_k &= \nabla\,\Gamma : \Omega^k_-(M,E) \;\longrightarrow\; \Omega^{d-k+1}_-(M,E). \end{aligned} \tag{6.75}$$

It follows from Assumption II of Subsection 6.5 that both B^+_k and B^-_k are invertible.

6.11. Graded determinant of the odd signature operator. Set

$$\Omega^{\mathrm{even}}_\pm(M,E) = \bigoplus_{p=0}^{r-1} \Omega^{2p}_\pm(M,E)$$

and let $B^\pm_{\mathrm{even}}$ denote the restriction of B_{even} to the space $\Omega^{\mathrm{even}}_\pm(M,E)$. Then

$$B^\pm_{\mathrm{even}} : \Omega^{\mathrm{even}}_\pm(M,E) \;\longrightarrow\; \Omega^{\mathrm{even}}_\pm(M,E).$$

We consider $\Omega^{\mathrm{even}}(M,E)$ as a graded vector space

$$\Omega^{\mathrm{even}}(M,E) = \Omega^{\mathrm{even}}_+(M,E) \,\oplus\, \Omega^{\mathrm{even}}_-(M,E).$$

By Definition 3.12, the graded determinant of the odd signature operator is

$$\operatorname{Det}_{\mathrm{gr},\theta}(B_{\mathrm{even}}) := e^{\operatorname{LDet}_{\mathrm{gr},\theta}(B_{\mathrm{even}})}, \tag{6.76}$$

where $\theta \in (-\pi, 0)$ is an Agmon angle for the operator $B = B_{\mathrm{even}} \oplus B_{\mathrm{odd}}$, cf. Definition 3.4, and

$$\operatorname{LDet}_{\mathrm{gr},\theta}(B_{\mathrm{even}}) := \operatorname{LDet}_\theta\left(B^+_{\mathrm{even}}\right) - \operatorname{LDet}_\theta\left(-B^-_{\mathrm{even}}\right) \;\in\; \mathbb{C}. \tag{6.77}$$

According to (3.27), $\mathrm{Det}_{\mathrm{gr},\theta}(B_{\mathrm{even}})$ is independent of the choice of the Agmon angle $\theta \in (-\pi, 0)$.

7. Relationship with the η-invariant

In this section we use the notations of the previous section and assume that, for a given pair (∇, g^M), Assumptions I and II of Subsection 6.5 are satisfied. In particular the operator $B = B(\nabla, g^M)$ is bijective. It follows that the operators B_{even} and $B_k^+ : \Omega_+^k(M, E) \to \Omega_+^{d-k-1}(M, E)$ $(0 \le k \le d-1)$ are also invertible.

7.1. Graded determinant and η-invariant. To simplify the notation set

$$\eta = \eta(\nabla, g^M) := \eta(B_{\mathrm{even}}), \tag{7.78}$$

and

$$\begin{aligned} \xi = \xi(\nabla, g^M, \theta) &:= \frac{1}{2} \sum_{k=0}^{d-1} (-1)^k \,\mathrm{LDet}_{2\theta}\left(B_{d-k-1}^+ \circ B_k^+\right) \\ &= \frac{1}{2} \sum_{k=0}^{d-1} (-1)^k \,\mathrm{LDet}_{2\theta}\left((\Gamma\,\nabla)^2\big|_{\Omega_+^k(M,E)}\right). \end{aligned} \tag{7.79}$$

Theorem 7.2. *Let $\nabla \in \mathrm{Flat}'(E, g^M)$ be a flat connection on a vector bundle E over a closed oriented Riemannian manifold (M, g^M) of odd dimension $d = 2r-1$. Let $\theta \in (-\pi/2, 0)$ be an Agmon angle for B such that there are no eigenvalues of the operator B in the solid angles $L_{(-\pi/2,\theta]}$ and $L_{(\pi/2,\theta+\pi]}$. Then*

$$\mathrm{LDet}_{\mathrm{gr},\theta}(B_{\mathrm{even}}) = \xi - i\pi\,\eta. \tag{7.80}$$

The rest of this section is devoted to the proof of Theorem 7.2. By (4.52), it is enough to show that

$$2\,\xi = \mathrm{LDet}_{2\theta}\,(B_{\mathrm{even}}^+)^2 - \mathrm{LDet}_{2\theta}\,(B_{\mathrm{even}}^-)^2; \tag{7.81}$$

$$\zeta_{2\theta}\big(0, (B_{\mathrm{even}}^+)^2\big) - \zeta_{2\theta}\big(0, (B_{\mathrm{even}}^-)^2\big) = 0. \tag{7.82}$$

Note that though the value at 0 of the ζ-function of a second order *differential* operator on an odd-dimensional manifold vanishes, [**39**], the operators $B_{\mathrm{even}}^{\pm}$ are *pseudo-differential* and, hence, the equality (7.82) is not trivial. In particular, the individual terms $\zeta_{2\theta}\big(0, (B_{\mathrm{even}}^{\pm})^2\big)$ need not vanish.

7.3. Calculation of $\zeta_{2\theta}\big(s, (B_{\mathrm{even}}^+)^2\big) - \zeta_{2\theta}\big(s, (B_{\mathrm{even}}^-)^2\big)$. Set

$$\begin{aligned} A_k^+ := B_k^+ + B_{d-k-1}^+ :\, &\Omega_+^k(M, E) \oplus \Omega_+^{d-k-1}(M, E) \\ &\longrightarrow\ \Omega_+^k(M, E) \oplus \Omega_+^{d-k-1}(M, E), \end{aligned}$$

for $k = 0, \dots, r-2$, and

$$A^+_{r-1} := B^+_{r-1} : \Omega^{r-1}_+(M,E) \longrightarrow \Omega^{r-1}_+(M,E).$$

Similarly, set

$$A^-_k := B^-_k + B^-_{d-k+1} : \Omega^k_-(M,E) \oplus \Omega^{d-k+1}_-(M,E) \longrightarrow \Omega^k_-(M,E) \oplus \Omega^{d-k+1}_-(M,E)$$

for $k = 1, \dots, r-1$, and

$$A^-_r := B^-_r : \Omega^r_-(M,E) \longrightarrow \Omega^r_-(M,E). \tag{7.83}$$

Then

$$\begin{aligned} (A^+_k)^2 &= (\Gamma\,\nabla)^2\big|_{\Omega^k_+(M,E)\oplus\Omega^{d-k-1}_+(M,E)}, \\ (A^-_k)^2 &= (\nabla\,\Gamma)^2\big|_{\Omega^k_-(M,E)\oplus\Omega^{d-k+1}_-(M,E)}. \end{aligned} \tag{7.84}$$

Hence,

$$\begin{aligned} \zeta_{2\theta}\big(s,(A^+_k)^2\big) &= \zeta_{2\theta}\big(s,(\Gamma\,\nabla)^2\big|_{\Omega^k_+(M,E)}\big) \\ &\quad + \zeta_{2\theta}\left(s,(\Gamma\,\nabla)^2\big|_{\Omega^{d-k-1}_+(M,E)}\right). \end{aligned} \tag{7.85}$$

From (6.66) and (6.73) we get

$$A^+_k = \Gamma \circ A^-_{k-1} \circ \Gamma. \tag{7.86}$$

Hence,

$$\zeta_{2\theta}\big(s,(A^+_k)^2\big) = \zeta_{2\theta}\big(s,(A^-_{k-1})^2\big). \tag{7.87}$$

Since $B^\pm_{\mathrm{even}}$ is a direct sum of the operators $A^\pm_{2p}$, we obtain from (7.87) that

$$\begin{aligned} &\zeta_{2\theta}\big(s,(B^+_{\mathrm{even}})^2\big) - \zeta_{2\theta}\big(s,(B^-_{\mathrm{even}})^2\big) \\ &= \sum_{p=0}^{[(r-1)/2]} \zeta_{2\theta}\big(s,(A^+_{2p})^2\big) - \sum_{p=1}^{[r/2]} \zeta_{2\theta}\big(s,(A^-_{2p})^2\big) \\ &= \sum_{p=0}^{[(r-1)/2]} \zeta_{2\theta}\big(s,(A^+_{2p})^2\big) - \sum_{p=1}^{[r/2]} \zeta_{2\theta}\big(s,(A^+_{2p-1})^2\big) \\ &= \sum_{k=0}^{r-1} (-1)^k\, \zeta_{2\theta}\big(s,(A^+_k)^2\big). \end{aligned}$$

Combining this equality with (7.85) we get

$$\begin{aligned} &\zeta_{2\theta}\big(s,(B^+_{\mathrm{even}})^2\big) - \zeta_{2\theta}\big(s,(B^-_{\mathrm{even}})^2\big) \\ &\quad = \sum_{k=0}^{d-1} (-1)^k\, \zeta_{2\theta}\left(s,(\Gamma\,\nabla)^2\big|_{\Omega^k_+(M,E)}\right). \end{aligned} \tag{7.88}$$

Lemma 7.4. *Let F_1, F_2 be vector bundles over M and let*

$$P : C^\infty(M, F_1) \to C^\infty(M, F_2) \quad \text{and} \quad Q : C^\infty(M, F_2) \to C^\infty(M, F_2)$$

be invertible elliptic pseudo-differential operators such that ϕ is an Agmon angle for PQ and QP. Then, every regular point $s \in \mathbb{C}$ of the function $s \mapsto \zeta_\phi(s, PQ)$ is also a regular point of $s \mapsto \zeta_\phi(s, QP)$ and

$$\zeta_\phi(s, PQ) = \zeta_\phi(s, QP). \tag{7.89}$$

In particular,

$$\zeta_\phi(0, PQ) = \zeta_\phi(0, QP). \tag{7.90}$$

Proof. For every elliptic operator D with Agmon angle ϕ

$$QD_\phi^{-s-1}Q^{-1} = \big(\, QDQ^{-1}\,\big)_\phi^{-s-1}.$$

Hence,

$$Q(PQ)_\phi^{-s-1} = \big[\, Q(PQ)_\phi^{-s-1}Q^{-1}\,\big]\, Q = (QP)_\phi^{-s-1}Q. \tag{7.91}$$

Recall that if T and S are operators such that the composition TS is of trace class, then ST is also of trace class and $\mathrm{Tr}(TS) = \mathrm{Tr}(ST)$, cf. [**22**, Ch. III, Th. 8.2]. Using this equality and (7.91) we obtain

$$\begin{aligned} \zeta_\phi(s, PQ) &= \mathrm{Tr}\,(PQ)_\phi^{-s} = \mathrm{Tr}\,\big[(PQ)_\phi^{-s-1}PQ\big] \\ &= \mathrm{Tr}\,\big[\, Q(PQ)_\phi^{-s-1}P\,\big] = \mathrm{Tr}\,(QP)_\phi^{-s} = \zeta_\phi(s, QP). \end{aligned} \tag{7.92}$$

Since both the left and the right hand sides of (7.92) are analytic in s, this equality holds for all regular points of the function $s \mapsto \zeta_\phi(s, PQ)$. q.e.d.

From (7.89), we conclude that for all regular points of the function $s \mapsto \zeta_{2\theta}\big(s, (\Gamma\nabla)^2\big|_{\Omega^k_+(M,E)}\big)$ the following equality holds

$$\begin{aligned} \zeta_{2\theta}\big(s, (\Gamma\,\nabla)^2\big|_{\Omega^k_+(M,E)}\big) &= \zeta_{2\theta}\big(s, (\Gamma\,\nabla\,\Gamma)\big|_{\Omega^{k+1}_-(M,E)}\nabla\big|_{\Omega^k_+(M,E)}\big) \\ &= \zeta_{2\theta}\big(s, \nabla\big|_{\Omega^k_+(M,E)}(\Gamma\,\nabla\,\Gamma)\big|_{\Omega^{k+1}_-(M,E)}\big) \\ &= \zeta_{2\theta}\big(s, (\nabla\,\Gamma)^2\big|_{\Omega^{k+1}_-(M,E)}\big). \end{aligned} \tag{7.93}$$

From (7.84), (6.70), and (6.71), we get

$$\begin{aligned} \zeta_{2\theta}\big(s, B^2\big|_{\Omega^k(M,E)}\big) &= \zeta_{2\theta}\Big(s, \big(\,(\Gamma\,\nabla)^2 + (\nabla\,\Gamma)^2\big)\big|_{\Omega^k(M,E)}\Big) \\ &= \zeta_{2\theta}\big(s, (\Gamma\,\nabla)^2\big|_{\Omega^k_+(M,E)}\big) + \zeta_{2\theta}\big(s, (\nabla\,\Gamma)^2\big|_{\Omega^k_-(M,E))}\big). \end{aligned}$$

Using this equality, (7.88), and (7.93), we obtain

$$\begin{aligned} &\zeta_{2\theta}\big(s, (B^+_{\mathrm{even}})^2\big) - \zeta_{2\theta}\big(s, (B^-_{\mathrm{even}})^2\big) \\ &\qquad = \sum_{k=0}^{d} (-1)^{k+1}\, k\ \zeta_{2\theta}\big(s, B^2\big|_{\Omega^k(M,E)}\big). \end{aligned} \tag{7.94}$$

7.5. Proof of Theorem 7.2. From (7.79) and (7.88) we conclude that

$$2\,\xi = \frac{d}{ds}\big|_{s=0}\Big[\zeta_{2\theta}\big(s,(B^+_{\rm even})^2\big) - \zeta_{2\theta}\big(s,(B^-_{\rm even})^2\big)\Big]. \tag{7.95}$$

Hence (7.81) is established.

Since the dimension of M is odd, the ζ-function of every elliptic differential operator of even order vanishes at 0, cf. [**39**]. Hence, by Lemma 6.4.1, the equality (7.94) implies (7.82). q.e.d.

8. Comparison with the Ray-Singer Torsion

8.1. Ray-Singer torsion. Let $E \to M$ be a complex vector bundle over a closed oriented manifold M of odd dimension $d = 2r-1$ and let ∇ be an acyclic flat connection on E. Fix a Riemannian metric g^M on M and a Hermitian metric h^E on E. Let ∇^* denote the adjoint of ∇ with respect to the scalar product $\langle\cdot,\cdot\rangle$ on $\Omega^\bullet(M,E)$ defined by h^E and the Riemannian metric g^M. If ∇ is acyclic (i.e., Assumption I of Subsection 6.5 is satisfied) the *Ray-Singer torsion* $T^{\rm RS}$ of E, [**35**, **5**, **12**], is defined by

$$\begin{aligned} T^{\rm RS} &= T^{\rm RS}(\nabla) \\ &:= \exp\Big(\frac{1}{2}\sum_{k=0}^{d}(-1)^{k+1}\,k\,\operatorname{LDet}_{-\pi}\big((\nabla^*\nabla+\nabla\nabla^*)\big|_{\Omega^k(M,E)}\big)\Big), \end{aligned} \tag{8.96}$$

where ∇^* denotes the adjoint of ∇ with respect to the scalar product $\langle\cdot,\cdot\rangle$ induced by g^M and h^E. Note that, as ∇ is assumed to be acyclic, $(\nabla^*\nabla+\nabla\nabla^*)_{|_{\Omega^k(M,E)}}$ is a strictly positive operator and, therefore, $\operatorname{LDet}_{-\pi}\big((\nabla^*\nabla+\nabla\nabla^*)\big|_{\Omega^k(M,E)}\big)$ is well defined.

The Ray-Singer torsion is a positive number, which, *in the case considered*, is independent of the the Hermitian metric h^E and the Riemannian metric g^M, cf. [**35**, **5**]. We denote by $\log T^{\rm RS}$ the value at $T^{\rm RS}$ of the principal branch of the logarithm.

The determinants in (8.96) are defined using the spectral cut $R_{-\pi}$ along the negative real axis. Since the spectrum of the operator $\nabla^*\nabla+\nabla\nabla^*$ lies on the positive real axis, we can replace it with a spectral cut R_ϕ for any $\phi\neq 0$ without changing the formula. In particular, we can take the spectral cut along $R_{2\theta}$, where $\theta\in(-\pi/2,0)$ is an Agmon angle for the odd signature operator B.

Using the decomposition (6.72), we have

$$\begin{aligned} \log T^{\rm RS} &= \frac{1}{2}\sum_{k=0}^{d}(-1)^k\,\operatorname{LDet}_{2\theta}\big(\nabla^*\nabla\big|_{\Omega^k_+(M,E)}\big) \\ &= \frac{1}{2}\sum_{k=0}^{d}(-1)^{k+1}\,\operatorname{LDet}_{2\theta}\big(\nabla\nabla^*\big|_{\Omega^k_-(M,E)}\big). \end{aligned} \tag{8.97}$$

This formula is proven, for example, on page 340 of [**12**].

Suppose now that the Hermitian metric h^E on E is invariant with respect to ∇. From (6.64), we obtain

$$\nabla^* \nabla\big|_{\Omega^k_+(M,E)} = (\Gamma \nabla)^2\big|_{\Omega^k_+(M,E)}.$$

Hence, we can rewrite (8.97) as

$$\begin{aligned} &\log T^{\mathrm{RS}} \\ &\quad = \frac{1}{2} \sum_{k=0}^{d} (-1)^k \operatorname{LDet}_{2\theta}\Big((\Gamma \nabla)^2\big|_{\Omega^k_+(M,E)}\Big) = \xi(\nabla, g^M, \theta), \end{aligned} \tag{8.98}$$

where the number $\xi = \xi(\nabla, g^M, \theta)$ is defined in (7.79).

The next theorem generalizes this result to the case where h^E is not necessarily invariant. Recall that the set $\mathrm{Flat}'(M, g^M)$ is defined in Subsection 6.7.

Theorem 8.2. *Assume that M is a closed oriented manifold and g^M is a Riemannian metric on M. Then there exists an C^0-open (cf. Subsection* 6.7*) neighborhood $U \subset \mathrm{Flat}(E)$ of the set of acyclic Hermitian connections on E, such that for every connection $\nabla \in U$ we have $\nabla \in \mathrm{Flat}'(M, g^M)$ and*

$$\begin{aligned} &\log T^{\mathrm{RS}}(\nabla) \\ &\quad = \frac{1}{2}\,\mathrm{Re} \sum_{k=0}^{d} (-1)^k \operatorname{LDet}_{2\theta}\Big((\Gamma \nabla)^2\big|_{\Omega^k_+(M,E)}\Big) = \mathrm{Re}\ \xi(\nabla, g^M, \theta). \end{aligned} \tag{8.99}$$

*Hence, in view of (*7.80*), for every $\nabla \in U$, we obtain*

$$\Big|\ \operatorname{Det}_{\mathrm{gr},\theta}(B_{\mathrm{even}})\ \Big| = T^{\mathrm{RS}}(\nabla) \cdot e^{\pi\, \mathrm{Im}\, \eta(\nabla, g^M)}. \tag{8.100}$$

If ∇ is an acyclic Hermitian connection, then the operator B_{even} is self-adjoint with respect to the inner product given by g^M and the invariant Hermitian metric h^E on E. Thus, the η-invariant $\eta = \eta(\nabla, g^M)$ is real. Hence, Theorem 8.2 implies the following

Corollary 8.3. *If ∇ is an acyclic Hermitian connection then*

$$\Big|\ \operatorname{Det}_{\mathrm{gr},\theta}(B_{\mathrm{even}})\ \Big| = T^{\mathrm{RS}}(\nabla). \tag{8.101}$$

In particular, Corollary 8.3 implies that, under the given assumptions, $\operatorname{Det}_{\mathrm{gr},\theta}(B_{\mathrm{even}})$ contains all the information about the Ray-Singer torsion, and, hence, "refines" it by having a phase.

