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ON THE ALGEBRAIC FUNDAMENTAL GROUP OF SURFACES WITH $K^2 \leq 3\chi$

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Abstract

Let S be a minimal complex surface of general type with $q(S) = 0$. We prove the following statements concerning the algebraic fundamental group $\pi_1^{\mathrm{alg}}(S)$:

- Assume that $K_S^2 \leq 3\chi(S)$. Then S has an irregular étale cover if and only if S has a free pencil of hyperelliptic curves of genus 3 with at least 4 double fibres.
- If $K_S^2 = 3$ and $\chi(S) = 1$, then S has no irregular étale cover.
- If $K_S^2 < 3\chi(S)$ and S does not have any irregular étale cover, then $|\pi_1^{\mathrm{alg}}(S)| \leq 9$. If $|\pi_1^{\mathrm{alg}}(S)| = 9$, then $K_S^2 = 2$, $\chi(S) = 1$.

1. Introduction

Every minimal surface S of general type satisfies the Noether inequality:

$$K_S^2 \geq 2\chi(S) - 6.$$

It has been clear for a long time that the closer a surface is to the Noether line $K^2 = 2\chi - 6$, the simpler its algebraic fundamental group is. In fact, Reid has conjectured that for $K^2 < 4\chi$ the algebraic fundamental group of S is either finite or it coincides, up to finite group extensions, with the fundamental group of a curve of genus $g \geq 1$, i.e., it is *commensurable* with the fundamental group of a curve, ([**Re1**, Conjecture 4], see also [**BHPV**], p. 294).

In the case of irregular surfaces or of regular surfaces having an irregular étale cover, Reid's conjecture follows from the Severi inequality, recently proved in [**Pa**], which states that the Albanese map of an irregular surface with $K^2 < 4\chi$ is a pencil.

Indeed, let S be an irregular surface satisfying $K^2 < 4\chi$, let $a\colon S \to B$ be the Albanese pencil of S and F a general fibre of a. The inclusion

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$F \hookrightarrow S$ induces a map $\psi\colon \pi_1^{\mathrm{alg}}(F) \to \pi_1^{\mathrm{alg}}(S)$. By [**X3**, Theorem 1] the image H of ψ is either 0 or $\mathbb{Z}_2$, and $H = \mathbb{Z}_2$ is possible only if F is hyperelliptic. The cokernel of ψ is the so-called *orbifold fundamental group* of the fibration a (cf. [**CKO**], [**Ca**, Lemma 4.2]). If a has no multiple fibres, then we have an exact sequence:

$$1 \to H \to \pi_1^{\mathrm{alg}}(S) \to \pi_1^{\mathrm{alg}}(B) \to 1. \tag{1.1}$$

If a has multiple fibres, then it is possible to find a Galois cover $B' \to B$ such that the fibration $a'\colon S' \to B'$ obtained from a by base change and normalization has no multiple fibres and the map $S' \to S$ is étale. Since $\pi_1^{\mathrm{alg}}(S')$ is a normal subgroup of $\pi_1^{\mathrm{alg}}(S)$ of finite index, it follows that in any case the algebraic fundamental group of an irregular surface satisfying $K^2 < 4\chi$ is commensurable with the fundamental group of a curve. Of course the same is true for a regular surface satisfying $K^2 < 4\chi$ and having an irregular étale cover.

Reid's conjecture is still open for surfaces not having an irregular cover. However, for surfaces satisfying $K^2 < 3\chi$, not only Reid's conjecture is true ([**Re1**] and [**Ho**]), but work by several authors gives more precise results on the algebraic fundamental group (cf. [**Bo**], [**Ho**], [**Re1**], [**Re2**], [**X2**], [**X3**]). The picture that emerges from their work is the following:

(I) If $K_S^2 < 2\chi(S)$, then S is regular and $\pi_1^{\mathrm{alg}}(S)$ is finite.