The rest of this section is occupied with the proof of Theorem 8.2.

8.4. Alternative formula for ξ. From (7.95) and (7.94) we obtain the following analogue of (8.96):

$$\xi(\nabla, g^M, \theta) = \frac{1}{2}\sum_{k=0}^{d}(-1)^{k+1}\, k\ \mathrm{LDet}_{2\theta}\Big[\Big((\Gamma\nabla)^2 + (\nabla\Gamma)^2\Big)\Big|_{\Omega^k(M,E)}\Big], \tag{8.102}$$

where $\theta \in (-\pi/2, 0)$ is an Agmon angle for B so that there are no eigenvalues of B in the solid angles $L_{(-\pi/2,\theta]}$ and $L_{(\pi/2,\theta+\pi]}$. Note that this condition implies that for all $k = 0, \ldots, d$, 2θ is an Agmon angle for $B^2\big|_{\Omega^k(M,E)}$.

8.5. Choice of the spectral cut. It follows from (3.27) that $\mathrm{Det}_{\mathrm{gr},\theta}(B_{\mathrm{even}})$ does not depend on the choice of the Agmon angle $\theta \in (-\pi, 0)$. It is convenient for us to work with an angle $\theta \in (-\pi/2, 0)$ such that there are no eigenvalues of the operator B in the solid angles $L_{(-\pi/2,\theta]}$, $L_{[-\theta,\pi/2)}$, $L_{(\pi/2,\theta+\pi]}$ and $L_{[-\theta-\pi,-\pi/2)}$. We will fix such an angle till the end of this section.

8.6. The dual connection. Fix a Hermitian metric h^E on E. Denote by ∇' the *connection on E dual to the connection ∇*. It is defined by the formula

$$dh^E(u, v) = h^E(\nabla u, v) + h^E(u, \nabla' v), \qquad u, v \in C^\infty(M, E).$$

From the definition of the scalar product $\langle\cdot,\cdot\rangle$ on $\Omega^\bullet(M, E)$ it then follows that

$$\nabla^* = \Gamma\,\nabla'\,\Gamma, \qquad (\nabla')^* = \Gamma\,\nabla\,\Gamma. \tag{8.103}$$

Since $\Gamma^2 = \mathrm{Id}$, (8.103) implies

$$\Big((\Gamma\nabla)^2\Big)^* = (\Gamma\nabla')^2, \quad \Big((\nabla\Gamma)^2\Big)^* = (\nabla'\Gamma)^2. \tag{8.104}$$

Let B' denote the odd signature operator associated to the connection ∇'. Using (6.75) and (8.103) one readily sees that

$$B^* = B'.$$

Therefore, if the connection ∇ satisfies Assumption I and II of Subsection 6.5, then so does the connection ∇'. Our choice of the angle θ in Subsection 8.5 guarantees that $\pm 2\theta$ are Agmon angles for the operator

$$(\Gamma\nabla')^2 = \big((\Gamma\nabla)^2\big)^*.$$

In particular, the number $\xi(\nabla', g^M, \theta)$ can be defined by formula (7.79), using the same angle θ and replacing everywhere ∇ by ∇'.

Lemma 8.7. *Using the notation introduced above, we have*

$$\xi(\nabla', g^M, \theta) \equiv \overline{\xi(\nabla, g^M, \theta)}, \qquad \mathrm{mod}\ \pi i, \tag{8.105}$$

where $\overline{z}$ denotes the complex conjugate of the number $z \in \mathbb{C}$.

Proof. Set

$$D_k := \Big((\Gamma\,\nabla)^2 + (\nabla\,\Gamma)^2\Big)\Big|_{\Omega^k(M,E)} : \Omega^k(M,E) \longrightarrow \Omega^k(M,E).$$

Then, by (8.104),

$$D_k^* = \Big((\Gamma\,\nabla')^2 + (\nabla'\,\Gamma)^2\Big)\Big|_{\Omega^k(M,E)} : \Omega^k(M,E) \longrightarrow \Omega^k(M,E).$$

With θ given as in Subsection 8.5, we have

$$\mathrm{LDet}_{2\theta}\ D_k^* = \overline{\mathrm{LDet}_{-2\theta}\ D_k}. \tag{8.106}$$

Note that D_k has a positive-definite leading symbol. Hence, at most finitely many eigenvalues of D_k lie in the solid angle $L_{[-2\theta,2\theta+2\pi]}$ (which contains the negative real axis).[8] Hence, by (3.26),

$$\mathrm{LDet}_{-2\theta}\, D_k \equiv \mathrm{LDet}_{2\theta}\, D_k \qquad \mod\ 2\pi i. \tag{8.107}$$

Using (8.106) and (8.107), we obtain now from (8.102), that

$$\begin{aligned}
&\xi(\nabla', g^M, \theta)\\
&= \frac{1}{2}\sum_{k=0}^{d} (-1)^{k+1} k\ \mathrm{LDet}_{2\theta}\ D_k^*\\
&\equiv \frac{1}{2}\sum_{k=0}^{d} (-1)^{k+1} k\ \overline{\mathrm{LDet}_{2\theta}\, D_k}\\
&= \overline{\xi(\nabla, g^M, \theta)}, \qquad \mod\ \pi i.
\end{aligned}$$

q.e.d.

Lemma 8.8. *For every* $\nabla \in \mathrm{Flat}'(M, g^M)$ *we have* $T^{\mathrm{RS}}(\nabla') = T^{\mathrm{RS}}(\nabla)$.

Proof. From (8.103), we obtain

$$\begin{aligned}
\nabla^*\nabla &= \Gamma\,\nabla'\,\Gamma\,\nabla = \Gamma\,(\,\nabla'\,\Gamma\,\nabla\,\Gamma\,)\,\Gamma = \Gamma\,\nabla'\,(\nabla')^*\,\Gamma;\\
\nabla\nabla^* &= \nabla\,\Gamma\,\nabla'\,\Gamma = \Gamma\,(\,\Gamma\,\nabla\,\Gamma\,\nabla'\,)\,\Gamma = \Gamma\,(\nabla')^*\,\nabla'\,\Gamma.
\end{aligned}$$

[8] By our assumptions on θ, all these eigenvalues must lie on the real axis. But we don't use this fact here.

From (8.96) we obtain

(8.108)

$$\begin{aligned}
&\log T^{\mathrm{RS}}(\nabla)\\
&= \frac{1}{2}\sum_{k=0}^{d}(-1)^{k+1}\,k\,\mathrm{LDet}_{-\pi}\left((\nabla^*\nabla+\nabla\nabla^*)\big|_{\Omega^k(M,E)}\right)\\
&= \frac{1}{2}\sum_{k=0}^{d}(-1)^{k+1}\,k\,\mathrm{LDet}_{-\pi}\left(\Gamma\left(\nabla'(\nabla')^*+(\nabla')^*\nabla'\right)\Gamma\big|_{\Omega^k(M,E)}\right)\\
&= \frac{1}{2}\sum_{k=0}^{d}(-1)^{k+1}\,k\,\mathrm{LDet}_{-\pi}\left(\left(\nabla'(\nabla')^*+(\nabla')^*\nabla'\right)\big|_{\Omega^{d-k}(M,E)}\right)\\
&= \frac{1}{2}\sum_{k=0}^{d}(-1)^{k}\,(d-k)\,\mathrm{LDet}_{-\pi}\left(\left(\nabla'(\nabla')^*+(\nabla')^*\nabla'\right)\big|_{\Omega^k(M,E)}\right).
\end{aligned}$$

By (8.97),

$$\begin{aligned}
&\sum_{k=0}^{d}(-1)^{k}\,\mathrm{LDet}_{-\pi}\left(\left(\nabla'(\nabla')^*+(\nabla')^*\nabla'\right)\big|_{\Omega^k(M,E)}\right)\\
&\quad=\sum_{k=0}^{d}(-1)^{k}\,\mathrm{LDet}_{-\pi}\left((\nabla')^*\nabla'\big|_{\Omega^k_+(M,E)}\right)\\
&\qquad+\sum_{k=0}^{d}(-1)^{k}\,\mathrm{LDet}_{-\pi}\left(\nabla'(\nabla')^*\big|_{\Omega^k_-(M,E)}\right)=0.
\end{aligned}$$

Hence, from (8.108), we obtain

$$\begin{aligned}
&\log T^{\mathrm{RS}}(\nabla)\\
&= \frac{1}{2}\sum_{k=0}^{d}(-1)^{k+1}\,k\,\mathrm{LDet}_{-\pi}\left(\left(\nabla'(\nabla')^*+(\nabla')^*\nabla'\right)\big|_{\Omega^k(M,E)}\right)\\
&= \log T^{\mathrm{RS}}(\nabla').
\end{aligned}$$

q.e.d.

8.9. Proof of Theorem 8.2. In the case ∇ is an acyclic Hermitian connection the statement has been already proved, cf. (8.98).

In the general case, let

$$\tilde{\nabla}=\begin{pmatrix}\nabla & 0\\ 0 & \nabla'\end{pmatrix}$$

denote the flat connection on $E\oplus E$ obtained as a direct sum of the connections ∇ and ∇'. From Lemmas 8.7 and 8.8 we obtain

$$T^{\mathrm{RS}}(\tilde{\nabla})=T^{\mathrm{RS}}(\nabla)\cdot T^{\mathrm{RS}}(\nabla')=\left(T^{\mathrm{RS}}(\nabla)\right)^2,$$

and

$$\xi(\tilde{\nabla}, g^M, \theta) = \xi(\nabla, g^M, \theta) + \xi(\nabla', g^M, \theta) \equiv 2 \text{ Re } \xi(\nabla, g^M, \theta) \tag{8.109}$$

modulo πi. Hence, to prove Theorem 8.2, it is enough to show that

$$\xi(\tilde{\nabla}, g^M, \theta) \equiv \log T^{\mathrm{RS}}(\tilde{\nabla}) \qquad \mod \pi i. \tag{8.110}$$

We will prove (8.109) by a deformation argument. For $t \in [-\pi/2, \pi/2]$ introduce the rotation U_t on

$$\Omega^\bullet := \Omega^\bullet(M, E) \oplus \Omega^\bullet(M, E)$$

given by

$$U_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Note that $U_t^{-1} = U_{-t}$.

Consider two one-parameter families of operators

$$\widetilde{B}(t), \widehat{B}(t) : \Omega^\bullet \to \Omega^\bullet, \qquad t \in [-\pi/2, \pi/2]$$

defined by

$$\widetilde{B}(t) := \Gamma\, U_t\, \tilde{\nabla}\, U_t^{-1} + \tilde{\nabla}\, \Gamma; \qquad \widehat{B}(t) := \Gamma\, \tilde{\nabla} + U_t\, \tilde{\nabla} U_t^{-1}\, \Gamma.$$

Note that $\widetilde{B}(0) = \widehat{B}(0) = B(\tilde{\nabla}, g^M)$. If the Hermitian metric h^E is invariant with respect to ∇ then $\nabla' = \nabla$ and

$$\widetilde{B}(t) = \widehat{B}(t) = B(\tilde{\nabla}, g^M) = B(\nabla, g^M) \oplus B(\nabla', g^M) \tag{8.111}$$

for all $t \in [-\pi/2, \pi/2]$. It follows then from Assumption II of Subsection 6.5 that the operator (8.111) is invertible.

Suppose now that ∇ is sufficiently close to an acyclic Hermitian connection ∇_0 in the C^0-topology, cf. Subsection 6.7, and that the metric h^E is chosen to be invariant with respect to the connection ∇_0. Then ∇' is also close to ∇_0. Since both $\widetilde{B}(t) - B(\tilde{\nabla}, g^M)$ and $\widehat{B}(t) - B(\tilde{\nabla}, g^M)$ are 0'th order differential operators, it follows that they are small in the standard operator norm (cf. proof of Proposition 6.8) for all $t \in [-\pi/2, \pi/2]$. Therefore the operators $\widetilde{B}(t), \widehat{B}(t)$ are invertible for all $t \in [-\pi/2, \pi/2]$.

We denote by $V \subset \mathrm{Flat}(E)$ the set of connections, which satisfy the following property: there exists a Hermitian metric h^E such that the operators $\widetilde{B}(t)$ and $\widehat{B}(t)$ are invertible for all $t \in [-\pi/2, \pi/2]$.[9] Then V is open in $\mathrm{Flat}(E)$. Moreover, since for every $\nabla \in V$ the operator $\widetilde{B}(0) = B(\nabla, g^M) \oplus B(\nabla', g^M)$ is invertible, it follows that $V \subset \mathrm{Flat}'(E, g^M)$.

The above discussion shows that V contains the set of acyclic Hermitian connections. In the rest of the proof we assume that $\nabla \in V$ and that h^E is chosen so that the operators $\widetilde{B}(t)$ and $\widehat{B}(t)$ are invertible.

[9]Recall that $\widetilde{B}(t)$ and $\widehat{B}(t)$ depend on h^E since the dual connection ∇' does.

Set

$$\Omega^\bullet_+(t) := \operatorname{Ker} U_t\tilde\nabla U_t^{-1}\,\Gamma = \left\{\,\Gamma\,U_t\,\binom{\omega}{\omega'} : \omega\in\operatorname{Ker}\nabla,\ \omega'\in\operatorname{Ker}\nabla'\,\right\};$$
$$\Omega^\bullet_- := \operatorname{Ker}\tilde\nabla = \operatorname{Ker}\nabla\oplus\operatorname{Ker}\nabla'.$$

Note that $\Omega^\bullet_-$ is independent of t.

Since the range of $\Gamma U_t\tilde\nabla U_t^{-1}$ is contained in $\Omega^\bullet_+(t)$ whereas the range of $\tilde\nabla\,\Gamma$ is contained in $\Omega^\bullet_-$, it follows from the surjectivity of $\widetilde{B}(t)$ that

$$\Omega^\bullet_+(t)+\Omega^\bullet_- = \Omega^\bullet, \qquad t\in[-\pi/2,\pi/2]. \tag{8.112}$$

Similarly, since, by definition, the kernel of $U_t\tilde\nabla U_t^{-1}\Gamma$ is equal to $\Omega^\bullet_+(t)$ whereas the kernel of $\Gamma\,\tilde\nabla$ is equal to $\Omega^\bullet_-$, it follows from injectivity of $\widehat{B}(t)$ that

$$\Omega^\bullet_+(t)\cap\Omega^\bullet_- = \{0\}, \qquad t\in[-\pi/2,\pi/2]. \tag{8.113}$$

Combining (8.112) and (8.113) we obtain

$$\Omega^\bullet = \Omega^\bullet_+(t)\oplus\Omega^\bullet_-, \qquad t\in[-\pi/2,\pi/2]. \tag{8.114}$$

For each $t\in[-\pi/2,\pi/2]$ define $\xi(t)\in\mathbb{C}/\pi i\mathbb{Z}$ by the formula

$$\xi(t)\equiv\frac{1}{2}\sum_{k=0}^{d}(-1)^k\,\mathrm{LDet}_{\theta'}\left(\Gamma\,U_t\,\tilde\nabla\,U_t^{-1}\,\Gamma\,\tilde\nabla\big|_{\Omega^k_+(t)}\right), \qquad \mod\ \pi i, \tag{8.115}$$

where $\theta'\in L_{(-2\theta,2\pi+2\theta)}$ is any Agmon angle for the operators[10]

$$\Gamma U_t\tilde\nabla U_t^{-1}\Gamma\tilde\nabla\big|_{\Omega^k_+(t)}, \qquad k=0,\dots,N.$$

Since

$$\Gamma\,U_0\,\tilde\nabla\,U_0^{-1}\,\Gamma\,\tilde\nabla\big|_{\Omega^k_+(0)} = \begin{pmatrix}\Gamma\nabla\Gamma\nabla\big|_{\Omega^k_+(M,E)} & 0\\ 0 & \Gamma\nabla'\Gamma\nabla'\big|_{\Omega^k_+(M,E)}\end{pmatrix},$$

for $t=0$, (8.115) coincides with (7.79) with ∇ replaced by $\tilde\nabla$.

Similarly, since

$$\Gamma\,U_{\pi/2}\,\tilde\nabla\,U_{\pi/2}^{-1}\,\Gamma\,\tilde\nabla\big|_{\Omega^k_+(\pi/2)} = \begin{pmatrix}\nabla^*\nabla\big|_{\Omega^k_+(M,E)} & 0\\ 0 & \nabla'^*\nabla'\big|_{\Omega^k_+(M,E)}\end{pmatrix}$$

for $t=\pi/2$ the right hand side of (8.115) coincides with (8.97). Summarizing, we conclude that

$$\xi(0)\equiv\xi(\tilde\nabla,g^M,\theta), \quad \xi(\pi/2)\equiv\log T^{\mathrm{RS}}(\tilde\nabla), \qquad \mod\ \pi i. \tag{8.116}$$

[10]Recall from Subsection 3.10 that a different choice of $\theta'\in L_{(-2\theta,2\pi+2\theta)}$ changes the number $\mathrm{LDet}_{\theta'}(\Gamma U_t\tilde\nabla U_t^{-1}\Gamma\tilde\nabla\big|_{\Omega^k_+(t)})$ by a multiple of $2\pi i$.