(II) If $K_S^2 < \frac{8}{3}\chi(S)$ and S is irregular, then the Albanese map of S is a pencil of curves of genus 2. If $K_S^2 < \frac{8}{3}\chi(S)$ and S is regular, then $\pi_1^{\mathrm{alg}}(S)$ is finite.

(III) If $K_S^2 < 3\chi(S)$ and S is irregular, then the Albanese map of S is a pencil of hyperelliptic curves of genus 2 or 3. If S is regular, then either $\pi_1^{\mathrm{alg}}(S)$ is finite or there exists an irregular étale cover $X \to S$. The Albanese map of X is a pencil of hyperelliptic curves of genus 3, which induces on S a free pencil of hyperelliptic curves of genus 3 with at least 4 double fibres. Conversely, if S has such a pencil, then it admits an irregular étale cover.

These results give a good understanding of the algebraic fundamental group of a surface S with $K^2 < 3\chi$ and infinite $\pi_1^{\mathrm{alg}}(S)$.

In fact, if S is irregular and the Albanese map $a\colon S \to B$ has multiple fibres, then by statement (III) and by the adjunction formula we have $g = 3$ and the multiple fibres are double fibres. Then there is a Galois cover $B' \to B$ with Galois group G such that the G-cover $S' \to S$ obtained by base change and normalization is étale and the induced fibration $a'\colon S' \to B'$ has no multiple fibres. One can show that G can be chosen to be a quotient of the dihedral group of order 8. So we have an exact sequence:

$$1 \to \pi_1^{\mathrm{alg}}(S') \to \pi_1^{\mathrm{alg}}(S) \to G \to 1$$

and the group $\pi_1^{\mathrm{alg}}(S')$ is described by sequence (1.1).

If S is a regular surface such that $K_S^2 < 3\chi(S)$ and $\pi_1^{\mathrm{alg}}(S)$ is infinite, then using (III), one constructs an irregular étale Galois cover $X \to S$ with Galois group $\mathbb{Z}_2$ or $\mathbb{Z}_2^2$ whose Albanese map is a pencil of curves of genus 3 without multiple fibres (more precisely, we have $\mathbb{Z}_2$ if the number k of double fibres of a is even and $\mathbb{Z}_2^2$ if k is odd). Then the group $\pi_1^{\mathrm{alg}}(X)$ is a normal subgroup of $\pi_1^{\mathrm{alg}}(S)$ of index 2 or 4 which can be described as explained above.

However, if the algebraic fundamental group of S is finite, then the above results give no additional information.

In this paper we give two improvements of the above results.

We first extend part of (III) to surfaces on the line $K^2 = 3\chi$:

Theorem 1.1. *Let S be a minimal complex surface of general type with $q(S) = 0$ and $K_S^2 \leq 3\chi(S)$.*

Then S has an irregular étale cover if and only if there exists a fibration $f\colon S \to \mathbb{P}^1$ such that:

(i) *the general fibre F of f is hyperelliptic of genus 3;*

(ii) *f has at least 4 double fibres.*

This improvement is made possible by the Severi inequality.

In the case $p_g(S) = 0$, Theorem 1.1 can be made more precise:

Theorem 1.2. *Let S be a smooth minimal surface of general type with $p_g(S) = 0$, $K_S^2 = 3$.*

Then S has no irregular étale cover.

Theorem 1.2 is sharp in a sense, since there are examples, due to Keum and Naie (cf. [**Na**]), of surfaces with $K^2 = 4$ and $p_g = 0$ that have an irregular cover.

On the other hand, it remains an open question whether the algebraic fundamental group of a surface with $K^2 = 3$ and $p_g = 0$ is finite or more generally whether the algebraic fundamental group of a surface with $K^2 = 3\chi$ that has no étale irregular cover is finite.

In even greater generality one would like to know whether the algebraic fundamental group of a surface with $K^2 < 4\chi$ that has no étale irregular cover is finite, deciding thus Reid's conjecture. This is a very challenging problem, which however does not seem possible to resolve with the methods of the present paper.