We will finish the proof of (8.110) (and, hence, of Theorem 8.2) by showing that

$$\frac{d}{dt}\,\xi(t)=0. \tag{8.117}$$

This is done by applying the arguments of the standard proof of the independence of the Ray-Singer torsion on the Hermitian metric. First we need the following notation (cf., for example, Section 2 of [**12**]): Suppose $f(s)$ is a function of a complex parameter s which is meromorphic near $s=0$. We call the zero order term in the Laurent expansion of f near $s=0$ the *finite part of f at 0* and denote it by $\mathrm{F.\,p._{s=0}}\, f(s)$. Then, cf. Lemma 3.7 of [**11**] or formula (1.13) of [**28**],

$$\begin{aligned}\frac{d}{dt}\,\mathrm{LDet}_{\theta'}\left(\Gamma\,U_t\,\tilde\nabla\,U_t^{-1}\,\Gamma\,\tilde\nabla\big|_{\Omega^k_+(t)}\right)\\ =\mathrm{F.\,p._{s=0}}\,\mathrm{Tr}\,\Big[\Big(\frac{d}{dt}\big(\Gamma\,U_t\,\tilde\nabla\,U_t^{-1}\Gamma\tilde\nabla\big)\Big)\left(\Gamma\,U_t\,\tilde\nabla\,U_t^{-1}\,\Gamma\,\tilde\nabla\right)^{-s-1}_{\theta'}\big|_{\Omega^k_+(t)}\Big].\end{aligned} \tag{8.118}$$

One has

$$\begin{aligned}\frac{d}{dt}\left(\Gamma\,U_t\,\tilde\nabla\,U_t^{-1}\,\Gamma\,\tilde\nabla\right)=\dot U_tU_t^{-1}\big(\Gamma\,U_t\tilde\nabla U_t^{-1}\Gamma\,\tilde\nabla\big)\big|_{\Omega^k_+(t)}\\ -\big(\Gamma\,U_t\tilde\nabla U_t^{-1}\Gamma\,\big)\big|_{\Omega^{k+1}_-}\dot U_tU_t^{-1}\tilde\nabla\big|_{\Omega^k_+(t)}.\end{aligned}$$

By Lemma 7.4, for $\mathrm{Re}\,s>d/2$,

$$\begin{aligned}&\mathrm{Tr}\left[\big(\Gamma\,U_t\tilde\nabla U_t^{-1}\Gamma\,\big)\big|_{\Omega^{k+1}_-}\dot U_tU_t^{-1}\,\tilde\nabla\big|_{\Omega^k_+(t)}\left(\Gamma\,U_t\tilde\nabla U_t^{-1}\Gamma\,\tilde\nabla\right)^{-s-1}_{\theta'}\big|_{\Omega^k_+(t)}\right]\\ &=\mathrm{Tr}\left[\dot U_tU_t^{-1}\tilde\nabla\big|_{\Omega^k_+(t)}\left(\Gamma\,U_t\tilde\nabla U_t^{-1}\Gamma\,\tilde\nabla\right)^{-s-1}_{\theta'}\big|_{\Omega^k_+(t)}\big(\Gamma\,U_t\tilde\nabla U_t^{-1}\Gamma\,\big)\big|_{\Omega^{k+1}_-}\right]\\ &=\mathrm{Tr}\left[\dot U_tU_t^{-1}\left(\tilde\nabla\,\Gamma\,U_t\tilde\nabla\,U_t^{-1}\Gamma)\big|_{\Omega^{k+1}_-}\right)^{-s}_{\theta'}\right].\end{aligned} \tag{8.119}$$

Hence, (8.118) implies that

$$\begin{aligned}\frac{d}{dt}\,\mathrm{LDet}_{\theta'}\left(\Gamma\,U_t\tilde\nabla U_t^{-1}\Gamma\,\tilde\nabla\big|_{\Omega^k_+(t)}\right)\\ =\mathrm{F.\,p._{s=0}}\,\mathrm{Tr}\,\Big[\dot U_tU_t^{-1}\left(\Gamma\,U_t\tilde\nabla U_t^{-1}\Gamma\,\tilde\nabla\big|_{\Omega^k_+(t)}\right)^{-s}_{\theta'}\\ -\dot U_tU_t^{-1}\left(\tilde\nabla\Gamma\,U_t\tilde\nabla U_t^{-1}\Gamma\big|_{\Omega^{k+1}_-}\right)^{-s}_{\theta'}\Big].\end{aligned} \tag{8.120}$$

Consider the operator

$$\widetilde\Delta_k(t):=\Gamma\,U_t\,\tilde\nabla\,U_t^{-1}\,\Gamma\,\tilde\nabla\,\big|_{\Omega^k_+(t)}+\tilde\nabla\,\Gamma\,U_t\,\tilde\nabla\,U_t^{-1}\,\Gamma\,\big|_{\Omega^k_-}. \tag{8.121}$$

It is a second order elliptic differential operator on $\Omega^k(M,E\oplus E)$, whose leading symbol is equal to the leading symbol of the Laplacian $\tilde\nabla^*\,\tilde\nabla+$

$\tilde{\nabla}\tilde{\nabla}^*$. In other words, $\widetilde{\Delta}_k(t)$ is a *generalized Laplacian* in the sense of [**4**].

The decomposition $\Omega^k(M, E \oplus E) = \Omega^k_+(t) \oplus \Omega^k_-$ implies that

$$\widetilde{\Delta}_k(t)^{-s}_{\theta'} = \Big(\Gamma\, U_t\, \tilde{\nabla}\, U_t^{-1}\, \Gamma\, \tilde{\nabla}\,\big|_{\Omega^k_+(t)}\Big)^{-s}_{\theta'} + \Big(\,\tilde{\nabla}\, \Gamma\, U_t\, \tilde{\nabla}\, U_t^{-1}\, \Gamma\,\big|_{\Omega^k_-}\,\Big)^{-s}_{\theta'}.$$

Hence, from (8.120), we obtain

$$\frac{d}{dt}\,\xi(t) = \frac{1}{2}\,\sum_{k=0}^{d}(-1)^k\ \mathrm{F.\,p.}_{s=0}\,\mathrm{Tr}\,\Big[\,\dot{U}_t U_t^{-1}\,\widetilde{\Delta}_k(t)^{-s}_{\theta'}\,\Big]. \tag{8.122}$$

By a slight generalization of a result of Seeley [**39**], which is discussed in [**47**], the right hand side of (8.122) is given by a *local formula*, i.e., by an integral

$$\int_M \phi \tag{8.123}$$

of a differential form ϕ, whose value at a point $x \in M$ depends only on the full symbol of $\widetilde{\Delta}$ and a finite number of its derivatives at the point x. Moreover, since the dimension of the manifold M is odd, the differential form ϕ vanishes identically. q.e.d.

9. Dependence of the Graded Determinant on the Riemannian Metric

As already mentioned, one can consider the graded determinant $\mathrm{Det}_{\mathrm{gr},\theta}(B_{\mathrm{even}})$, defined in (6.76), as a refinement of the Ray-Singer torsion. However, in general, $\mathrm{Det}_{\mathrm{gr},\theta}(B_{\mathrm{even}})$ depends on the choice of the Riemannian metric g^M on M. In this section we investigate this dependence. In particular, we show that, if $\dim M = 2r - 1 \equiv 1 \pmod 4$, then $\mathrm{Det}_{\mathrm{gr},\theta}(B_{\mathrm{even}})$ is independent of g^M. Later we will use the results of this section to construct a refinement of the Ray-Singer torsion which is a diffeomorphism invariant of the pair (E, ∇) (i.e., is independent of the metric).

9.1. The η-invariant of the trivial bundle. Let

$$B_{\mathrm{trivial}} = B_{\mathrm{trivial}}(g^M) : \Omega^{\mathrm{even}}(M) \ \to\ \Omega^{\mathrm{even}}(M)$$

denote the even part of the odd signature operator corresponding to the trivial line bundle over M endowed with the trivial connection. We denote by

$$\eta_{\mathrm{trivial}} = \eta_{\mathrm{trivial}}(g^M) := \frac{1}{2}\,\eta\big(0, B_{\mathrm{trivial}}(g^M)\big)$$

the η-invariant of $B_{\mathrm{trivial}}(g^M)$. Since the operator B_{trivial} is self-adjoint, η_{trivial} is a real number, cf. Theorem A.2. Also, if $\dim M \equiv 1 (\mathrm{mod}\, 4)$, then $\eta_{\mathrm{trivial}} = 0$, cf. [**2**].

Definition 9.2. A Riemannian metric g^M on M is called *admissible for a given acyclic connection* ∇ if the odd signature operator $B = B(\nabla, g^M)$ satisfies Assumption II of Subsection 6.5. We denote the set of admissible metrics by $\mathcal{M}(\nabla)$.

We are now ready to formulate the main result of this section.

Theorem 9.3. *Let E be a flat vector bundle over a closed oriented odd-dimensional manifold M and let ∇ be the flat connection on E. For each admissible Riemannian metric $g^M \in \mathcal{M}(\nabla)$ consider the number*

$$\mathrm{Det}_{\mathrm{gr},\theta}\big(B_{\mathrm{even}}(\nabla, g^M)\big) \cdot e^{i\pi\,(\mathrm{rank}\,E)\;\eta\left(B_{\mathrm{trivial}}(g^M)\right)} \in \mathbb{C}\backslash\{0\}, \tag{9.124}$$

where $\theta \in (-\pi/2, 0)$ is an Agmon angle for $B_{\mathrm{even}}(\nabla, g^M)$. Then the number (9.124) is independent of $g^M \in \mathcal{M}(\nabla)$ and $\theta \in (-\pi/2, 0)$.

In particular, if $\dim M \equiv 1 (\mathrm{mod}\, 4)$, then $\eta\big(B_{\mathrm{trivial}}(g^M)\big) = 0$, cf. [**2**]*, and, hence, $\mathrm{Det}_{\mathrm{gr},\theta}\big(B_{\mathrm{even}}(\nabla, g^M)\big)$ is independent of g^M.*

The rest of this section is dedicated to the proof of Theorem 9.3.

9.4. Dependence of the η-invariant on the metric. Recall from Theorem 7.2 that, for $g^M \in \mathcal{M}(\nabla)$,

$$\mathrm{LDet}_{\mathrm{gr},\theta}\big(B_{\mathrm{even}}(\nabla, g^M)\big) = \xi(\nabla, g^M, \theta) - i\pi\,\eta(\nabla, g^M), \tag{9.125}$$

where $\theta \in (-\pi/2, 0)$ is an Agmon angle for B such that there are no eigenvalues of the operator B in the solid angles $L_{(-\pi/2,\theta]}$ and $L_{(\pi/2,\theta+\pi]}$.

The following proposition is proven on page 52 of [**21**]. See also Theorem 2.4 of [**3**] where the result in the case of a unitary connection is established.

Proposition 9.5. *Modulo $\mathbb{Z}$, the difference*

$$\eta(\nabla, g^M) - (\mathrm{rank}\,E)\;\eta_{\mathrm{trivial}}(g^M)$$

is independent of the Riemannian metric g^M. In particular, the imaginary part $\mathrm{Im}\,\eta(\nabla, g^M)$ of $\eta(\nabla, g^M)$ is independent of g^M.

To prove Theorem 9.3 we need to study the dependence of $\xi = \xi(\nabla, g^M, \theta)$ on g^M.

9.6. Dependence of ξ on the Riemannian metric. For (9.125) to hold we need to assume that there are no eigenvalues of the operator B in the solid angles $L_{(-\pi/2,\theta]}$ and $L_{(\pi/2,\theta+\pi]}$. However, for the study of the dependence of ξ on g^M it will be convenient for us to work with $\xi(\nabla, g^M, \theta)$ with the only assumption that both $\theta \in (-\pi, 0)$ and $\theta + \pi$ are Agmon angles for B_{even}. If θ_1 and θ_2 are two such angles then, by (3.26) and (7.79),

$$\xi(\nabla, g^M, \theta_1) \equiv \xi(\nabla, g^M, \theta_2) \qquad \mathrm{mod}\;\; \pi i. \tag{9.126}$$

Proposition 9.7. *Suppose g_0^M, $g_1^M \in \mathcal{M}(\nabla)$ are admissible Riemannian metrics on M and let $\theta_0, \theta_1 \in (-\pi/2, 0)$ be such that, for $j = 0, 1$, both θ_j and $\theta_j + \pi$ are Agmon angles for $B(\nabla, g_j^M)$. Then*

$$\xi(\nabla, g_1^M, \theta_1) \equiv \xi(\nabla, g_0^M, \theta_0) \qquad \mod \ \pi i. \tag{9.127}$$

Remark 9.8. If, in addition, ∇ is Hermitian and h^E is a ∇-invariant Hermitian metric on E, then, from (6.64) we obtain $(\Gamma\,\nabla)^2 = \nabla^*\,\nabla$. By (8.97), $\xi(t, \theta_0)$ coincides, in this case, with the Ray-Singer torsion. Hence, in the case of a Hermitian connection, the statement of the proposition reduces to the classical result about the independence of the Ray-Singer torsion on the Riemannian metric.

For the proof of the proposition we first consider the case when g_0^M and g_1^M belong to the same path-connected component of the set $\mathcal{M}(\nabla)$ of admissible metrics.

Suppose that $g_t^M \in \mathcal{M}(\nabla)$, $t \in \mathbb{R}$, is a smooth family of admissible Riemannian metrics on M and let $B_t = B(\nabla, g_t^M)$ be the corresponding odd signature operator. To simplify the notation set

$$\xi(t, \theta) := \xi(\nabla, g_t^M, \theta).$$

Fix $t_0 \in \mathbb{R}$ and let $\theta_0 \in (-\pi/2, 0)$ be an Agmon angle for B_{t_0} such that there are no eigenvalues of B_{t_0} in the solid angles $L_{(-\pi/2, \theta_0]}$ and $L_{(\pi/2, \theta_0+\pi)}$. Choose $\delta > 0$ so that for every $t \in (t_0 - \delta, t_0 + \delta)$ both θ_0 and $\theta_0 + \pi$ are Agmon angles of B_t. For $t \neq t_0$ it might happen that there are eigenvalues of B_t in $L_{(-\pi/2, \theta_0)}$ and/or $L_{(\pi/2, \theta_0+\pi)}$. Hence, (9.125) is not necessarily true, in general, for $t \neq t_0$. However, from (9.126), we conclude that for every $t \in (t_0 - \delta, t_0 + \delta)$ and $\theta \in (-\pi/2, 0)$, such that θ and $\theta + \pi$ are Agmon angles for B_t,

$$\xi(t, \theta) \equiv \xi(t, \theta_0) \qquad \mod \ \pi i. \tag{9.128}$$

Lemma 9.9. *Under the above assumptions, $\xi(t, \theta_0)$ is independent of $t \in (t_0 - \delta, t_0 + \delta)$.*

Proof. Let Γ_t denote the chirality operator corresponding to the metric g_t^M. Then, cf. Lemma 3.7 of [**11**] or formula (1.13) of [**28**],

$$\begin{aligned} \frac{d}{dt}\, \mathrm{LDet}_{2\theta_0} \left((\Gamma_t \nabla)^2 \big|_{\Omega^k_+(M,E)} \right) \\ = \mathrm{F.\,p.}_{s=0}\, \mathrm{Tr} \left[\left(\frac{d}{dt}\, (\Gamma_t \nabla)^2 \right) \left((\Gamma_t \nabla)^2 \right)^{-s-1}_{2\theta_0} \big|_{\Omega^k_+(M,E)} \right], \end{aligned} \tag{9.129}$$

where we use the notation $\mathrm{F.\,p.}_{s=0}$ introduced in Subsection 8.9.

We denote by $\dot\Gamma_t$ the derivative of Γ_t with respect to the parameter t. Then

$$\frac{d}{dt}\,(\Gamma_t\nabla)^2\big|_{\Omega^k_+(M,E)} = \dot\Gamma_t\,\Gamma_t\,(\Gamma_t\nabla)^2\big|_{\Omega^k_+(M,E)} + (\Gamma_t\nabla)\big|_{\Omega^{d-k-1}_+(M,E)}\dot\Gamma_t\Gamma_t(\Gamma_t\nabla)\big|_{\Omega^k_+(M,E)}, \tag{9.130}$$

where we used that $\Gamma_t^2 = 1$. Using (9.130) and the equality $\operatorname{Tr} AB = \operatorname{Tr} BA$, we obtain from (9.129) that

$$\begin{aligned}\frac{d}{dt}\,&\operatorname{LDet}_{2\theta_0}\Big((\Gamma_t\,\nabla)^2\big|_{\Omega^k_+(M,E)}\Big)\\ &= \mathrm{F.\,p.}_{s=0}\operatorname{Tr}\Big[\dot\Gamma_t\Gamma_t\,\big((\Gamma_t\,\nabla)^2\big|_{\Omega^k_+(M,E)}\big)^{-s}_{2\theta_0}\\ &\quad+\dot\Gamma_t\,\Gamma_t\,\big((\Gamma_t\,\nabla)^2\big|_{\Omega^{d-k-1}_+(M,E)}\big)^{-s}_{2\theta_0}\Big].\end{aligned} \tag{9.131}$$

Hence,

$$\begin{aligned}\frac{d}{dt}\,&\sum_{k=0}^{d}(-1)^k\operatorname{LDet}_{2\theta_0}\Big((\Gamma_t\,\nabla)^2\big|_{\Omega^k_+(M,E)}\Big)\\ &= 2\sum_{k=0}^{d}(-1)^k\ \mathrm{F.\,p.}_{s=0}\operatorname{Tr}\Big[\dot\Gamma_t\Gamma_t\,\big((\Gamma_t\,\nabla)^2\big|_{\Omega^k_+(M,E)}\big)^{-s}_{2\theta_0}\Big].\end{aligned} \tag{9.132}$$

Similarly,

$$\begin{aligned}\frac{d}{dt}\,&\sum_{k=0}^{d}(-1)^{k-1}\operatorname{LDet}_{2\theta_0}\Big((\nabla\Gamma_t)^2\big|_{\Omega^k_-(M,E)}\Big)\\ &= 2\sum_{k=0}^{d}(-1)^{k-1}\ \mathrm{F.\,p.}_{s=0}\operatorname{Tr}\Big[\Gamma_t\dot\Gamma_t\,\big((\nabla\Gamma_t)^2\big|_{\Omega^k_-(M,E)}\big)^{-s}_{2\theta_0}\Big].\end{aligned} \tag{9.133}$$

From (7.79), we see that (9.132) is equal to $2\frac{d}{dt}\xi(t,\theta_0)$. By (7.93), the left hand sides of (9.132) and (9.133) are equal. Hence (9.133) is also equal to $2\frac{d}{dt}\xi(t,\theta_0)$. We conclude that

$$\begin{aligned}\frac{d}{dt}\,\xi(t,\theta_0) &= \sum_{k=0}^{d}(-1)^k\ \mathrm{F.\,p.}_{s=0}\operatorname{Tr}\Big[\dot\Gamma_t\Gamma_t\,\big((\Gamma_t\,\nabla)^2\big|_{\Omega^k_+(M,E)}\big)^{-s}_{2\theta_0}\Big]\\ &= \sum_{k=0}^{d}(-1)^{k-1}\,\mathrm{F.\,p.}_{s=0}\operatorname{Tr}\Big[\Gamma_t\dot\Gamma_t\,\big((\nabla\Gamma_t)^2\big|_{\Omega^k_-(M,E)}\big)^{-s}_{2\theta_0}\Big].\end{aligned} \tag{9.134}$$

Hence,

$$
\begin{aligned}
(9.135)\qquad & 2\,\frac{d}{dt}\,\xi(t,\theta_0) \\
&= \sum_{k=0}^{d} (-1)^k \ \mathrm{F.\,p.}_{s=0}\,\mathrm{Tr}\,\Big[\dot\Gamma_t\Gamma_t\,\big((\Gamma_t\,\nabla)^2\big|_{\Omega^k_+(M,E)}\big)^{-s}_{2\theta_0} \\
&\qquad - \Gamma_t\dot\Gamma_t\,\big((\nabla\Gamma_t)^2\big|_{\Omega^k_-(M,E)}\big)^{-s}_{2\theta_0}\Big].
\end{aligned}
$$

Since $\Gamma_t^2 = 1$, we obtain $\dot\Gamma_t\Gamma_t + \Gamma_t\dot\Gamma_t = \frac{d}{dt}\Gamma_t^2 = 0$. Hence, (9.135) can be rewritten as

$$
(9.136)\qquad 2\,\frac{d}{dt}\,\xi(t,\theta_0) = \sum_{k=0}^{d} (-1)^k \ \mathrm{F.\,p.}_{s=0}\,\mathrm{Tr}\,\Big[\dot\Gamma_t\Gamma_t\,\widetilde\Delta_k(t)^{-s}_{2\theta_0}\Big],
$$

where $\widetilde\Delta_k(t) = (\Gamma_t\nabla)^2 + (\nabla\Gamma_t)^2$ (k=0,. . . ,d). By a slight generalization of a result of Seeley [**39**], which is discussed in [**47**], the right hand side of (9.136) is given by a *local formula*, i.e., by the integral (8.123) of a differential form ϕ_t, whose value at any point $x \in M$ depends only on the values of the components of the metric tensor g_t^M and a finite number of their derivatives at x. Moreover, since the dimension of the manifold M is odd, the differential form ϕ_t vanishes identically. Hence, $\frac{d}{dt}\xi(t,\theta_0) = 0$ for all $t \in (t_0-\delta, t_0+\delta)$. q.e.d.

9.10. Proof of Proposition 9.7. Set

$$
g_t^M = (1-t)\,g_0^M + t\,g_1^M, \qquad t \in [0,1],
$$

and let Γ_t denote the chirality operator corresponding to the metric g_t^M. The operators Γ_t depend real analytically on t and we can extend their definition to all t in some connected open connected neighborhood $U \subset \mathbb{C}$ of $[0,1]$. Hence, the operator

$$
B_t := \Gamma_t\,\nabla + \nabla\,\Gamma_t
$$

is well defined for all $t \in U$ and is holomorphic on U in the sense of Subsection 5.4. If we choose the neighborhood U to be small enough then $B_t^2 \in \mathrm{Ell}_{m,(-5\pi/4,-3\pi/4)}(M,E)$ for all $t \in U$. By Corollary 5.10 the set

$$
\Sigma := \big\{t \in U :\ B_t \text{ is not invertible}\big\}
$$

is a complex analytic subset of U. Thus $U\backslash\Sigma$ is connected. Since the metrics g_0^M and g_1^M are admissible, it follows that 0 and 1 are in $U\backslash\Sigma$. For $\theta \in (-\pi/2, 0)$ such that both θ and $\theta+\pi$ are Agmon angles for B_t, set

$$
(9.137)\qquad \xi(t,\theta) := \frac{1}{2}\sum_{k=0}^{d} (-1)^{k+1}\,k\ \mathrm{LDet}_{2\theta}\,\Big[B_t^2\big|_{\Omega^k(M,E)}\Big].
$$

If $t \in [0,1]$ is such that the metric g_t^M is admissible, then $\xi(t,\theta) = \xi(\nabla, g_t^M, \theta)$.

By Theorem 5.7.a, the function $t \mapsto \xi(t,\theta)$ is holomorphic on the open set

$$U_\theta := \big\{ t \in U\backslash\Sigma : \quad \theta,\ \theta+\pi \text{ are Agmon angles for } B_t \big\}.$$

By (9.128), if $t \in U_{\theta_1} \cap U_{\theta_2}$ then $\xi(t,\theta_1) \equiv \xi(t,\theta_2)$ modulo $\pi\mathbb{Z}$. Hence, we can define a multivalued analytic function on $U\backslash\Sigma$ by the formula

$$t \ \mapsto \ \xi(t,\theta_t) + \pi\mathbb{Z},$$

where $\theta_t \in (-\pi/2, 0)$ is any angle such that $t \in U_{\theta_t}$.

Since the metric g_0^M is admissible there exists $\varepsilon > 0$ such that for all real $t \in [0,\varepsilon]$ the metric g_t^M is admissible and θ_0 and $\theta_0 + \pi$ are Agmon angles for B_t. Hence, by Lemma 9.9, the holomorphic function $\xi(t,\theta_0)$ is constant on $[0,\varepsilon]$. Thus, since the set $U\backslash\Sigma$ is connected, our multivalued analytic function $t \mapsto \xi(t,\theta)$ is constant on $U\backslash\Sigma$. q.e.d.

9.11. Proof of Theorem 9.3. The fact that (9.124) is independent of θ follows immediately from (3.26). Let us prove that it is independent of $g^M \in \mathcal{M}(\nabla)$. Suppose g_0^M and g_1^M are admissible metrics. We shall use the notation introduced in Subsection 9.10. For $t \in U\backslash\Sigma$, fix $\theta_t \in (-\pi/2, 0)$ such that there are no eigenvalues of B_t in the solid angles $L_{(-\pi/2,\theta_t]}$ and $L_{(\pi/2,\theta_t+\pi)}$. As t is not necessarily real, in general, B_t is not an odd signature operator associated to a Riemannian metric. Hence, to calculate $\mathrm{LDet}_{\mathrm{gr},\theta_t}(B_t)$ we can not use Theorem 7.2. However, a verbatim repetition of the proof of this theorem shows that

$$\mathrm{LDet}_{\mathrm{gr},\theta_t}\big(B_t\big) = \xi(t,\theta_t) - i\pi\,\eta(B_t),$$

where $\xi(t,\theta_t)$ is defined by (9.137). It follows now from Propositions 9.5 and 9.7 that

$$\begin{aligned} (9.138) \quad & \mathrm{Det}_{\mathrm{gr},\theta_0}\big(B_{\mathrm{even}}(\nabla, g_0^M)\big) \cdot e^{i\pi\,(\mathrm{rank}\,E)\ \eta\left(B_{\mathrm{trivial}}(g^M)\right)} \\ & \qquad = \pm\ \mathrm{Det}_{\mathrm{gr},\theta_1}\big(\,B_{\mathrm{even}}(\nabla, g_1^M)\,\big) \cdot e^{i\pi\,(\mathrm{rank}\,E)\ \eta\left(B_{\mathrm{trivial}}(g^M)\right)}, \end{aligned}$$

where θ_j $(j = 0, 1)$ is an Agmon angle for $B_{\mathrm{even}}(\nabla, g_j^M)$.

Since the function

$$t \ \mapsto \ \mathrm{Det}_{\mathrm{gr},\theta_t}\big(\,B_{\mathrm{even}}(\nabla, g_t^M)\,\big) \cdot e^{i\pi\,(\mathrm{rank}\,E)\ \eta\left(B_{\mathrm{trivial}}(g^M)\right)}$$

(which is independent of θ_t) is continuous on the connected set $U\backslash\Sigma$, the sign on the right hand side of (9.138) must be positive. The theorem is proven. q.e.d.

10. Refined Analytic Torsion

Theorem 9.3 justifies the following

Definition 10.1. Let M be a closed oriented manifold of odd dimension $\dim M = 2r-1$. Suppose that there exists a Riemannian metric g^M such that $\nabla \in \mathrm{Flat}'(M, g^M)$. The *refined analytic torsion* $T(\nabla)$ is defined by the formula

$$T(\nabla) = T(M, E, \nabla) := \mathrm{Det}_{\mathrm{gr},\theta}(B_{\mathrm{even}}) \cdot e^{i\pi\,(\mathrm{rank}\,E)\;\eta_{\mathrm{trivial}}(g^M)} \in \mathbb{C}\backslash\{0\}, \tag{10.139}$$

where $\theta \in (-\pi, 0)$ is an Agmon angle for the operator $B_{\mathrm{even}} = B_{\mathrm{even}}(\nabla, g^M)$.

In particular, if $\dim M \equiv 1 (\mathrm{mod}\, 4)$, then $\eta_{\mathrm{trivial}} = 0$, cf. [**2**] and $T(\nabla)$ is equal to the graded determinant of the odd signature operator B_{even}.

Note that $T(\nabla) \neq 0$ and does not depend on the choice of $\theta \in (-\pi, 0)$, cf. (3.27). Further, by Corollary 8.3, if ∇ is a Hermitian connection then

$$|T(\nabla)| = T^{\mathrm{RS}}(\nabla). \tag{10.140}$$

Substituting (9.125) into (10.139) we obtain

$$T(\nabla) = e^{\xi} \cdot e^{-i\pi\left(\eta - (\mathrm{rank}\,E)\eta_{\mathrm{trivial}}\right)}. \tag{10.141}$$

The expression $\eta - (\mathrm{rank}\,E)\eta_{\mathrm{trivial}}$ is known as the ρ-invariant of ∇ and is independent of the metric g^M modulo $\mathbb{Z}$.

10.2. Example. We now calculate the refined analytic torsion in the simplest possible example, when $M = \mathbb{R}/2\pi\mathbb{Z}$ is the circle and $E = M \times \mathbb{C}$ is the trivial line bundle over M. Fix a number $a \in \mathbb{C}\backslash\mathbb{Z}$ and define the connection ∇_a on E by the formula

$$\nabla_a : f \;\mapsto\; df + iafdx, \qquad f \in \Omega^0(M, E) = \Omega^0(M),$$

where $x \in [0, 2\pi)$ is the coordinate on M. According to the formula (6.61), the odd signature operator is

$$B_{\mathrm{even}} = B_0 : f \;\mapsto\; -i * \nabla_a f = -if' + af.$$

As $a \in \mathbb{C}\backslash\mathbb{Z}$, the operator $B_0 : \Omega^0(M) \to \Omega^0(M)$ is invertible and its eigenvalues are given by $a + n$, $n \in \mathbb{Z}$. Since $\mathrm{Re}\, a \in \mathbb{R}\backslash\mathbb{Z}$, the angle $\theta = -\pi/2$ is an Agmon angle for B_0.

The refined analytic torsion $T(\nabla_a) = \mathrm{Det}_\theta(B_0)$ can be easily calculated using, for example, the general formula for determinants of elliptic operators on the circle, obtained in [**10**] (see also [**48**] for an alternative way of calculation). As $\dim M \equiv 1 (\mathrm{mod}\, 4)$ we obtain the following formula for the refined analytic torsion

$$T(\nabla_a) = 1 - e^{2a\pi i} = 2\,(\sin \pi a)\, e^{i\frac{\pi}{2}(2a-1)}.$$

Note that if $a \in \mathbb{R}\backslash\mathbb{Z}$ then ∇_a is a Hermitian connection. We conclude that, even for a Hermitian connection, the refined analytic torsion is a complex number, which, depending on the value of a, can have an arbitrary phase, aside from $\pm\pi/2$.

11. An Alternative Definition of the Refined Analytic Torsion

The correction term $\exp(i\pi(\operatorname{rank} E)\eta_{\text{trivial}})$ in Definition 10.1 is hard to compute. In this section we suggest an alternative definition of the refined analytic torsion, where we replace η_{trivial} with an expression, which depends on some choices, but is much easier to compute.

11.1. Dependence of η_{trivial} on the metric. First, we describe the dependence of $\eta_{\text{trivial}}(g^M)$ on the Riemannian metric g^M.

11.1.1. Case when M bounds an oriented manifold N'. Suppose, first, that M is the oriented boundary of a smooth compact oriented manifold N'. Let $\operatorname{sign}(N')$ denote the signature of N', cf. [**2**]. This is an integer defined in purely cohomological terms. In particular, it is independent of the metric. The signature theorem for manifolds with boundary (cf. Theorem 4.14 of [**2**] and Theorem 2.2 of [**3**]) states that

$$\operatorname{sign}(N') = \int_{N'} L(p) - \eta(B_{\text{trivial}}), \tag{11.142}$$

where $L(p) := L_{N'}(p)$ is the Hirzebruch L-polynomial in the Pontrjagin forms of a Riemannian metric on N' which is a product near M. It follows from (11.142) that $\int_{N'} L(p)$ is independent of the choice of the Riemannian metric on N' among those that near $\partial N'$ are equal to the product of the given metric g^M on M and a metric on the interval. Note that if $\dim M \equiv 1 \pmod 4$ then $L(p)$ does not have a term of degree $\dim N'$ and, hence, $\int_{N'} L(p) = 0$.

Combining (11.142) with the metric independence of $\operatorname{sign}(N')$ and Proposition 9.5, we conclude that, modulo $\mathbb{Z}$,

$$\eta - (\operatorname{rank} E) \int_{N'} L(p) \tag{11.143}$$

is independent of the metric g^M. Since for different choices of N', the integral $\int_{N'} L(p)$ differs by an integer, the expression (11.143), modulo $\mathbb{Z}$, is also independent of N'.

11.1.2. General case (M does not necessarily bound an oriented manifold). In general, there might be no smooth oriented manifold whose oriented boundary is diffeomorphic to M. However, since $\dim M$ is odd, there exists an oriented manifold N whose oriented boundary is the disjoint union of two copies of M (with the same orientation), cf. [**44**], [**38**, Th. IV.6.5]. Then the same arguments as above show that, modulo $\mathbb{Z}$,

$$\eta - \frac{\operatorname{rank} E}{2} \int_{N} L(p) \tag{11.144}$$

is metric independent. In particular, if $\dim M \equiv 1 \pmod 4$, then the reduction of η modulo $\mathbb{Z}$ is metric independent.

Remark 11.2. Note again that replacing η by (11.144) removes the dependence on the metric but creates a new dependence on the choice of the manifold N. For different choices of N the integrals $\int_N L(p)$ might differ by an integer.

11.3. Alternative definition of the refined analytic torsion. Let M be a closed oriented manifold of dimension $\dim M = 2r-1$. Assume that there exists a Riemannian metric g^M on M such that $\nabla \in \mathrm{Flat}'(M, g^M)$. Let $\theta \in (-\pi, 0)$ be an Agmon angle for $B_{\mathrm{even}} = B_{\mathrm{even}}(\nabla, g^M)$. Choose a smooth compact oriented manifold N whose oriented boundary is diffeomorphic to two disjoint copies of M. Then one can define a version of the refined analytic torsion

$$T'(\nabla) = T'(M, E, \nabla, N) := \mathrm{Det}_{\mathrm{gr},\theta}(B_{\mathrm{even}}) \cdot \exp\Big(i\pi \frac{\operatorname{rank} E}{2} \int_N L(p)\Big), \tag{11.145}$$

where $L(p) = L_N(p)$ is the Hirzebruch L-polynomial in the Pontrjagin forms of a Riemannian metric on N which is a product near ∂N.

Remark 11.4. It follows from the above discussion and Theorem 9.3 that $T'(\nabla)$ is independent of the angle $\theta \in (-\pi, 0)$ and of the metric. But *it does depend on the choice of the manifold* N. However, from Proposition 9.5, we conclude that $T'(\nabla)$ is independent of the choice of N up to multiplication by $i^{k \cdot \operatorname{rank} E}$ $(k \in \mathbb{Z})$. If $\operatorname{rank} E$ is even then $T'(\nabla)$ is well defined up to a sign, and if $\operatorname{rank} E$ is divisible by 4, then $T'(\nabla)$ is a well defined complex number.

(Here a quantity being well defined means that it depends only on M, E and ∇.)

Remark 11.5. If M is the oriented boundary of a smooth compact oriented manifold N', one can define still another version of the refined analytic torsion:

$$T^{\#}(\nabla) = T^{\#}(M, E, \nabla, N') := \mathrm{Det}_{\mathrm{gr},\theta}(B_{\mathrm{even}}) \cdot \exp\Big(i\pi \cdot \operatorname{rank} E \int_{N'} L(p)\Big). \tag{11.146}$$

Note that the indeterminacy in the definition of $T^{\#}(\nabla)$ is smaller than the indeterminacy in the definition of $T^{\#}(\nabla)$, cf. Remark 11.4, as $T^{\#}(\nabla)$ is well defined up to a sign. If $\operatorname{rank} E$ is even, then $T^{\#}(\nabla)$ is a well defined complex number.

12. Comparison Between the Refined Analytic and the Ray-Singer Torsions

Assume that M is a closed oriented odd-dimensional manifold and g^M is a Riemannian metric on M. By Theorem 8.2, there exists a C^0-open neighborhood $U \subset \mathrm{Flat}(E)$ of the set of acyclic Hermitian connections on E, such that, for every $\nabla \in U$,

$$\left| \operatorname{Det}_{\mathrm{gr},\theta}(B_{\mathrm{even}}) \right| = T^{\mathrm{RS}}(\nabla) \cdot e^{\pi \operatorname{Im} \eta(\nabla, g^M)}. \tag{12.147}$$

Combining this equality with (7.80) and the definition of the refined analytic torsion we obtain

Theorem 12.1. *Assume that M is a closed oriented odd-dimensional manifold and g^M is a Riemannian metric on M. Then there exists a C^0-open neighborhood $U \subset \mathrm{Flat}(E)$ of the set of acyclic Hermitian connections on E, such that $U \subset \mathrm{Flat}'(E, g^M)$ and for all $\nabla \in U$*

$$\log \frac{|T(\nabla)|}{T^{\mathrm{RS}}(\nabla)} = \pi \operatorname{Im} \eta(\nabla, g^M). \tag{12.148}$$

In this section we present a local expression for the right hand side of (12.148).

12.2. Dependence of the η-invariant on the connection. Suppose that $t \mapsto \nabla_t$, $t \in [0,1]$, is a smooth path of connections in $\mathrm{Flat}'(E, g^M)$. We shall need the following result of Gilkey [**21**, Th. 3.7][11] (see also Theorem 7.6 of [**18**][12]):

Theorem 12.3. *Let $\overline{\eta}(\nabla_t, g^M) \in \mathbb{C}/\mathbb{Z}$ denote the reduction of $\eta(\nabla_t, g^M)$ modulo $\mathbb{Z}$. Then $\overline{\eta}(\nabla_t, g^M)$ depends smoothly on t, cf.* [**21**, §1].

1. *If* $\dim M \equiv 3 \pmod 4$ *then* $\overline{\eta}(\nabla_t, g^M)$ *is independent of* $t \in [0,1]$.
2. *Suppose* $\dim M \equiv 1 \pmod 4$. *Set*

$$\psi_t := \frac{d}{dt} \nabla_t \in \Omega^1(M, \operatorname{End} E). \tag{12.149}$$

Then

$$\frac{d}{dt} \overline{\eta}(\nabla_t, g^M) = \frac{i}{2\pi} \int_M L(p) \wedge \operatorname{Tr}(\psi_t), \tag{12.150}$$

where $L(p) = L_M(p)$ is the Hirzebruch L-polynomial in the Pontrjagin forms of g^M.

[11] Note that Gilkey considered the η-invariant of the full odd signature operator $B = B_{\mathrm{even}} \oplus B_{\mathrm{odd}}$. Hence, our $\eta(\nabla, g^M)$ is equal to one half of the invariant considered in [**21**].

[12] Note, however, that in the formula of Theorem 7.6 of [**18**] the sign has to be replaced by the opposite one.

12.4. Cohomology class $\mathbf{Arg}_\nabla$. Following Farber, [**17**], we denote by $\mathbf{Arg}_\nabla$ the unique cohomology class $\mathbf{Arg}_\nabla \in H^1(M, \mathbb{C}/\mathbb{Z})$ such that for every closed curve $\gamma \in M$ we have

$$\det\big(\mathrm{Mon}_\nabla(\gamma)\big) = \exp\big(2\pi i \langle \mathbf{Arg}_\nabla, [\gamma]\rangle\big), \tag{12.151}$$

where $\mathrm{Mon}_\nabla(\gamma)$ denotes the monodromy of the flat connection ∇ along the curve γ and $\langle\cdot,\cdot\rangle$ denotes the natural pairing

$$H^1(M, \mathbb{C}/\mathbb{Z}) \times H_1(M, \mathbb{Z}) \longrightarrow \mathbb{C}/\mathbb{Z}.$$

Remark 12.5. The notation $\mathbf{Arg}_\nabla$ is motivated by the case where ∇ is a Hermitian connection. In this case, $\mathrm{Mon}_\nabla(\gamma)$ is unitary and $\mathbf{Arg}_\nabla \in H^1(M, \mathbb{R}/\mathbb{Z})$. Therefore, the expression $2\pi\langle \mathbf{Arg}_\nabla, [\gamma]\rangle$ is equal to the phase of the complex number $\det(\mathrm{Mon}_\nabla(\gamma))$.

Lemma 12.6. *Assume that ∇_t ($t \in [0,1]$) is a smooth path of connections. Then, using the notation introduced in Theorem* 12.3.2, *we have*

$$2\pi i \frac{d}{dt}\, \mathbf{Arg}_{\nabla_t} = -\big[\ \mathrm{Tr}\ \psi_t\ \big] \ \in\ H^1(M, \mathbb{C}), \tag{12.152}$$

where $\big[\ \mathrm{Tr}\ \psi_t\ \big]$ denotes the cohomology class of the closed differential form $\mathrm{Tr}\ \psi_t$.