Finally, we bound the cardinality of $\pi_1^{\mathrm{alg}}(S)$ in the case when it is a finite group:

Theorem 1.3. *Let S be a minimal surface of general type such that $K_S^2 < 3\chi(S)$. If S has no irregular étale cover, then $\pi_1^{\mathrm{alg}}(S)$ is a finite group of order ≤ 9.*

Moreover, if $\pi_1^{\mathrm{alg}}(S)$ has order 9, then $\chi(S) = 1$ and $K_S^2 = 2$, namely S is a numerical Campedelli surface.

This bound is sharp, since there are examples of surfaces with $p_g = 0$, $K^2 = 2$ and $\pi_1^{\mathrm{alg}}(S) = \mathbb{Z}_9$, $\mathbb{Z}_3^2$ (cf. [**X1**, Ex. 4.11], [**MP1**]).

By this theorem only a very short list of finite groups can occur as the algebraic fundamental groups of surfaces with $K^2 \leq 3\chi - 1$. The list is even more restricted if $K^2 \leq 3\chi - 2$: in [**MP2**] it is shown that in this case $|\pi_1^{\mathrm{alg}}(S)| \leq 5$, with equality holding only for surfaces with $K_S^2 = 1$ and $p_g(S) = 0$. Moreover $|\pi_1^{\mathrm{alg}}(S)| = 3$ is possible only for $2 \leq \chi(S) \leq 4$ and $K^2 = 3\chi - 3$.

Notation and conventions. We work over the complex numbers. All varieties are projective algebraic. We denote by χ or $\chi(S)$ the holomorphic Euler characteristic of the structure sheaf of the surface S.

2. The proof of Theorem 1.1

In this section we assume that S is a minimal complex surface of general type with $q(S) = 0$ and $K_S^2 \leq 3\chi(S)$. In order to prove Theorem 1.1 we need some intermediate steps.

Lemma 2.1. *Let $\rho\colon Z \to S$ be an étale cover such that $q(Z) > 0$.*

Then the Albanese pencil $a\colon Z \to A$ induces a fibration $f\colon S \to \mathbb{P}^1$ such that:

(i) *the general fibre F of f is a curve of genus* 3*;*

(ii) *f has at least* 4 *double fibres.*

Moreover, all irregular étale covers of S induce the same fibration $f\colon S \to \mathbb{P}^1$.

Proof. If $\rho\colon Z \to S$ is an irregular étale cover, then the Galois closure of ρ is an irregular Galois étale cover. We denote by $\pi\colon Y \to S$ a minimal element of the set of irregular Galois étale covers of S.

Denote by d the degree of π. The surface S is minimal of general type with $K_Y^2 = dK_S^2$, $\chi(Y) = d\chi(S)$. Hence $K_Y^2 \leq 3\chi(Y) < 4\chi(Y)$ and therefore, by the Severi inequality ([**Pa**]), the image of the Albanese map of Y is a curve. Write $a\colon Y \to B$ for the Albanese pencil, and let b be the genus of B and g the genus of the general fibre F of a. The Galois group G of π acts on the curve B. This action is effective by the assumption that π is minimal among the irregular étale covers of S. Hence we have a commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & S \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{\bar{\pi}} & \mathbb{P}^1 \end{array} \tag{2.1}$$

The map $\bar{\pi}$ is a Galois cover with group G and the general fibre of f is also equal to F. Since the map π is obtained from f by taking base change with $\bar{\pi}$ and normalizing, the fibre of f over a point x of $\mathbb{P}^1$ has multiplicity equal to the ramification order of $\bar{\pi}$ over x. Notice that, since $\mathbb{P}^1$ is simply connected, the branch divisor of $\bar{\pi}$ is nonempty and therefore the fibration f always has multiple fibres. Notice also that, since S is of general type, the existence of multiple fibres implies $g \geq 3$.