Proof. Let S^1 be the standard circle and let $x \in [0, 2\pi)$ be the coordinate on S^1. Let $\gamma : S^1 \to M$ be a closed curve. Fix a trivialization of the bundle $\gamma^* E \to S^1$. Let $A_t(x)$ denote the (periodically extended to $\mathbb{R}$) connection form on $\gamma^* E$ induced by ∇_t. Then, for each $t \in [0,1]$, the monodromy $\mathrm{Mon}_{\nabla_t}(\gamma)$ along γ is given by the matrix $\Phi_t(2\pi)$ where $\Phi_t(x)$ is the matrix function solving the following initial value problem

$$\begin{aligned} &\frac{\partial}{\partial x}\,\Phi_t(x) + A_t(x)\Phi_t(x) = 0, \qquad x \in \mathbb{R};\\ &\Phi_t(0) = \mathrm{Id}\,. \end{aligned} \tag{12.153}$$

Let $\dot\Phi_t(x)$ and $\dot A_t(x)$ denote the derivative with respect to t of the matrices $\Phi_t(x)$ and $A_t(x)$ respectively.

We are interested in computing

$$\begin{aligned} 2\pi i \frac{d}{dt}\,\big\langle\ \mathbf{Arg}_{\nabla_t}, [\gamma]\ \big\rangle &= \frac{d}{dt}\ \log\det\ \Phi_t(2\pi)\\ &= \frac{d}{dt}\ \mathrm{Tr}\log\ \Phi_t(2\pi)\\ &= \mathrm{Tr}\ \dot\Phi_t(2\pi)\,\Phi_t(2\pi)^{-1}. \end{aligned} \tag{12.154}$$

Note that, though log is a multivalued function, the derivatives

$$\frac{\partial}{\partial t}\ \log\det\ \Phi_t(2\pi) \quad \text{and} \quad \frac{\partial}{\partial t}\ \mathrm{Tr}\log\ \Phi_t(2\pi)$$

are unambiguously defined complex numbers.

From (12.153), we obtain

$$\frac{\partial}{\partial x}\,\dot\Phi_t(x) + \dot A_t(x)\,\Phi_t(x) + A_t(x)\,\dot\Phi_t(x) = 0. \tag{12.155}$$

Hence,

$$\left(\frac{\partial}{\partial x}\,\dot\Phi_t(x)\right)\,\Phi_t(x)^{-1} = -\dot A_t(x) - A_t(x)\,\dot\Phi_t(x)\,\Phi_t(x)^{-1}. \tag{12.156}$$

On the other side,

(12.157)

$$\begin{aligned}
&\mathrm{Tr}\,\frac{\partial}{\partial x}\left(\dot\Phi_t(x)\,\Phi_t(x)^{-1}\right)\\
&= \mathrm{Tr}\,\Big(\frac{\partial}{\partial x}\,\dot\Phi_t(x)\Big)\Phi_t(x)^{-1} - \mathrm{Tr}\;\dot\Phi_t(x)\,\Phi_t(x)^{-1}\Big(\frac{\partial}{\partial x}\,\Phi_t(x)\Big)\,\Phi_t(x)^{-1}\\
&= \mathrm{Tr}\,\Big(\frac{\partial}{\partial x}\,\dot\Phi_t(x)\Big)\Phi_t(x)^{-1} + \mathrm{Tr}\;A_t(x)\,\dot\Phi_t(x)\,\Phi_t(x)^{-1},
\end{aligned}$$

where in the last equality we used (12.153).

Combining (12.157) with (12.156) and using (12.149) we get

$$\frac{\partial}{\partial x}\;\mathrm{Tr}\;\dot\Phi_t(x)\,\Phi_t(x)^{-1} = -\;\mathrm{Tr}\;\dot A_t(x) = -\;\mathrm{Tr}\;\gamma^*\psi_t(x). \tag{12.158}$$

Here $\gamma^*\psi_t$ denotes the pull-back of the differential form ψ_t under the map $\gamma : S^1 \to M$.

From (12.154) and (12.158) we obtain

$$\begin{aligned}
\frac{d}{dt}\,2\pi i\langle \mathbf{Arg}_{\nabla_t}, [\gamma]\rangle &= \frac{d}{dt}\Big(\log\det\,\Phi_t(2\pi) - \log\det\,\Phi_t(0)\Big)\\
&= -\int_0^{2\pi}\mathrm{Tr}\;\dot A_t(x)\,dx\\
&= -\big\langle\,[\mathrm{Tr}\;\psi_t], [\gamma]\,\big\rangle.
\end{aligned}$$

q.e.d.

From Lemma 12.6 and (12.150) we obtain

$$\begin{aligned}
\frac{d}{dt}\,\overline{\eta}(\nabla_t, g^M) &= \big\langle\,[L(p)]\cup\frac{i}{2\pi}[\mathrm{Tr}\;\psi_t], [M]\,\big\rangle\\
&= \big\langle\,[L(p)]\cup\frac{d}{dt}\,\mathbf{Arg}_{\nabla_t}, [M]\,\big\rangle,
\end{aligned} \tag{12.159}$$

where $\cup$ denotes the cup-product in cohomology.

Assume that ∇_t ($t\in[0,1]$) is a smooth path of acyclic connections and that the connection ∇_0 is Hermitian. By Remark 12.5, $\mathrm{Im}\,\mathbf{Arg}_{\nabla_0} = 0$ and, thus, (12.152) leads to

$$\Big[\int_0^t \mathrm{Tr}\,(\mathrm{Re}\,\psi_t)\,dt\Big] = 2\pi\;\mathrm{Im}\,\mathbf{Arg}_{\nabla_t} \in H^1(M,\mathbb{R}). \tag{12.160}$$

12.7. Comparison with the Ray-Singer torsion. Let

$$U \ \subset \ \mathrm{Flat}'(E, g^M)$$

be as in Theorem 12.1. Denote by $U' \subset U$ the set of flat connections satisfying the following condition: for every $\nabla \in U'$ there exists a smooth path $t \mapsto \nabla_t \in U$, $t \in [0,1]$, of connections such that ∇_0 is Hermitian, and $\nabla_1 = \nabla$. Then $U' \subset \mathrm{Flat}'(E, g^M)$ is an open neighborhood of the set of acyclic Hermitian connections.

Theorem 12.8. *Let M be a closed oriented odd-dimensional manifold and let g^M be a Riemannian metric on M. Suppose $\nabla \in U'$. Then, with $L(p) = L_M(p)$ denoting the Hirzebruch L-polynomial in the Pontrjagin forms of a Riemannian metric on M,*

$$\log \frac{|T(\nabla)|}{T^{\mathrm{RS}}(\nabla)} = \pi \left\langle\, [L(p)] \cup \mathrm{Im}\, \mathbf{Arg}_\nabla, [M] \,\right\rangle. \tag{12.161}$$

In the case $\dim M \equiv 3 \pmod 4$

$$|T(\nabla)| = T^{\mathrm{RS}}(\nabla). \tag{12.162}$$

Remark 12.9.

1. The advantage of Theorem 12.8 over Theorem 12.1 is that the right hand side of (12.161) is given by a local formula. Hence, it might be possible to effectively compute it in some examples.
2. When $\dim M \equiv 3 \pmod 4$ the right hand side of (12.161) vanishes since $L(p)$ has no component of degree $\dim M - 1$ and, hence,

$$[L(p)] \cup \mathrm{Im}\, \mathbf{Arg}_\nabla$$

does not have a component of degree $\dim M$.

Proof. Let $\nabla_t \in U'$ $(0 \le t \le 1)$ be a smooth path of connections such that ∇_0 is a Hermitian connection and $\nabla_1 = \nabla$. Then, by Theorem 12.1,

$$\log \frac{|T(\nabla_t)|}{T^{\mathrm{RS}}(\nabla_t)} = \pi \ \mathrm{Im}\ \eta(\nabla_t, g^M), \qquad \text{for every} \quad t \in [0,1]. \tag{12.163}$$

As the number $\overline{\eta}(\nabla_t, g^M)$ is defined modulo integers, its imaginary part is a well defined real number and

$$\mathrm{Im}\ \overline{\eta}(\nabla_t, g^M) = \mathrm{Im}\ \eta(\nabla_t, g^M).$$

Since the connection ∇_0 is Hermitian, $\mathrm{Im}\, \eta(\nabla_0, g^M) = 0$. Hence, from Theorem 12.3.1 we conclude that $\mathrm{Im}\ \eta(\nabla, g^M) = 0$ if $\dim M \equiv 3 \pmod 4$. From Theorem 12.3.2 and (12.160) we see that

$$\begin{aligned} \mathrm{Im}\ \eta(\nabla, g^M) &= \frac{1}{2\pi} \int_0^1 \Big(\int_M L(p) \wedge \mathrm{Tr}(\mathrm{Re}\, \psi_t) \Big)\, dt \\ &= \left\langle\, [L(p)] \cup \mathrm{Im}\, \mathbf{Arg}_\nabla, [M] \,\right\rangle, \end{aligned}$$

if $\dim M \equiv 1 \pmod 4$. Theorem 12.8 follows now from (12.163). q.e.d.

13. Graded Determinant as a Holomorphic Function of a Representation of $\pi_1(M)$

In this section we show, first, that the graded determinant $\mathrm{Det}_{\mathrm{gr},\theta}(B_{\mathrm{even}}(\nabla, g^M))$ of the odd signature operator is, in an appropriate sense, a holomorphic function of the connection ∇. Then we change the point of view and consider the graded determinant as a function of the representation of the fundamental group $\pi_1(M)$ of M. More precisely, each representation α of $\pi_1(M)$ induces a flat vector bundle $(E_\alpha, \nabla_\alpha)$ over M and we denote by $B_\alpha = B(\nabla_\alpha, g^M)$ the corresponding odd signature operator. The space $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$ of all complex n-dimensional representations of $\pi_1(M)$ has a natural structure of a complex algebraic variety. We show that $\mathrm{Det}_{\mathrm{gr},\theta}(B_{\alpha,\mathrm{even}})$ is a well defined holomorphic function on an open subset of this variety. Throughout the section we assume that the dimension of M is odd.

13.1. Graded determinant as a holomorphic function. Let M be a closed oriented manifold of odd dimension $d = 2r - 1$ and let $E \to M$ be a flat vector bundle over M. Every (not necessarily flat) connection on E can be viewed as a first order differential operator on $\Omega^\bullet(M, E)$. Thus the space $\mathcal{C}(E)$ of all connections on E is an affine subspace of the space $\mathrm{Diff}_1(M, \Lambda^\bullet T^*M \otimes E)$ of first order differential operators on the complex vector bundle $\Lambda^\bullet T^*M \otimes E \to M$ and, hence, inherits from $\mathrm{Diff}_1(M, \Lambda^\bullet T^*M \otimes E)$ the structure of a Fréchet space. See Subsection 5.3 for the definition of the Fréchet topology on $\mathrm{Diff}_1(M, \Lambda^\bullet T^*M \otimes E)$.

Fix a Riemannian metric g^M on M. Recall that we denote by $\mathrm{Flat}'(E, g^M)$ the set of flat connections ∇ on E such that the pair (∇, g^M) satisfies Assumption I and II of Subsection 6.5. By (3.27), the graded determinant $\mathrm{Det}_{\mathrm{gr},\theta}\big(B_{\mathrm{even}}(\nabla, g^M)\big)$ is independent of the choice of the Agmon angle $\theta \in (-\pi, 0)$. Thus one obtains a function

$$\begin{aligned} &\mathrm{Det}_{\mathrm{gr}} : \mathrm{Flat}'(E, g^M) \longrightarrow \mathbb{C}\backslash\{0\}, \\ &\mathrm{Det}_{\mathrm{gr}} : \nabla \mapsto \mathrm{Det}_{\mathrm{gr},\theta}\big(B_{\mathrm{even}}(\nabla, g^M)\big), \end{aligned} \tag{13.164}$$

where θ is any Agmon angle of $B(\nabla, g^M)$ in the interval $(-\pi, 0)$. Recall that the notion of a holomorphic curve has been introduced in Subsection 5.1.

Proposition 13.2. *Suppose E is a vector bundle over a closed oriented odd-dimensional Riemannian manifold (M, g^M). Let $\mathcal{O} \subset \mathbb{C}$ be an open set and let $\gamma : \mathcal{O} \to \mathrm{Flat}'(E, g^M)$ be a holomorphic curve in $\mathrm{Flat}'(E, g^M)$. Then the function $\lambda \mapsto \mathrm{Det}_{\mathrm{gr}}\big(B_{\mathrm{even}}(\gamma(\lambda), g^M)\big)$ is holomorphic on $\mathcal{O}$.*

In fact, we will need a slightly more general statement. Thus we will, first, generalize Proposition 13.2 and, then, prove this more general version.

13.3. Extension of the graded determinant to non-flat connections. Recall from Theorem 7.2 that, for every connection $\nabla \in \mathrm{Flat}'(E, g^M)$,

$$\mathrm{Det}_{\mathrm{gr},\theta}\big(B_{\mathrm{even}}(\nabla, g^M)\big) = e^{\xi(\nabla, g^M, \theta)} \cdot e^{-i\pi\eta(\nabla, g^M)}, \tag{13.165}$$

where $\theta \in (-\pi/2, 0)$ is any an Agmon angle for $B(\nabla, g^M)$ such that there are no eigenvalues of the operator $B(\nabla, g^M)$ in the solid angles $L_{(-\pi/2,\theta]}$ and $L_{(\pi/2,\theta+\pi]}$.

Let $\nabla_0 \in \mathrm{Flat}'(M, g^M)$. Then $B(\nabla_0, g^M)$ is invertible. Formula (6.61) defines the odd signature operator $B(\nabla, g^M)$ for an arbitrary, not necessarily flat, connection. We wish to extend the notion of the graded determinant to operators $B(\nabla, g^M)$ with ∇ in some open neighborhood of $\mathrm{Flat}'(E, g^M)$ in $\mathcal{C}(E)$.

The same arguments as in the proof of Proposition 6.8 show that there exists a C^0-neighborhood $\mathcal{U}$ of ∇_0 in the space $\mathcal{C}(E)$ of all connections such that $B(\nabla, g^M)$ is invertible for all $\nabla \in \mathcal{U}$. As in Lemma 6.4, the leading symbol of $B(\nabla, g^M)$ is symmetric and, hence, $B(\nabla, g^M)$ admits an Agmon angle for $B(\nabla, g^M)$ such that there are no eigenvalues of the operator $B(\nabla, g^M)$ in the solid angles $L_{(-\pi/2,\theta]}$, $L_{(\pi/2,\theta+\pi]}$. Thus we can use formula (4.33) to define $\eta(\nabla, g^M) = \eta\big(B_{\mathrm{even}}(\nabla, g^M)\big)$ for all $\nabla \in \mathcal{U}$. Similarly, we can use the expression (8.102) for ξ to define $\xi(\nabla, g^M, \theta)$ to all $\nabla \in \mathcal{U}$. We now use (13.165) as the *definition* of $\mathrm{Det}_{\mathrm{gr},\theta}\big(B_{\mathrm{even}}(\nabla, g^M)\big)$ for $\nabla \in \mathcal{U}$.

Proposition 13.4. *Suppose E is a complex vector bundle over a closed oriented odd-dimensional Riemannian manifold (M, g^M) and let $\mathcal{U} \subset \mathcal{C}(E)$ be the C^0-open set defined above. Let $\mathcal{O} \subset \mathbb{C}$ be an open set and let $\gamma : \mathcal{O} \to \mathcal{U}$ be a holomorphic curve in $\mathcal{C}(E)$ such that there exists $\lambda_0 \in \mathcal{O}$ with $\gamma(\lambda_0) \in \mathrm{Flat}'(E, g^M)$. Then the function $\lambda \mapsto \mathrm{Det}_{\mathrm{gr},\theta}\big(B_{\mathrm{even}}(\gamma(\lambda), g^M)\big)$ is holomorphic in a neighborhood of λ_0. Here $\theta \in (-\pi/2, 0)$ is any Agmon angle for $B_{\mathrm{even}}(\gamma(\lambda), g^M)$ such that there are no eigenvalues of the operator $B_{\mathrm{even}}(\gamma(\lambda), g^M)$ in the solid angles $L_{(-\pi/2,\theta]}$ and $L_{(\pi/2,\theta+\pi]}$.*

Proof. Fix an Agmon angle $\theta \in (-\pi/2, 0)$ for $B_{\mathrm{even}}(\gamma(\lambda_0), g^M)$ such that there are no eigenvalues of the operator $B_{\mathrm{even}}(\gamma(\lambda_0), g^M)$ in the solid angles $L_{(-\pi/2,\theta]}$ and $L_{(\pi/2,\theta+\pi]}$. Then, for all λ in a small neighborhood of λ_0, θ is also an Agmon angle for $B_{\mathrm{even}}(\gamma(\lambda), g^M)$ and there are no eigenvalues of $B_{\mathrm{even}}(\gamma(\lambda), g^M)$ in the solid angles $L_{(-\pi/2,\theta]}$ and $L_{(\pi/2,\theta+\pi]}$.

By Corollary 5.9, the function

$$\mathcal{O} \longrightarrow \mathbb{C}, \qquad \lambda \mapsto e^{-2i\pi\eta(\gamma(\lambda), g^M)}$$

is holomorphic on $\mathcal{O}$. Similarly, Theorem 5.7.a and the expression (8.102) for ξ imply that the function $\lambda \mapsto e^{2\xi(\gamma(\lambda),g^M,\theta)}$ also is holomorphic on $\mathcal{O}$. Hence,

$$F(\lambda) := \mathrm{Det}_{\mathrm{gr},\theta}\big(B_{\mathrm{even}}(\gamma(\lambda),g^M)\big)^2 = e^{2\xi(\gamma(\lambda),g^M,\theta)} \cdot e^{-2i\pi\eta(\gamma(\lambda),g^M)}$$

is a non-vanishing holomorphic function on $\mathcal{O}$.

Since $F(\lambda)$ is a continuous function of λ and $F(\lambda_0) \neq 0$, we can find a neighborhood $\mathcal{O}' \subset \mathcal{O}$ of λ_0 such that for all $\lambda \in \mathcal{O}'$ we have

$$\big|\, F(\lambda) - F(\lambda_0)\,\big| \leq \frac{1}{2}\,\big|\, F(\lambda_0)\,\big|.$$

Then $\mathrm{Det}_{\mathrm{gr},\theta}\big(B_{\mathrm{even}}(\lambda,g^M)\big)$ coincides on $\mathcal{O}'$ with one of the two analytic square roots of $F(\lambda)$. q.e.d.

Remark 13.5. The above arguments show a very close relationship between $e^{\xi(\nabla,g^M,\theta)}$ and $e^{-i\pi\eta(\nabla,g^M)}$. Each of these numbers by itself depends on the choice of the Agmon angle θ. But their product is a well defined holomorphic function. This relationship plays a very important role in the whole paper since it explains many features of the refined analytic torsion.