We remark that the fibration a is not smooth and isotrivial. In fact, if this were the case then Y would be a free quotient of a product of curves, hence it would satisfy $K_Y^2 = 8\chi(Y)$. Hence we may define the slope of a (cf. [**X3**]):

$$\lambda(a) := \frac{K_Y^2 - 8(b-1)(g-1)}{\chi(Y) - (b-1)(g-1)}.$$

The slope inequality ([**X3**], cf. also [**CH**], [**St**]) gives

$$4(g-1)/g \leq \lambda(a) \leq K_Y^2/\chi(Y) = K_S^2/\chi(S) \leq 3, \tag{2.2}$$

where the second inequality is a consequence of $b > 0$. Hence we get $g = 3$ or $g = 4$.

Assume $g = 4$. In this case (2.2) becomes:

$$3 \leq \lambda(a) \leq K_S^2/\chi(S) \leq 3.$$

It follows that the slope inequality is sharp in this case and $K_S^2 = 3\chi(S)$. By [**Ko2**, Prop. 2.6], this implies that F is hyperelliptic. Let σ be the involution of S induced by the hyperelliptic involution on the fibres of f. The divisorial part R of the fixed locus of σ satisfies $FR = 10$. As remarked above, f has at least a fibre of multiplicity $m > 1$, that we denote by mA. Since $g = 4$, by the adjunction formula $\frac{6}{m}$ is divisible by 2, yielding $m = 3$. Hence $3AR = 10$, a contradiction. So we have proved $g = 3$.

Using the adjunction formula again, we see that the multiple fibres of f are double fibres, hence all the branch points of $\bar{\pi}$ have ramification order equal to 2. Let k be the number of branch points of $\bar{\pi}$. By applying the Hurwitz formula to $\bar{\pi}$, we get $k \geq 4$.

Given an irregular étale cover $\rho\colon Z \to S$, we can always find an étale cover $W \to S$ which dominates both Z and Y. The Albanese pencil of W is a pullback both from Y and from Z, hence the fibrations induced on S by the Albanese pencils of Z, W and Y are the same. q.e.d.

We introduce some more notation. Assume that $f\colon S \to \mathbb{P}^1$ is the fibration defined in Lemma 2.1. Let $\bar{\pi}\colon B \to \mathbb{P}^1$ be the double cover branched on 4 points corresponding to double fibres $2F_1, \ldots, 2F_4$ of f and $\pi\colon Y \to S$ the étale double cover obtained by base change with $\bar{\pi}$ and normalization, as in diagram (2.1). Then $K_Y^2 = 2K_S^2$, $\chi(Y) = 2\chi(S)$

and $q(Y) = 1$. We write $\eta := F_1 + F_2 - F_3 - F_4$. Clearly, η has order 2 in $\mathrm{Pic}(S)$ and π is the étale double cover corresponding to η.

Lemma 2.2. *The general fibre F of f is hyperelliptic.*

Proof. Assume by contradiction that F is not hyperelliptic and consider the pencil $a\colon Y \to B$, whose general fibre is also equal to F. Set $\mathcal{E} := a_*\omega_Y$ and denote by $\psi\colon Y \to \mathbb{P}(\mathcal{E})$ the relative canonical map, which is a morphism by Remark 2.4 of [**Ko2**]. Let V be the image of ψ. The surface V is a relative quartic in $\mathbb{P}(\mathcal{E})$ and, by Lemma 3.1 and Theorem 3.2 of [**Ko2**], its singularities are at most rational double points. The map ψ is birational and it contracts precisely the nodal curves of Y, which are all vertical since B has genus 1. Hence V is the canonical model of Y.

Let ι be the involution associated to the cover $Y \to S$. This involution induces automorphisms of B, $\mathcal{E}$, $\mathbb{P}(E)$ and V (that we denote again by ι) compatible with a, ψ and the inclusion $V \subset \mathbb{P}(\mathcal{E})$. Given $b \in B$, write $\mathbb{P}^2_b$ for the fiber of $\mathbb{P}(\mathcal{E})$ over b and $V_b := V \cap \mathbb{P}^2_b$. The curve V_b is a plane quartic inside $\mathbb{P}^2_b$. For every $b \in B$, the map ι induces a projective isomorphism between $\mathbb{P}^2_b$ and $\mathbb{P}^2_{\iota(b)}$ that restricts to an isomorphism of V_b with $V_{\iota(b)}$. In particular, if b is one of the four fixed points of ι on B, then ι induces an involution of $\mathbb{P}^2_b$ that preserves the quartic V_b. Since the fixed locus of an involution of the plane contains a line, it follows that ι has at least a fixed point on V_b. In particular, the action of ι on V is not free.

On the other hand, one checks that a fixed point free automorphism of a minimal surface of general type induces a fixed point free automorphism of the canonical model. So we have a contradiction. q.e.d.

We can now give:

Proof of Theorem 1.1. The "if" part is a consequence of Lemma 2.1 and Lemma 2.2. Conversely, if S has a fibration with 4 double fibres $2F_1, \dots, 2F_4$ then the étale double cover associated with $\eta := F_1 + F_2 - F_3 - F_4$ has irregularity equal to 1. q.e.d.

3. The proof of Theorem 1.2

In this section we let S denote a smooth minimal surface of general type with $p_g(S) = 0$, $K_S^2 = 3$. To prove Theorem 1.2 we argue by contradiction.

Thus assume that S has an irregular étale cover. Then by Theorem 1.1 there exists a fibration $f\colon S \to \mathbb{P}^1$ whose general fibre is hyperelliptic of genus 3 and with at least 4 double fibres $2F_1, \dots, 2F_4$. As before, denote by $\pi\colon Y \to S$ the étale double cover given by $\eta = F_1 + F_2 - F_3 - F_4$ and by ι the involution associated with π. The invariants of Y are: $q(Y) = 1$, $p_g(Y) = 2$, $K_Y^2 = 6$.

The hyperelliptic involution on the fibres of $a\colon Y \to B$ and $f\colon S \to Y$ induces involutions τ of Y and σ of S. By construction, these involutions are compatible with the map $\pi\colon Y \to S$; namely, we have $\pi \circ \tau = \sigma \circ \pi$. We denote by $p\colon S \to \Sigma := S/\sigma$ the quotient map.

Lemma 3.1. *The involutions τ and ι of Y commute.*

Proof. Denote by h the composite map $Y \to S \to \Sigma$. By construction, both ι and τ belong to the Galois group G of h. Since h has degree 4 and ι and τ are involutions, the group G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and ι and τ commute. q.e.d.

Lemma 3.2. *The involution $\iota\tau$ has at least* 16 *isolated fixed points on Y.*

Proof. Let $q\colon Y \to Z := Y/\iota\tau$ be the quotient map. The surface Z is nodal. The regular 1-forms and 2-forms of Z correspond to the elements of $H^0(Y, \Omega^1_Y)$, respectively $H^0(Y, \omega_Y)$, that are invariant under the action of $\iota\tau$. By the same argument, since $p_g(S) = p_g(Y/\tau) = 0$, both ι and τ act on $H^0(Y, \omega_Y)$ as multiplication by -1. It follows that $\iota\tau$ acts trivially on $H^0(Y, \omega_Y)$ and $p_g(Z) = 2$. Since ι acts on B as an involution with quotient $\mathbb{P}^1$ and τ acts trivially on B, it follows that the action of $\iota\tau$ on B is equal to the action of ι and that $q(Z) = 0$.

Let D be the divisorial part of the fixed locus of $\iota\tau$ on Y and let k be the number of isolated fixed points of $\iota\tau$. We recall the Holomorphic Fixed Point formula (see [**AS**], p. 566):

$$\sum_i (-1)^i \operatorname{Tr}(\iota\tau | H^i(Y, \mathcal{O}_Y)) = (k - K_Y D)/4.$$

By the above considerations, this can be rewritten as:

$$k = 16 + K_Y D.$$

The statement now follows from the fact that K_Y is nef. q.e.d.