13.6. Space of representations of the fundamental group. Let M be a closed oriented manifold of odd dimension $d = 2r-1$, where $r \geq 1$. Denote by $\widetilde{M}$ the universal cover of M and by $\pi_1(M)$ the fundamental group of M, viewed as the group of deck transformations of $\widetilde{M} \to M$. The set $\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$ of all n-dimensional complex representations of $\pi_1(M)$ has a natural structure of a complex algebraic variety. Indeed, $\pi_1(M)$ is a finitely presented group, i.e., it is generated by a finite number of elements $\gamma_1,\dots,\gamma_L$, which satisfy finitely many relations. Hence, a representation $\alpha \in \mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$ is given by $2L$ invertible $n\times n$-matrices $\alpha(\gamma_1),\dots,\alpha(\gamma_L),\ \alpha(\gamma_1^{-1}),\dots,\alpha(\gamma_L^{-1})$ with complex coefficients satisfying finitely many polynomial equations. In other words, a representation α is given by a point of the direct product $\mathrm{Mat}_{n\times n}(\mathbb{C})^{2L}$ of $2L$ copies of the space $\mathrm{Mat}_{n\times n}$ of $n\times n$-matrices with complex coefficients.

In the sequel, we fix generators $\gamma_1,\dots,\gamma_L$ of $\pi_1(M)$ and view $\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$ as an algebraic subset of $\mathrm{Mat}_{n\times n}(\mathbb{C})^{2L}$ with the induced topology. For $\alpha \in \mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$, we denote by

$$E_\alpha := \widetilde{M} \times_\alpha \mathbb{C}^n \ \longrightarrow M$$

the flat vector bundle induced by α. Let ∇_α be the flat connection on E_α induced from the trivial connection on $\widetilde{M}\times\mathbb{C}^n$. We will also denote by ∇_α the induced differential

$$\nabla_\alpha : \Omega^\bullet(M,E_\alpha) \ \longrightarrow \Omega^{\bullet+1}(M,E_\alpha),$$

where $\Omega^\bullet(M, E_\alpha)$ denotes the space of smooth differential forms of M with values in E_α.

For each connected component[13] bundles E_α are isomorphic, see e.g., [**23**].

Let $\mathrm{Rep}_0(\pi_1(M), \mathbb{C}^n) \subset \mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$ denote the (possibly empty) set of all representations $\alpha \in \mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$ such that the connection ∇_α is acyclic. A representation $\alpha \in \mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$ is called *unitary* if there exists a Hermitian scalar product $(\cdot, \cdot)$ on $\mathbb{C}^n$ which is preserved by the matrices $\alpha(\gamma)$ for all $\gamma \in \pi_1(M)$. The scalar product $(\cdot, \cdot)$ induces a flat Hermitian metric h^{E_α} on the bundle E_α. We denote the set of unitary representations by $\mathrm{Rep}^u(\pi_1(M), \mathbb{C}^n)$. One might think of

$$\mathrm{Rep}^u(\pi_1(M), \mathbb{C}^n) \ \subset \ \mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$$

as *the real locus* of the complex algebraic variety $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$. Set

$$\mathrm{Rep}^u_0(\pi_1(M), \mathbb{C}^n) := \mathrm{Rep}^u(\pi_1(M), \mathbb{C}^n) \cap \mathrm{Rep}_0(\pi_1(M), \mathbb{C}^n).$$

13.7. Graded determinant of the odd signature operator as a function on the space of representations. Fix a Riemannian metric g^M on M. Let

$$B_\alpha := B(\nabla_\alpha, g^M) : \Omega^\bullet(M, E_\alpha) \ \longrightarrow \ \Omega^\bullet(M, E_\alpha)$$

and let $B_{\alpha,\mathrm{even}}$ denote the restriction of B_α to $\Omega^{\mathrm{even}}(M, E_\alpha)$.

Suppose that for some representation $\alpha_0 \in \mathrm{Rep}_0(\pi_1(M), \mathbb{C}^n)$ the operator B_{α_0} is invertible (in other words, we assume that (∇_{α_0}, g^M) satisfies Assumptions I and II of Subsection 6.5). Then there exists an open neighborhood (in classical topology) $V \subset \mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$ of the set of acyclic unitary representations such that, for all $\alpha \in V$ the pair (∇_α, g^M) satisfies Assumptions I and II of Subsection 6.5. Thus, for all $\alpha \in V$, the graded determinant $\mathrm{Det}_{\mathrm{gr},\theta}(B_{\alpha,\mathrm{even}})$ is defined, where $\theta \in (-\pi, 0)$ is an Agmon angle for B_α.

Theorem 13.8. *Let M be a closed oriented odd-dimensional manifold and let g^M be a Riemannian metric on M. Let $\mathcal{O} \subset \mathbb{C}$ be a connected open set and let $\gamma : \mathcal{O} \to \mathrm{Rep}_0(\pi_1(M), \mathbb{C}^n)$ be a holomorphic curve. Assume that for $\lambda_0 \in \mathcal{O}$ the connection $\nabla_{\gamma(\lambda_0)}$ on $E_{\gamma(\lambda_0)} \to M$ satisfies Assumption* II *of Subsection* 6.5 *(with respect to the given metric g^M). Then the function*

$$\lambda \ \mapsto \mathrm{Det}_{\mathrm{gr},\theta}\left(B_{\gamma(\lambda),\mathrm{even}}\right) \tag{13.166}$$

is holomorphic in a neighborhood of λ_0.

Proof. First, we need to introduce some additional notations. Let E be a vector bundle over M and let ∇ be a (not necessarily flat)

[13]In this paper we always consider the classical (not the Zariski) topology on the complex analytic space $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$.

connection on E. Fix a base point $x_* \in M$ and let E_{x_*} denote the fiber of E over x_*. We will identify E_{x_*} with $\mathbb{C}^n$ and $\pi_1(M, x_*)$ with $\pi_1(M)$.

For a closed path $\phi : [0,1] \to M$ with $\phi(0) = \phi(1) = x_*$, we denote by $\mathrm{Mon}_\nabla(\phi) \in \mathrm{End}\, E_{x_*} \simeq \mathrm{Mat}_{n\times n}(\mathbb{C})$ the monodromy of ∇ along ϕ, cf. (12.153). Note that, if ∇ is flat then $\mathrm{Mon}_\nabla(\phi)$ depends only on the class $[\phi]$ of ϕ in $\pi_1(M)$. Hence, if ∇ is flat, then the map $\phi \mapsto \mathrm{Mon}_\nabla(\phi)$ defines an element of $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$, called the *monodromy representation* of ∇.

Suppose now that $\mathcal{O} \subset \mathbb{C}$ is a connected open set. Let

$$\gamma : \mathcal{O} \ \to \ \mathrm{Rep}_0(\pi_1(M), \mathbb{C}^n)$$

be a holomorphic curve. By Proposition 4.5 of [**23**], all the bundles $E_{\gamma(\lambda)},\ \lambda \in \mathcal{O}$, are isomorphic to each other. In other words, there exists a vector bundle $E \to M$ and a family of flat connections $\nabla_\lambda,\ \lambda \in \mathcal{O}$, on E, such that the monodromy representation of ∇_λ is isomorphic to $\gamma(\lambda)$ for all $\lambda \in \mathcal{O}$. Moreover, the family ∇_λ can be chosen to be real differentiable, i.e., such that for every $\lambda \in \mathcal{O}$ there exist $\omega_1, \omega_2 \in \Omega^1(M, \mathrm{End}\, E)$ with

$$\nabla_\mu = \nabla_\lambda + \mathrm{Re}(\mu - \lambda) \cdot \omega_1 + \mathrm{Im}(\mu - \lambda) \cdot \omega_2 + o(\mu - \lambda), \tag{13.167}$$

where $o(\mu - \lambda)$ is understood in the sense of the Fréchet topology on $\mathcal{C}(E)$ introduced in Subsection 13.1.

By Lemma B.6 there exist a smooth form $\omega \in \Omega^1(M, \mathrm{End}\, E)$ with $\nabla_\lambda \omega = 0$ and a family $G(\mu) \in \mathrm{End}\, E$ ($\mu \in \mathcal{O}$) of gauge transformations such that $G(\lambda) = \mathrm{Id}$ and

$$\nabla_\lambda + (\mu - \lambda)\, \omega = G(\mu) \cdot \nabla_\mu \cdot G(\mu)^{-1} + o(\mu - \lambda).$$

Note that the connection $\nabla_\lambda + (\mu - \lambda)\, \omega$ is not necessarily flat.

From the definition of the the odd signature operator it then follows that

$$B\left(\nabla_\lambda + (\mu - \lambda)\, \omega, g^M\right) = G(\mu) \cdot B(\nabla_\mu, g^M) \cdot G(\mu)^{-1} + o(\mu - \lambda), \tag{13.168}$$

where $o(\mu - \lambda)$ is understood in the sense of the Fréchet topology introduced in Subsection 5.3.

Suppose now that λ is close enough to λ_0 so that the connection $\nabla_{\gamma(\lambda)}$ satisfies Assumption II of Subsection 6.5 (with respect to the metric g^M). Recall that in Subsection 13.3 we extended the definition of the graded determinant of $B_{\mathrm{even}}(\nabla, g^M)$ to the case when the connection ∇ is not necessarily flat. Thus

$$\mathrm{Det}_{\mathrm{gr},\theta}\left(\, B_{\mathrm{even}}(\nabla_\lambda + (\mu - \lambda)\, \omega, g^M)\,\right)$$

is defined for all $\mu \in \mathbb{C}$ close enough to λ.

By Proposition 13.4, the map

$$\mu \ \mapsto \ \mathrm{Det}_{\mathrm{gr},\theta}\left(B_{\mathrm{even}}(\nabla_\lambda + (\mu - \lambda)\omega, g^M)\right)$$

is holomorphic near λ. Hence, there exists a number $a \in \mathbb{C}$ such that

$$\begin{aligned}(13.169)\quad &\mathrm{Det}_{\mathrm{gr},\theta}\left(B_{\mathrm{even}}(\nabla_\lambda + (\mu-\lambda)\,\omega, g^M)\right)\\ &\qquad = \mathrm{Det}_{\mathrm{gr},\theta}\left(B_{\mathrm{even}}(\nabla_\lambda, g^M)\right) + a\cdot(\mu-\lambda) + o(\mu-\lambda).\end{aligned}$$

On the other side, (13.168) implies that

$$\begin{aligned}(13.170)\qquad &\mathrm{Det}_{\mathrm{gr},\theta}\left(B_{\mathrm{even}}(\nabla_\lambda + (\mu-\lambda)\,\omega, g^M)\right)\\ &= \mathrm{Det}_{\mathrm{gr},\theta}\left(G(\mu)\cdot B_{\mathrm{even}}(\nabla_\mu, g^M)\cdot G(\mu)^{-1}\right) + o(\mu-\lambda)\\ &= \mathrm{Det}_{\mathrm{gr},\theta}\left(B_{\mathrm{even}}(\nabla_\mu, g^M)\right) + o(\mu-\lambda).\end{aligned}$$

Combining (13.169) with (13.170) we obtain

$$\begin{aligned}&\mathrm{Det}_{\mathrm{gr},\theta}\left(B_{\mathrm{even}}(\nabla_\mu, g^M)\right)\\ &\qquad = \mathrm{Det}_{\mathrm{gr},\theta}\left(B_{\mathrm{even}}(\nabla_\lambda, g^M)\right) + a\cdot(\mu-\lambda) + o(\mu-\lambda).\end{aligned}$$

Since the above equality holds for all λ close enough to λ_0 the theorem is proven. q.e.d.

Corollary 13.9. *Let M be a closed oriented odd-dimensional manifold. Let $V \subset \mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$ be an open set such that for every $\alpha \in V$ there exists a Riemannian metric g^M such that the connection $\nabla_\alpha \in \mathrm{Flat}'(E_\alpha, g^M)$ (cf. Subsection 6.7). Assume, further, that all the points of V are regular points of the complex algebraic set $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$. Then the map*

$$\mathrm{Det} : V \longrightarrow \mathbb{C}, \qquad \mathrm{Det} : \alpha \mapsto \mathrm{Det}(\alpha) := \mathrm{Det}_{\mathrm{gr},\theta}(B_{\alpha,\mathrm{even}})$$

is holomorphic. Here $\theta \in (-\pi, 0)$ is an Agmon angle for $B_{\alpha,\mathrm{even}}$.

Proof. By Hartogs' theorem (cf., for example, [**26**, Th. 2.2.8]), a function on a smooth algebraic variety is holomorphic if its restriction to each holomorphic curve is holomorphic. Hence, the corollary follows immediately from Theorem 13.8. q.e.d.

Remark 13.10. In Section 14 below we mostly view the graded determinant of the odd signature operator as a function on the space of representations rather than as a function on the space of flat connections. As $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$ is a *finite dimensional* algebraic variety, we can use the methods of complex analysis of holomorphic functions on finite dimensional varieties.

By the definition of the refined analytic torsion, cf. Definition 10.1, Theorem 13.8 and Corollary 13.9 imply now the following

Corollary 13.11. *Let M be a closed oriented odd-dimensional manifold.*

1. *Let $\mathcal{O} \subset \mathbb{C}$ be an open set and let $\gamma : \mathcal{O} \to \mathrm{Rep}_0(\pi_1(M), \mathbb{C}^n)$ be a holomorphic curve. Assume that for $\lambda_0 \in \mathcal{O}$ there exists a Riemannian metric g^M so that the connection $\nabla_{\gamma(\lambda_0)}$ satisfies Assumption* II *of Subsection* 6.5. *Then the function*

$$\lambda \mapsto T(\nabla_{\gamma(\lambda)}) \tag{13.171}$$

is holomorphic in a neighborhood of λ_0.

2. *Let $V \subset \mathrm{Rep}_0(\pi_1(M), \mathbb{C}^n)$ denote the open set of all representations α such that for some Riemannian metric g^M on M the connection $\nabla_\alpha \in \mathrm{Flat}'(E_\alpha, g^M)$. Let $\Sigma \subset \mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$ denote the set of singular points of the complex algebraic variety $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$. Then $\alpha \mapsto T(\nabla_\alpha)$ is a holomorphic function on $V \backslash \Sigma$.*

14. Comparison with Turaev's Refinement of the Combinatorial Torsion

In [**42**, **43**], Turaev introduced a refinement $T^{\mathrm{comb}}_\alpha(\varepsilon, \mathfrak{o})$ of the combinatorial torsion associated to an acyclic representation α of $\pi_1(M)$. This refinement depends on an additional combinatorial data, denoted by ε and called the *Euler structure* as well as on the *cohomological orientation* of M, i.e., on the orientation $\mathfrak{o}$ of the determinant line of the cohomology $H^\bullet(M, \mathbb{R})$ of M. There are two versions of the Turaev torsion – the homological and the cohomological one. In this paper it is more convenient for us to use the cohomological Turaev torsion as it is defined by Farber and Turaev in Section 9.2 of [**20**]. Given $\alpha \in \mathrm{Rep}_0(\pi_1(M), \mathbb{C}^n)$, the cohomological Turaev torsion $T^{\mathrm{comb}}_\alpha(\varepsilon, \mathfrak{o})$ is a non-vanishing complex number. If $\alpha \in \mathrm{Rep}^u_0(\pi_1(M), \mathbb{C}^n)$ the absolute value of the Turaev torsion is equal to the Reidemeister torsion.[14] One can view Theorem 8.2 as an analytic analogue of this result, where the role of the Reidemeister torsion is played by the Ray-Singer torsion. Another property of the Turaev torsion is that it is a holomorphic function of $\alpha \in \mathrm{Rep}_0(\pi_1(M), \mathbb{C}^n)$. In Corollary 13.11 we established the same property for the refined analytic torsion.

Though, in general, the refined analytic torsion $T_\alpha = T(\nabla_\alpha)$ and the Turaev torsion $T^{\mathrm{comb}}_\alpha(\varepsilon, \mathfrak{o})$ are not equal they are very closely related. In this section we establish this relationship. As an application we strengthen and generalize a theorem of Farber [**17**] about the relationship between the Turaev torsion and the η-invariant.

14.1. Notation. Let M be a closed oriented odd-dimensional manifold. In this section we view the refined analytic torsion as a function of a representation of $\pi_1(M)$. Let $V \subset \mathrm{Rep}_0(\pi_1(M), \mathbb{C}^n)$ be an open set

[14] Here the Reidemeister torsion is understood as the positive real number defined, for example, in Definition 1.1 of [**35**].

consisting of representations α such that $\nabla_\alpha \in \mathrm{Flat}'(E, g^M)$. Let $V' \subset V$ be the open subset of V such that, for all $\alpha \in V'$, the connection ∇_α belongs to the open set U' defined in Subsection 12.7. For every $\alpha \in V$ we set $T_\alpha := T(\nabla_\alpha)$, $T^{\mathrm{RS}}_\alpha := T^{\mathrm{RS}}(\nabla_\alpha)$, $\eta_\alpha := \eta(\nabla_\alpha, g^M)$, etc.

14.2. Comparison between the Turaev and the Ray-Singer Torsion. Theorem 10.2 of [**20**] establishes a relationship between the Turaev and the Ray-Singer torsions for real representations α. The following result is an immediate extension of this result to complex acyclic representations.

Theorem 14.3. *Suppose M is a closed oriented odd-dimensional manifold. Let $c(\varepsilon) \in H_1(M, \mathbb{Z})$ denote the characteristic class of the Euler structure ε, cf.* [**43**] *or Section* 5.2 *of* [**20**]*. Then, for every $\alpha \in \mathrm{Rep}_0(\pi_1(M), \mathbb{C}^n)$,*

$$\log \frac{|T^{\mathrm{comb}}_\alpha(\varepsilon, \mathfrak{o})|}{T^{\mathrm{RS}}_\alpha} = -\,\pi\, \langle \mathrm{Im}\, \mathbf{Arg}_\alpha, c(\varepsilon)\rangle, \tag{14.172}$$

where the cohomology class $\mathbf{Arg}_\alpha := \mathbf{Arg}_{\nabla_\alpha} \in H^1(M, \mathbb{C}/\mathbb{Z})$ is defined in Subsection 12.4 *and $\langle \cdot, \cdot \rangle$ denotes the natural pairing*

$$H^1(M, \mathbb{C}/\mathbb{Z}) \times H_1(M, \mathbb{Z}) \;\longrightarrow\; \mathbb{C}/\mathbb{Z}.$$

In particular, if $\alpha \in \mathrm{Rep}^u_0(\pi_1(M), \mathbb{C}^n)$ then

$$\big|\, T^{\mathrm{comb}}_\alpha(\varepsilon, \mathfrak{o}) \,\big| = T^{\mathrm{RS}}_\alpha. \tag{14.173}$$

Note that, though $\langle \mathbf{Arg}_\alpha, c(\varepsilon)\rangle$ is defined only modulo $\mathbb{Z}$, its imaginary part $\mathrm{Im}\langle \mathbf{Arg}_\alpha, c(\varepsilon)\rangle$ is a well defined complex number.