Proof of Theorem 1.2. By Lemma 3.1, the involution $\iota\tau$ of Y induces σ on S. By Lemma 3.2, $\iota\tau$ has at least 16 isolated fixed points. Since the images on S of these points are isolated fixed points of σ, the involution σ has at least 8 isolated fixed points. On the other hand, by [**CCM**, Prop. 3.3] there are at most $K_S^2 + 4 = 7$ isolated fixed points of σ. So we have a contradiction, and thus S has no irregular étale cover. q.e.d.

4. The proof of Theorem 1.3

To prove Theorem 1.3 we will use the following two results proved in [**Be**, Cor. 5.8], although not stated explicitly.

Proposition 4.1. *Let Y be a surface of general type such that the canonical map of Y has degree 2 onto a rational surface. If G is a group that acts freely on Y, then $G = \mathbb{Z}_2^r$, for some r.*

Proof. The group G is finite, since a surface of general type has finitely many automorphisms.

Let T be the quotient of Y by the canonical involution. The surface T is rational, with canonical singularities, and G acts on T.

Since T is rational, each element $g \in G$ acts with fixed points. The argument in the proof of [**Be**, Cor. 5.8] shows that each g has order 2, hence $G = \mathbb{Z}_2^r$. q.e.d.

Corollary 4.2. *Let S be a minimal surface of general type such that $K_S^2 < 3\chi(S)$, and S has no irregular étale cover. If $Y \to S$ is an étale G-cover, then either $|G| \leq 10$ or $G = \mathbb{Z}_2^r$, for some $r \geq 4$.*

Proof. Let $\pi\colon Y \to S$ be an étale G-cover of degree $d > 10$. By assumption we have $q(Y) = 0$ and $K_Y^2 < 3p_g(Y) - 7$, and therefore the canonical map of Y is 2-to-1 onto a rational surface by [**Be**, Theorem 5.5]. Hence $G = \mathbb{Z}_2^r$ for some $r \geq 4$ by Proposition 4.1. q.e.d.

For related statements see the results of [**X2**] on hyperelliptic surfaces and the results of [**AK**] and [**Ko1**].

We remark that the next result is well known for the cases $\chi(S) = 1$ and $K_S^2 = 1$ or 2 ([**Re2**]).

Proposition 4.3. *Let S be a minimal surface of general type with $K_S^2 < 3\chi(S)$. If S has no irregular étale cover, then $|\pi_1^{\mathrm{alg}}(S)| \leq 9$.*

Proof. Let $Y \to S$ be an étale G-cover. By Corollary 4.2, it is enough to exclude the following possibilities: a) $G = \mathbb{Z}_2^r$ for some $r \geq 4$, and b) $|G| = 10$.

Consider case a) and assume by contradiction that $\pi\colon Y \to S$ is a Galois étale cover with Galois group $G = \mathbb{Z}_2^4$. By [**Miy**], $\chi(S) \geq 2$. We have $\chi(Y) = 16\chi(S) \geq 32$ and $K_Y^2 < 3(\chi(Y) - 5)$. Notice that, since $K_Y^2 < 3\chi(Y) - 10$, by [**Be**, Theorem 5.5] the surface Y has a pencil of hyperelliptic curves. Hence Y satisfies the assumptions of [**X2**, Theorem 1] and there exists a unique free pencil $|F|$ of hyperelliptic curves of genus $g \leq 3$ on Y. The action of G preserves $|F|$ by the uniqueness of $|F|$. Since $\mathrm{Aut}(\mathbb{P}^1)$ does not contain a subgroup isomorphic to $\mathbb{Z}_2^3$, there is a subgroup $H < G$ of order ≥ 4 that maps every curve of $|F|$ to itself. Since the action of G on Y is free, this implies that $g - 1$ is divisible by 4, contradicting $g \leq 3$ and S of general type.