Proof. Let $\alpha^{\mathbb{R}}$ denote the representation α considered as a real representation. Then, for every closed curve γ in M, we have

$$\det \mathrm{Mon}_{\alpha^{\mathbb{R}}}(\gamma) = \big|\det \mathrm{Mon}_\alpha(\gamma)\big|^2.$$

Define $\mathbf{Arg}_{\alpha^{\mathbb{R}}} \in H^1(M, \mathbb{R}/\mathbb{Z})$ by

$$\det\, \mathrm{Mon}_{\alpha^{\mathbb{R}}}(\gamma) = \exp\, \big(2\pi i \langle \mathbf{Arg}_{\alpha^{\mathbb{R}}}, [\gamma]\rangle\big).$$

Then, from (12.151), we obtain

$$\exp\, \big(2\pi i \langle \mathbf{Arg}_{\alpha^{\mathbb{R}}}, [\gamma]\rangle\big) = \exp\, \big(2\pi i \langle 2i\, \mathrm{Im}\, \mathbf{Arg}_\alpha, [\gamma]\rangle\big).$$

Hence,

$$\langle \mathbf{Arg}_{\alpha^{\mathbb{R}}}, [\gamma]\rangle \equiv 2i\ \mathrm{Im}\, \langle \mathbf{Arg}_\alpha, [\gamma]\rangle \mod \mathbb{Z}. \tag{14.174}$$

Let $T^{\mathrm{comb}}_{\alpha^{\mathbb{R}}}(\varepsilon, \mathfrak{o})$ and $T^{\mathrm{RS}}_{\alpha^{\mathbb{R}}}$ denote the Turaev and the Ray-Singer torsions associated to the representation $\alpha^{\mathbb{R}}$. Then

$$|T^{\mathrm{comb}}_\alpha(\varepsilon, \mathfrak{o})|^2 = T^{\mathrm{comb}}_{\alpha^{\mathbb{R}}}(\varepsilon, \mathfrak{o}), \qquad (T^{\mathrm{RS}}_\alpha)^2 = T^{\mathrm{RS}}_{\alpha^{\mathbb{R}}}. \tag{14.175}$$

By formula (10.3) of [**20**], we have

$$\left(\frac{T^{\rm comb}_{\alpha^{\mathbb{R}}}(\varepsilon, \mathfrak{o})}{T^{\rm RS}_{\alpha^{\mathbb{R}}}} \right)^2 = \left| \exp\left(2\pi i \langle \mathbf{Arg}_{\alpha^{\mathbb{R}}}, c(\varepsilon) \rangle \right) \right| .$$

Combining this equality with (14.174) and (14.175), we obtain (14.172).

If α is unitary, then $\operatorname{Im} \mathbf{Arg}_\alpha = 0$ and (14.173) follows. q.e.d.

14.4. The homology class β_ε. We need the following

Lemma 14.5. *Let M be a closed oriented manifold of odd dimension $d = 2n-1$. Let $L_{d-1}(p) \in H^{d-1}(M, \mathbb{Q})$ denote the component in dimension $d-1$ of the Hirzebruch L-polynomial $L(p)$ in the Pontrjagin classes of M. Then there exists a lift $\widetilde{L}_{d-1} \in H^{d-1}(M, \mathbb{Z})$ of $L_{d-1}(p)$ to the integer cohomology and the reduction of $\widetilde{L}_{d-1}$ modulo 2 is equal to the $(d-1)$-Stiefel-Whitney class $w_{d-1}(M) \in H^{d-1}(M, \mathbb{Z}_2)$ of M.*

Proof. For any homology class $\xi \in H_{d-1}(M, \mathbb{Z})$ there exists a smooth oriented submanifold $X_\xi \subset M$, representing ξ. Then $\langle L_{d-1}(p), \xi \rangle$ is equal to the signature $\sigma(X_\xi)$ of X_ξ. In particular, it is an integer; hence, the lift $\widetilde{L}_{d-1} \in H^{d-1}(M, \mathbb{Z})$ exists.

The parity of $\sigma(X_\xi)$ is equal to the parity of the Euler characteristic $\chi(X_\xi)$ of X_ξ, which, in turn, is equal to

$$\langle w_{d-1}(M), X_\xi \rangle = \langle w_{d-1}(X_\xi), X_\xi \rangle .$$

Thus we conclude that

$$\langle \widetilde{L}_{d-1} - w_{d-1}(M), \xi \rangle = 0 \qquad \mod 2,$$

for any homology class $\xi \in H_{d-1}(M, \mathbb{Z})$. q.e.d.

Note that a lift of $L_{d-1}(p)$ to the integer cohomology is not uniquely defined if $H^{d-1}(M, \mathbb{Z})$ has a torsion. From now on we fix such a lift $\widetilde{L}_{d-1} \in H^{d-1}(M, \mathbb{Z})$ and we denote by $\widehat{L}_1 \in H_1(M, \mathbb{Z})$ the Poincaré dual of $\widetilde{L}_{d-1}$. Let $c(\varepsilon) \in H_1(M, \mathbb{Z})$ denote the characteristic class of the Euler structure ε, cf. [**43**] or Section 5.2 of [**20**].

Corollary 14.6. *The class $\widehat{L}_1(p) + c(\varepsilon) \in H_1(M, \mathbb{Z})$ is divisible by 2, i.e., there exists a (not necessarily unique) homology class $\beta_\varepsilon \in H_1(M, \mathbb{Z})$ such that*

$$-2\,\beta_\varepsilon = \widehat{L}_1(p) + c(\varepsilon). \tag{14.176}$$

Proof. It is shown on page 209 of [**20**] that the reduction of $c(\varepsilon)$ modulo 2 is equal to the Poincaré dual of the Stiefel-Whitney class $w_{d-1}(M)$. Hence, it follows from Lemma 14.5 that the reduction of $\widehat{L}_1(p) + c(\varepsilon)$ is the zero element of $H_1(M, \mathbb{Z}_2)$. q.e.d.

The equality (14.176) defines β_ε modulo two-torsion elements in $H_1(M, \mathbb{Z})$. We fix a solution of (14.176) and for the rest of the paper β_ε denotes this solution.

14.7. Comparison between the Turaev and the refined analytic torsions. To simplify the notation let us denote by $\widehat{L}(p) \in H_\bullet(M,\mathbb{Z})$ the Poincaré dual of the cohomology class $[L(p)]$. Let $\widehat{L}_1 \in H_1(M,\mathbb{Z})$ denote the component of $\widehat{L}(p)$ in $H_1(M,\mathbb{Z})$. Then

$$\langle\, [L(p)] \cup \mathbf{Arg}_\alpha, [M]\,\rangle = \langle \mathbf{Arg}_\alpha, \widehat{L}_1\rangle \in \mathbb{C}/\mathbb{Z}.$$

Recall that the neighborhood V' of $\mathrm{Rep}_0^u(\pi_1(M),\mathbb{C}^n)$ was defined in Subsection 14.1. If $\alpha \in V'$ then by Theorem 12.8 and (14.172)

$$\begin{aligned} \mathrm{Re}\,\log \frac{T_\alpha}{T_\alpha^{\mathrm{comb}}(\varepsilon,\mathfrak{o})} &= \log \frac{|T_\alpha|}{|T_\alpha^{\mathrm{comb}}(\varepsilon,\mathfrak{o})|} \\ &= \pi \left\langle \mathrm{Im}\,\mathbf{Arg}_\alpha, c(\varepsilon) + \widehat{L}_1 \right\rangle \\ &= -2\pi \left\langle \mathrm{Im}\,\mathbf{Arg}_\alpha, \beta_\varepsilon \right\rangle, \end{aligned} \tag{14.177}$$

where $\beta_\varepsilon \in H_1(M,\mathbb{Z})$ is the homology class defined in (14.176).

Let Σ denote the set of singular points of the complex analytic set $\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$. By Corollary 13.11, the refined analytic torsion T_α is a non-vanishing holomorphic function of $\alpha \in V\backslash\Sigma$. By the very construction [**42**, **43**, **20**] the Turaev torsion is a non-vanishing holomorphic function of $\alpha \in \mathrm{Rep}_0(\pi_1(M),\mathbb{C}^n)$. Hence,

$$\frac{T_\alpha}{T_\alpha^{\mathrm{comb}}(\varepsilon,\mathfrak{o})}$$

is a holomorphic function on $V'\backslash\Sigma$.

By construction of the cohomology class $\mathbf{Arg}_\alpha$, for every homology class $z \in H_1(M,\mathbb{Z})$, the expression

$$e^{2\pi\, i\, \langle \mathbf{Arg}_\alpha, z\rangle}$$

is a holomorphic function on $\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$.

Now the expression (14.177) can be rewritten as

$$\left| \frac{T_\alpha}{T_\alpha^{\mathrm{comb}}(\varepsilon,\mathfrak{o})} \right| = \left| e^{2\pi i \langle \mathbf{Arg}_\alpha, \beta_\varepsilon\rangle} \right|, \qquad \alpha \in V'.$$

If the absolute values of two non-vanishing holomorphic functions are equal on a connected open set then the functions must be equal up to a factor $a \in \mathbb{C}$ with $|a| = 1$. Hence, on each connected component $C \subset V'\backslash\Sigma$, there exists a constant $\phi_C(\varepsilon,\mathfrak{o}) \in \mathbb{R}$, depending on ε and $\mathfrak{o}$, so that

$$\frac{T_\alpha}{T_\alpha^{\mathrm{comb}}(\varepsilon,\mathfrak{o})}\, e^{-i\phi_C(\varepsilon,\mathfrak{o})} = e^{2\pi i \langle \mathbf{Arg}_\alpha, \beta_\varepsilon\rangle}, \qquad \alpha \in C. \tag{14.178}$$

Note that the constants $\phi_C(\varepsilon,\mathfrak{o})$ are defined up to an additive multiple of 2π. Since T_α, $T_\alpha^{\mathrm{comb}}(\varepsilon,\mathfrak{o})$, and $e^{2\pi i\langle \mathbf{Arg}_\alpha, \beta_\varepsilon\rangle}$ depend continuously on $\alpha \in V'$, we can choose these constants so that

$$\phi_{C_1}(\varepsilon,\mathfrak{o}) = \phi_{C_2}(\varepsilon,\mathfrak{o})$$

whenever C_1 and C_2 are contained in the same connected component of V'. Thus (14.178) remains valid if C is a connected component of V'.

Thus we have proven the following extension of the Cheeger-Müller theorem about the equality between the Reidemeister and the Ray-Singer torsions.

Theorem 14.8. *Suppose M is a closed oriented odd dimensional manifold. Let ε be an Euler structure on M and let $\mathfrak{o}$ be a cohomological orientation of M. Let $V' \subset \mathrm{Rep}_0(\pi_1(M), \mathbb{C}^n)$ be as in Subsection* 14.1. *Then, for each connected component C of V', there exists a constant $\phi_C = \phi_C(\varepsilon, \mathfrak{o}) \in \mathbb{R}$, depending on ε and $\mathfrak{o}$, such that*

$$\frac{T_\alpha}{T_\alpha^{\mathrm{comb}}(\varepsilon, \mathfrak{o})} = e^{i\phi_C}\, e^{2\pi i \langle \mathbf{Arg}_\alpha, \beta_\varepsilon \rangle}. \tag{14.179}$$

Remark 14.9. It would be very interesting to calculate the constant $\phi_C(\varepsilon, \mathfrak{o})$. In particular, it would be interesting to know whether it actually depends on the connected component C of V'. Another interesting question is for which acyclic representations α one can find an Euler structure ε and a cohomological orientation $\mathfrak{o}$ such that $T_\alpha = T_\alpha^{\mathrm{comb}}(\varepsilon, \mathfrak{o})$.[15]

14.10. Comparison with the Farber-Turaev absolute torsion. An immediate application of Theorem 14.8 concerns the notion of the *absolute torsion* introduced by Farber and Turaev in [**19**]. Suppose that the Stiefel-Whitney class $w_{d-1}(M) \in H^{d-1}(M, \mathbb{Z}_2)$ vanishes, a condition always satisfied if $\dim M \equiv 3 (\mathrm{mod}\, 4)$, cf. [**32**]. Then, by [**19**, §3.2], there exists an Euler structure ε such that $c(\varepsilon) = 0$. Assume, in addition, that the first Stiefel-Whitney class $w_1(E_\alpha)$, viewed as a homomorphism $H_1(M, \mathbb{Z}) \to \mathbb{Z}_2$, vanishes on the 2-torsion subgroup of $H_1(M, \mathbb{Z})$. In this case there is also a canonical choice of the cohomological orientation $\mathfrak{o}$, cf. [**19**, §3.3]. Then the Turaev torsion $T_\alpha^{\mathrm{comb}}(\varepsilon, \mathfrak{o})$ corresponding to any ε with $c(\varepsilon) = 0$ and the canonically chosen $\mathfrak{o}$ will be the same.

If the above assumptions on $w_{d-1}(M)$ and $w_1(E_\alpha)$ are satisfied, then the number

$$T_\alpha^{\mathrm{abs}} := T_\alpha^{\mathrm{comb}}(\varepsilon, \mathfrak{o}) \in \mathbb{C}, \qquad (c(\varepsilon) = 0), \tag{14.180}$$

is canonically defined, i.e., is *independent of any choices*. It was introduced by Farber and Turaev, [**19**], who called it the *absolute torsion*.[16]

[15] Added in proof: Recently, Rung-Tzung Huang [**27**] showed by an explicit calculation for lens spaces that, in general, the constant $\phi_C(\varepsilon, \mathfrak{o})$ depends on the connected component C. He also showed that it is independent of the Euler structure ε.

[16] Farber and Turaev, [**19**], also defined the absolute torsion in the case when the representation α is not acyclic, in which case the absolute torsion is not a number but an element of the determinant line $\mathrm{Det}(H^\bullet(M, E_\alpha)$ of the cohomology of M with coefficients in E_α.

Using (14.176) and the fact that $\widehat{L}_1$ vanishes if $\dim M \equiv 3 \pmod 4$, Theorem 14.8 leads to the following

Corollary 14.11. *In addition to the assumptions made in Theorem* 14.8 *suppose that* $\dim M \equiv 3 \pmod 4$ *and that the* 2*-torsion subgroup of* $H_1(M,\mathbb{Z})$ *is trivial. Then the ratio* $T_\alpha/T_\alpha^{\rm abs}$ *is locally constant on* V' *and its absolute value is equal to 1.* [17]

14.12. Phase of the Turaev torsion of a unitary representation. As an application of our study of the refined analytic torsion we obtain a result about the phase of the Turaev torsion which improves and generalizes a theorem of Farber [**17**], cf. Remark 14.16 below.

We denote the phase of a complex number z by $\mathbf{Ph}(z) \in [0, 2\pi)$ so that $z = |z| e^{i\,\mathbf{Ph}(z)}$.

Suppose $\alpha \in \mathrm{Rep}_0^u(\pi_1(M), \mathbb{C}^n)$ is a unitary representation. Then the number $\xi_\alpha = \xi(\nabla_\alpha, g^M, \theta)$, defined in (7.79), is real (in fact, in this case, ξ_α coincides with $\log T_\alpha^{\rm RS}$, cf. (8.98)). Moreover, the η-invariant η_α is real, cf. Subsection 4.8. Thus, (7.80) and the definition of the refined analytic torsion (Definition 10.1) imply

$$\mathbf{Ph}(T_\alpha) = -\,\pi\,\eta_\alpha + \pi\,(\operatorname{rank}\alpha)\,\eta_{\rm trivial} \qquad \mod 2\pi\,\mathbb{Z}. \tag{14.181}$$

The second term on the right hand side of (14.181) vanishes if $\dim M \equiv 1 \pmod 4$.

Combining (14.181) with Theorem 14.8 we obtain the following

Theorem 14.13. *Under the assumptions of Theorem* 14.8 *suppose that* α_1, $\alpha_2 \in \mathrm{Rep}_0^u(\pi_1(M), \mathbb{C}^n)$ *are unitary representations which lie in the same connected component of* V'*. In particular, they have the same rank. Then, modulo* $2\pi\,\mathbb{Z}$*,*

$$\begin{aligned}\mathbf{Ph}(T^{\rm comb}_{\alpha_1}(\varepsilon,\mathfrak{o})) + \pi\,\eta_{\alpha_1} + 2\pi\,\big\langle\,\mathbf{Arg}_{\alpha_1}, \beta_\varepsilon\big\rangle \\ \equiv\ \mathbf{Ph}(T^{\rm comb}_{\alpha_2}(\varepsilon,\mathfrak{o})) + \pi\,\eta_{\alpha_2} + 2\pi\,\big\langle\,\mathbf{Arg}_{\alpha_2}, \beta_\varepsilon\big\rangle.\end{aligned} \tag{14.182}$$

14.14. Sign of the absolute torsion. Suppose that the Stiefel-Whitney class $w_{d-1}(M) = 0$ and that the first Stiefel-Whitney class $w_1(E_\alpha)$, viewed as a homomorphism $H_1(M,\mathbb{Z}) \to \mathbb{Z}_2$, vanishes on the 2-torsion subgroup of $H_1(M,\mathbb{Z})$. Then the Farber-Turaev absolute torsion (14.180) is defined. If $\alpha \in \mathrm{Rep}_0^u(\pi_1(M), \mathbb{C}^n)$, then $T_\alpha^{\rm abs}$ is real, cf. Theorem 3.8 of [**19**] and, hence,

$$e^{i\,\mathbf{Ph}(T_\alpha^{\rm abs})} = \operatorname{sign}(T_\alpha^{\rm abs}).$$

From Theorem 14.8 and Theorem 14.13 we now obtain the following

[17] Added in proof: Recently, Huang [**27**] proved that, under the assumptions of Corollary 14.11, $T_\alpha/T_\alpha^{\rm abs} = \pm e^{-i\pi\rho_\alpha}$, where $\rho_\alpha = \eta_\alpha - (\operatorname{rank}\alpha)\eta_{\rm trivial}$ is the ρ-invariant of E_α.

Theorem 14.15. *Under the assumptions of Theorem* 14.8 *suppose that* α_1, $\alpha_2 \in \mathrm{Rep}_0^u(\pi_1(M), \mathbb{C}^n)$ *are unitary representations which lie in the same connected component of* V'.

1) *Let* $\dim M \equiv 3 \pmod 4$. *Assume that the first Stiefel-Whitney class* $w_1(E_{\alpha_1}) = w_1(E_{\alpha_2})$ *vanishes on the* 2*-torsion subgroup of* $H_1(M, \mathbb{Z})$. *Then*

$$\operatorname{sign}\big(T^{\mathrm{abs}}_{\alpha_1}\big) \cdot e^{i\pi\eta_{\alpha_1}} = \operatorname{sign}\big(T^{\mathrm{abs}}_{\alpha_2}\big) \cdot e^{i\pi\eta_{\alpha_2}}.$$

2) *Let* $\dim M \equiv 1 \pmod 4$. *Assume that* $w_{d-1}(M) = 0$ *and the group* $H_1(M, \mathbb{Z})$ *has no* 2*-torsion. Then*

$$\operatorname{sign}\big(T^{\mathrm{abs}}_{\alpha_1}\big) \cdot e^{i\pi\left(\eta_{\alpha_1} - \langle [L(p)] \cup \mathbf{Arg}_{\alpha_1}, [M]\rangle\right)} = \operatorname{sign}\big(T^{\mathrm{abs}}_{\alpha_2}\big) \cdot e^{i\pi\left(\eta_{\alpha_2} - \langle [L(p)] \cup \mathbf{Arg}_{\alpha_2}, [M]\rangle\right)}.$$

Remark 14.16. Note that the unitary representations α_1 and α_2 in Theorem 14.15 are assumed to be connected by a path in V'. For the special case when there is a real analytic path α_t of *unitary* representations connecting α_1 and α_2 such that the twisted deRham complex (6.63) is acyclic for all but finitely many values of t, Theorem 14.15 was established by Farber, using a completely different method.

Appendix A. Determinant of an Operator with the Spectrum Symmetric about the Real Axis

In this appendix we show that for a wide and important class of differential operators, including the self-adjoint ones, formula (4.34) represents $\mathrm{LDet}_\theta(D)$ as a sum of its real and imaginary parts.

Definition A.1. The spectrum of D is *symmetric with respect to the real axis* if the following condition holds: if λ is an eigenvalue of D, then $\overline{\lambda}$ also is an eigenvalue of D and has the same algebraic multiplicity as λ.

Note that every operator with *real coefficients* has this property. See [**1**] for examples of other interesting operators with symmetric spectrum.[18]

Theorem A.2. *Let* $D : C^\infty(M, E) \to C^\infty(M, E)$ *be an injective elliptic differential operator of order* m *with self-adjoint leading symbol, whose spectrum is symmetric about the real axis. Let* $\theta \in (-\pi/2, 0)$ *be an Agmon angle for* D. *Then the numbers* $\zeta_{2\theta}(0, D^2)$, $\eta(D)$, *and*

[18] All the operators considered in [**1**] have spectrum symmetric about the *imaginary axis*. However, the spectrum of the operator considered in Section 5 of [**1**] is also symmetric about the real axis. Further, the spectrum of the operator $\Gamma_L \cdot D_{mn}$, discussed at the end of Section 6.8 of [**1**], is symmetric about the real axes.

$\mathrm{Det}_{2\theta}(D^2) = e^{-\zeta'_{2\theta}(0,D^2)}$ *are real. In particular, the following analogue of* (4.45) *and* (4.46) *holds:*

$$\mathrm{Det}_\theta(D) = (-1)^{m_-} \cdot \sqrt{\left|\mathrm{Det}_{2\theta}(D^2)\right|} \cdot e^{-i\pi\left(\eta(D) - \frac{1}{2}\zeta_{2\theta(0,D^2)}\right)}, \tag{A.183}$$

where $m_- = \operatorname{rank} P_-$ *is the number of the eigenvalues of* D (*counted with their algebraic multiplicities*) *on the negative part of the imaginary axis, cf. Subsection* 4.1.

Corollary A.3. *If, in addition to the assumptions of Theorem* A.2, D *is self-adjoint, then* $m_- = 0$ *and* $\mathrm{Det}_{2\theta}(D^2)$ *is real and positive. Hence, as expected, formulas* (4.45) *and* (4.46) *hold.*

Proof of Corollary A.3. If D is self-adjoint, the spectrum of D lies on the real line. Hence, in particular, $m_- = 0$. It follows from (4.41) together with (A.184) and (A.186) below that $\mathrm{Im}\left(\zeta'_{2\theta}(0, D^2)\right) = 0$. Hence, $\mathrm{Det}_{2\theta}(D^2) > 0$. q.e.d.

Remark A.4. It is interesting to compare (A.183) with Theorem 3.2 of [**1**]. Suppose that the spectrum of D is also symmetric about the imaginary axis. Then $\eta(D) = 0$. If, in addition, $\dim M$ is odd, then $\zeta_{2\theta}(0, D^2) = 0$, cf. Remark 4.7.c. Hence, (A.183) imply that, in this case, $\mathrm{Det}_\theta(D)$ is real and its sign is equal to $(-1)^{m_-}$. Theorem 3.2 of [**1**] states that this is true without the assumption that the spectrum of D is symmetric about the real axis[19] (i.e., for every invertible elliptic operator with self-adjoint leading symbol, whose spectrum is symmetric about the imaginary axis).

Proof of Theorem A.2. In view of (3.26), it is enough to consider the case when θ is sufficiently close to $-\pi/2$ so that there are no eigenvalues of D in the solid angles $L_{(-\pi/2,\theta]}$ and $L_{(\pi/2,\theta+\pi]}$, which we will henceforth assume. By (4.41), (4.42), and (4.44) it suffices to show that the numbers

$$\begin{aligned}\zeta_\theta(0, \tilde{\Pi}_+, D) \;\pm\; \zeta_\theta(0, \tilde{\Pi}_-, -D) \\ = \left(\zeta_\theta(0, \Pi_+, D) \;\pm\; \zeta_\theta(0, \Pi_-, -D)\right) \\ + \left(\zeta_\theta(0, P_+, D) \;\pm\; \zeta_\theta(0, P_-, -D)\right)\end{aligned}$$

are real and that the imaginary part of the number

$$\begin{aligned}\zeta'_\theta(0, \tilde{\Pi}_+, D) + \zeta'_\theta(0, \tilde{\Pi}_-, -D) \\ = \left(\zeta'_\theta(0, \Pi_+, D) + \zeta'_\theta(0, \Pi_-, -D)\right) \\ + \left(\zeta'_\theta(0, P_+, D) + \zeta'_\theta(0, P_-, -D)\right)\end{aligned}$$

is equal to $-\pi m_-$.

[19] In [**1**] the spectral cut was taken in the *upper half-plane*. Consequently, m_- is replaced there by m_+.

Since the projections $P_\pm$ have finite rank, one has

$$\zeta_\theta(0, P_\pm, \pm D) = \operatorname{rank} P_\pm.$$

Thus these numbers are real.

Because the spectrum of D is symmetric about the real axis,

$$\operatorname{rank} P_+ = \operatorname{rank} P_-.$$

As, for every $r > 0$ one has

$$\frac{d}{ds}|_{s=0}\, (ir)_\theta^{-s} = -\log r - i\frac{\pi}{2},$$

we conclude that

$$\operatorname{Im}\, \zeta'_\theta(0, P_+, D) = \operatorname{Im}\, \zeta'_\theta(0, P_-, -D) = -\frac{\pi}{2}\, \operatorname{rank} P_-.$$

Hence,

$$\operatorname{Im}\, \Big(\zeta'_\theta(0, P_+, D) + \zeta'_\theta(0, P_-, -D)\Big) = -\pi\, \operatorname{rank} P_- \in \pi\, \mathbb{Z}. \tag{A.184}$$

It remains to show that

$$\zeta_\theta(0, \Pi_\pm, \pm D),\ \zeta'_\theta(0, \Pi_\pm, \pm D)\ \in\ \mathbb{R}.$$

We will show that the numbers $\zeta_\theta(0, \Pi_+, D)$ and $\zeta'_\theta(0, \Pi_+, D)$ are real. The fact that the other two numbers are real as well follows then by replacing D with $-D$.

Let

$$\lambda_j > 0, \qquad j \in I_1 \subset \mathbb{N}$$

be all the positive real eigenvalues of D and let

$$\lambda_j\ =\ \rho_j e^{i\alpha_j}, \qquad j \in I_2 \subset \mathbb{N}$$

be all the eigenvalues of D which lie in the solid angle $L_{(0,\pi/2)}$. Let m_j denote the algebraic multiplicity of λ_j, cf. Subsection 3.9. Since the spectrum of D is symmetric about the real axis,

$$\rho_j e^{-i\alpha_j}, \qquad j \in I_2,$$

are all the eigenvalues of D in the solid angle $L_{(-\pi/2,0)}$ and

$$\begin{aligned}
&\zeta_\theta(s, \Pi_+, D)\\
&= \sum_{j\in I_1} m_j\, \lambda_j^{-s} + \sum_{j\in I_2} m_j\, \rho_j^{-s}\, (e^{-is\alpha_j} + e^{is\alpha_j})\\
&= \sum_{j\in I_1} m_j\, \lambda_j^{-s} + 2\sum_{j\in I_2} m_j\, \rho_j^{-s}\, \cos(s\alpha_j), \qquad \operatorname{Re} s > \frac{\dim M}{m}.
\end{aligned}$$

Hence,

$$\zeta_\theta(s, \Pi_+, D) = \overline{\zeta_\theta(\bar s, \Pi_+, D)}, \qquad \operatorname{Re} s > \frac{\dim M}{m}. \tag{A.185}$$

Since both sides of (A.185) are holomorphic functions of s, the equality (A.185) holds for all regular points of $\zeta_\theta(s, \Pi_+, D)$. In particular,

$\zeta_\theta(s, \Pi_+, D)$ is real for all real regular points. Hence, $\zeta_\theta(0, \Pi_+, D) \in \mathbb{R}$. Since (A.185) implies

$$\zeta'_\theta(s, \Pi_+, D) = \overline{\zeta'_\theta(\bar{s}, \Pi_+, D)}, \tag{A.186}$$

we conclude that the number $\zeta'_\theta(0, \Pi_+, D)$ is also real. q.e.d.

Appendix B. Families of flat connections

In this appendix we review some of the results of [**23**] and reformulate them in a form convenient for our purposes. These results are used in Section 13.

B.1. Connections flat modulo lower order terms. First, we introduce some definitions from [**23**], but we formulate them in a slightly different form which is more convenient for our purposes.

Let $k[t] = k[t_1, \dots, t_r]$ denote the polynomial ring in r variables over a field k. Let $\mathfrak{m} \subset k[t]$ denote the unique maximal ideal of $k[t]$ (the *augmentation ideal*), i.e., the ideal generated by $t_1, \dots, t_r$. Let $A_m = k[t]/\mathfrak{m}^{m+1}$. We denote by $G_m = \mathrm{GL}(n, A_m)$ the group of matrices with entries in A_m.

Let M be a manifold and let E be a complex vector bundle over M. Suppose ∇ is a flat connection on E. Let

$$\nabla(t) = \nabla + \sum_{0<|\alpha|\le m} \omega_\alpha t^\alpha, \qquad t \in k^r, \tag{B.187}$$

be a family of connections. Here $\alpha \in (\mathbb{Z}_\ge)^r$ is a multi-index, $|\alpha| = \alpha_1 + \dots + \alpha_r$, $t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_r^{\alpha_r}$, and ω_α are smooth 1-forms with values in End E. We say that the family $\nabla(t)$ is *flat modulo* t^{m+1} if $\nabla(t)^2 \in \mathfrak{m}^{m+1}$.

Fix a base point $x_* \in M$. Given a continuous path $\phi : [0,1] \to M$, $\phi(0) = \phi(1) = x_*$, for any $t \in k^r$, we denote by $\mathrm{Mon}_{\nabla(t)}(\phi)$ the monodromy of $\nabla(t)$ along ϕ, cf. (12.153). If the family $\nabla(t)$ is flat modulo t^{m+1} then, for any homotopic paths $\phi_i : [0,1] \to M$, $\phi_i(0) = \phi_i(1) = x_*$ $(i = 1, 2)$,

$$\mathrm{Mon}_{\nabla(t)}(\phi_1) \equiv \mathrm{Mon}_{\nabla(t)}(\phi_2) \qquad \mod \mathfrak{m}^{m+1}.$$

Hence, we have a well defined representation

$$\mathrm{Mon}_{\nabla(t)} : \pi_1(M, x_*) \longrightarrow G_m. \tag{B.188}$$

One says that two families of connections $\nabla_1(t)$ and $\nabla_2(t)$, which are flat modulo t^{m+1}, are *A_m-gauge equivalent* if there exists a family of gauge transformations

$$\mathfrak{g}(t) = \mathfrak{g}_0 + \sum_{0<|\alpha|\le m} \mathfrak{g}_\alpha t^\alpha \tag{B.189}$$

where each $\mathfrak{g}_\alpha$ is a gauge transformation of E, such that

$$\nabla_2(t) \equiv \mathfrak{g}(t) \cdot \nabla_1(t) \cdot \mathfrak{g}(t)^{-1} \qquad \text{mod } \mathfrak{m}^{m+1}.$$

B.2. Relationship between families of connections and families of representations of the fundamental group in G_m. Proposition 6.3 of [**23**] states that there is a one-to-one correspondence between the A_m-gauge equivalence classes of connections $\nabla(t)$ and the isomorphism classes of representations $\gamma(t)$ of $\pi_1(M)$ in G_m given by the monodromy representation (B.188). In other words, we have the following

Lemma B.3.

(i) *For every family of representations*

$$\gamma(t) : \pi_1(M, x_*) \to G_m,$$

there exists a flat modulo t^{m+1} family of connections $\nabla(t)$ such that

$$\text{Mon}_{\nabla(t)} \equiv \gamma(t) \qquad \text{mod } \mathfrak{m}^{m+1}. \tag{B.190}$$

(ii) *Every two connections $\nabla_1(t)$ and $\nabla_2(t)$ which are of the form* (B.187)*, are flat modulo t^{m+1}, and satisfy* (B.190) *are A_m-gauge equivalent, i.e., there exists a family of gauge transformations* (B.189) *such that*

$$\nabla_2(t) \equiv \mathfrak{g}(t) \cdot \nabla_1(t) \cdot \mathfrak{g}(t)^{-1} \qquad \text{mod } \mathfrak{m}^{m+1}. \tag{B.191}$$

Moreover, if $\nabla_1(0) = \nabla_2(0)$, then one can choose

$$\mathfrak{g}(t) = \mathfrak{g}_0 + \sum_{0<|\alpha|\le m} \mathfrak{g}_\alpha t^\alpha$$

so that $\mathfrak{g}_0 = \text{Id}$.

B.4. The case when $k = \mathbb{C}$ or $\mathbb{R}$. Suppose now that $k = \mathbb{C}$ or $\mathbb{R}$ and $r \in \mathbb{Z}_{\ge 1}$. Let $\mathcal{O} \subset k^r$ be an open set and let ∇_μ ($\mu \in \mathcal{O}$) be a family of connections such that for some $\lambda \in \mathcal{O}$ we have

$$\nabla_\mu = \nabla_\lambda + \sum_{0<|\alpha|\le m} \omega_\alpha (\mu - \lambda)^\alpha + o(|\mu - \lambda|^m), \qquad \mu \in \mathcal{O}, \tag{B.192}$$

where $o(|\mu - \lambda|^m)$ is understood in the sense of the Fréchet topology on $\mathcal{C}(E)$ introduced in Subsection 13.1.

Denote $t = \mu - \lambda$ and set

$$\overline{\nabla}(t) = \nabla_\lambda + \sum_{0<|\alpha|\le m} \omega_\alpha t^\alpha. \tag{B.193}$$

Then $\nabla_\mu = \overline{\nabla}(\mu - \lambda) + o(|\mu - \lambda|^m)$. Hence, for every closed path $\phi : [0, 1] \to M$, $\phi(0) = \phi(1) = x_*$ we have

$$\text{Mon}_{\nabla_\mu}(\phi) = \text{Mon}_{\overline{\nabla}(\mu-\lambda)}(\phi) + o(|\mu - \lambda|^m), \tag{B.194}$$

where $o(|\mu-\lambda|^m)$ is understood in the sense of the Fréchet topology introduced in Subsection 5.3. If the family $\overline{\nabla}(t)$ is flat modulo t^{m+1}, we will view $\mathrm{Mon}_{\nabla_\mu}$ as a map $\pi_1(M,x_*)\to G_m$ by identifying it with $\mathrm{Mon}_{\overline{\nabla}(\mu-\lambda)}$.

B.5. Application to real differentiable families of flat connections. Let $\mathcal{O}\subset\mathbb{C}$ be an open set. A family ∇_μ ($\mu\in\mathcal{O}$) of flat connections on E is called *real differentiable* at $\lambda\in\mathcal{O}$ if there exist $\omega_1,\omega_2\in\Omega^1(M,\mathrm{End}\ E)$ with

$$\nabla_\mu=\nabla_\lambda+\mathrm{Re}(\mu-\lambda)\cdot\omega_1+\mathrm{Im}(\mu-\lambda)\cdot\omega_2+o(\mu-\lambda). \tag{B.195}$$

(Again, $o(\mu-\lambda)$ is understood in the sense of the Fréchet topology on $\mathcal{C}(E)$ introduced in Subsection 13.1.)

Lemma B.6. *Let $\lambda\in\mathbb{C}$ and let $\mathcal{O}\subset\mathbb{C}$ be an open neighborhood of λ in $\mathbb{C}$. Suppose that ∇_μ ($\mu\in\mathcal{O}$) is a family of flat connections which is real differentiable at λ, cf.* (B.195)*. Assume that the map*

$$\mathcal{O}\ \to\ \mathrm{Rep}(\pi_1(M),\mathbb{C}^n),\qquad \mu\ \mapsto\ \mathrm{Mon}_{\nabla_\mu}$$

is a holomorphic curve in $\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$*. Then the following statements hold:*

(i) *There exists a smooth form $\omega\in\Omega^1(M,\mathrm{End}\ E)$ such that $\nabla_\lambda\omega=0$ and*

$$\mathrm{Mon}_{\nabla_\lambda+(\mu-\lambda)\omega}(\phi)=\mathrm{Mon}_{\nabla_\mu}(\phi)+o(\mu-\lambda), \tag{B.196}$$

for every closed path $\phi:[0,1]\to M,\ \phi(0)=\phi(1)=x_*$.

(ii) *There exists a family of gauge transformations $G(\mu)\in\mathrm{End}\ E$ ($\mu\in\mathcal{O}$) such that $G(\lambda)=\mathrm{Id}$ and*

$$\nabla_\lambda+(\mu-\lambda)\,\omega=G(\mu)\cdot\nabla_\mu\cdot G(\mu)^{-1}+o(\mu-\lambda). \tag{B.197}$$

Proof. To prove part (i) of the lemma we apply Lemma B.3 with $k=\mathbb{C}$, $t=t_1$ (i.e., $r=1$), $m=1$, and $t=\mu-\lambda$. Since $\mu\to\mathrm{Mon}_{\nabla_\mu}$ is a holomorphic curve, its Taylor expansion at λ up to first order, $\gamma(t)$, defines a map $\mathcal{O}\to G_1$. Then

$$\mathrm{Mon}_{\nabla_{\lambda+t\omega}}=\gamma(t)+o(t). \tag{B.198}$$

By Lemma B.3(i), there exists a flat modulo t^2 family of connections $\tilde\nabla(t)=\tilde\nabla(0)+t\widetilde{\omega}$ such that

$$\mathrm{Mon}_{\tilde\nabla(t)}\equiv\gamma(t)\qquad \mathrm{mod}\ t^2. \tag{B.199}$$

Since $\mathrm{Mon}_{\tilde\nabla(0)}=\gamma(0)$ there exists a gauge transformation $\mathfrak{g}\in\mathrm{End}\ E$ such that its restriction to the fiber of E over the base point x_* is the identity map and

$$\nabla_\lambda=\mathfrak{g}^{-1}\cdot\tilde\nabla(0)\cdot\mathfrak{g}.$$

Then $\omega := \mathfrak{g}^{-1}\,\widetilde{\omega}\,\mathfrak{g}$ is a smooth End E-valued 1-form and (B.199) takes the form

$$\text{Mon}_{\nabla_\lambda + t\omega} \equiv \gamma(t) \qquad \text{mod } t^2, \tag{B.200}$$

which together with (B.198) implies (B.196). Note that $\nabla_\lambda + t\omega$ is a flat modulo t^2 connection and, hence, $\nabla_\lambda \omega = 0$.

For part (ii) let us set $k = \mathbb{R}$, $t = (t_1, t_2)$ (i.e., $r = 2$), and $m = 1$. Denote $t_1 := \text{Re}(\mu - \lambda)$, $t_2 := \text{Im}(\mu - \lambda)$. Then, by the assumption of real differentiability, ∇_μ is of the form

$$\nabla_\lambda + t_1\,\omega_1 + t_2\,\omega_2 = \nabla_\mu + o(\mu - \lambda).$$

Note that both $\nabla_\lambda + t_1\omega_1 + t_2\omega_2$ and $\nabla_\lambda + (t_1 + it_2)\omega$ are flat modulo t^2 connections which, by (B.198), induce the same monodromy representation $\gamma(t) : \pi_1(M, x_*) \to G_1$. Hence, (B.197) follows from Lemma B.3(ii). q.e.d.

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