Consider now case b) and assume by contradiction that $\pi\colon Y \to S$ is a Galois cover with Galois group G of order 10. For $K_S^2 < 3\chi(S) - 1$, we have $K_Y^2 < 3\chi(Y) - 10$ and, as in the proof of Corollary 4.2, G is of the form $\mathbb{Z}_2^a$, a contradiction. So we have $K_S^2 = 3\chi(S) - 1$, $K_Y^2 = 3\chi(Y) - 10$,

$q(Y) = 0$ and so, by [**AK**], the canonical map of Y is either birational or 2-to-1 onto a rational surface. By Proposition 4.1, the last possibility does not occur, since G has order 10.

The surface Y satisfies $p_g(Y) = 10\chi(S) - 1 \geq 9$. Surfaces on the Castelnuovo line $K^2 = 3\chi - 10$ with birational canonical map are classified (cf. [**Ha**], [**Mir**] and [**AK**]): for $p_g(Y) \geq 8$, the canonical model V of Y is a relative quartic inside a $\mathbb{P}^2$-bundle

$$\mathbb{P} := \mathrm{Proj}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c)),$$

where $0 \leq a \leq b \leq c$ and $a + b + c = p_g(Y) + 3$.

If the Galois group G preserves the fibration $f \colon V \to \mathbb{P}^1$ induced by the projection $\mathbb{P} \to \mathbb{P}^1$, then, as in Lemma 2.2, we obtain a contradiction by considering the action on V of an element of order 2 of G.

So, to conclude the proof we just have to show that G preserves f. Let W be the image of $\mathbb{P}$ via the tautological linear system. By the results of [**AK**], [**Ha**], [**Mir**], the threefold W is the intersection of all the quadrics that contain the canonical image of Y and therefore it is preserved by the automorphisms of V. One checks that W has a unique ruling by planes which induces the fibration f on V. Therefore every automorphism of V preserves the fibration f. q.e.d.

To obtain the statement of Theorem 1.3 we now show the following:

Proposition 4.4. *Let S be a minimal surface of general type with $K_S^2 < 3\chi(S)$. If $|\pi_1^{\mathrm{alg}}(S)| = 9$, then $\chi(S) = 1$ and $K_S^2 = 2$, namely S is a numerical Campedelli surface.*

Proof. Suppose that $|\pi_1^{\mathrm{alg}}(S)| = 9$ and $\chi(S) \geq 2$. The argument in the proof of Proposition 4.3 shows that $K_S^2 = 3\chi(S) - 1$. Let $\pi \colon Y \to S$ be the universal cover. We have $K_Y^2 = 3p_g(Y) - 6$, $p_g(Y) = 9\chi(Y) - 1 \geq 17$. By [**Ko1**, Lem. 2.2] the bicanonical map of Y has degree 1 or 2. Arguing as in the proof of Proposition 4.3, one shows that the bicanonical map of Y is birational. Then, since $p_g(Y) \geq 11$, by the results of [**Ko1**] the situation is analogous to the case of a surface with $K^2 = 3p_g - 7$ and birational canonical map. Namely, the intersection of all the quadrics through the canonical image of Y is a threefold W, which is the image of a $\mathbb{P}^2$-bundle $\mathbb{P} := \mathrm{Proj}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c))$ via the tautological linear system, and Y is birational to a relative quartic of $\mathbb{P}$. In particular, there is a fibration $f \colon Y \to \mathbb{P}^1$ with general fibre a nonhyperelliptic curve of genus 3. One can show as above that the Galois group $G = \pi_1^{\mathrm{alg}}(S)$ of π preserves f. Then we obtain a contradiction, since the multiple fibres of a genus 3 fibration are double fibres and a smooth genus 3 curve does not admit a free action of a group of order 9. q.e.d.

Remark. Numerical Campedelli surfaces with fundamental group $\mathbb{Z}_9$ and $\mathbb{Z}_3^2$ do exist (cf. [**X1**, Ex. 4.11], [**MP1**]).

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