

## A DIFFEOMORPHISM CLASSIFICATION OF MANIFOLDS WHICH ARE LIKE PROJECTIVE PLANES

LINUS KRAMER & STEPHAN STOLZ

### Abstract

We give a complete diffeomorphism classification of 1-connected closed manifolds  $M$  with integral homology  $H_*(M) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ , provided that  $\dim(M) \neq 4$ .

The integral homology of an oriented closed manifold<sup>1</sup>  $M$  contains at least two copies of  $\mathbb{Z}$  (in degree 0 resp.  $\dim M$ ). If  $M$  is simply connected and its homology has minimal size (i.e.,  $H_*(M) \cong \mathbb{Z} \oplus \mathbb{Z}$ ), then  $M$  is a homotopy sphere (i.e.,  $M$  is homotopy equivalent to a sphere). It is well-known from the proof of the (generalized) Poincaré conjecture that any homotopy sphere is homeomorphic to the standard sphere  $S^n$  of dimension  $n$ . By contrast, the cardinality of the set  $\Theta_n$  of diffeomorphism classes of homotopy spheres of dimension  $n$  can be very large (but finite except possibly for  $n = 4$ ) [7]. In fact, the connected sum of homotopy spheres gives  $\Theta_n$  the structure of an abelian group which is closely related to the stable homotopy group  $\pi_{n+k}(S^k)$ ,  $k \gg n$  (currently known approximately in the range  $n \leq 100$ ).

Somewhat surprisingly, it is easier to obtain an explicit diffeomorphism classification of 1-connected closed manifolds whose integral homology consists of *three* copies of  $\mathbb{Z}$ . Examples of such manifolds are the 1-connected projective planes (i.e., the projective planes over the complex numbers, the quaternions or the octonions). Eells and Kuiper pioneered the study of these ‘projective plane like’ manifolds [4] and obtained many important and fundamental results. For example, they show that the integral cohomology ring of such a manifold  $M$  is isomorphic to the cohomology ring of a projective plane, i.e.,  $H^*(M) \cong \mathbb{Z}[x]/(x^3)$ . This in turn implies that the dimension of  $M$  must be  $2m$  with  $m = 2, 4$  or  $8$  (cf. [4, §5]).

---

This paper was supported by a research grant in the *Schwerpunktprogramm Globale Differentialgeometrie* by the Deutsche Forschungsgemeinschaft; the second author was partially supported by NSF grant DMS 0104077.

Received 07/07/2005.

<sup>1</sup>All manifolds are assumed to be smooth.

We remark that a 1-connected closed manifold  $M$  of dimension  $n \geq 5$  with  $H_*(M) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$  admits a Morse function with three critical points, which is the assumption that Eells-Kuiper work with. Any 1-connected projective plane like manifold of dimension 4 is homeomorphic to the complex projective plane by Freedman's homeomorphism classification of simply connected smooth 4-manifolds [5].

Eells and Kuiper prove that there are six (resp. sixty) homotopy types of projective plane like manifolds of dimension  $2m$  for  $m = 4$  (resp.  $m = 8$ ) [4, §5]. They get close to obtaining a classification of these manifolds up to homeomorphism resp. diffeomorphism. One way to phrase their result is the following. If  $M$  is a smooth manifold of this type, let  $p_m^2(M)[M] \in \mathbb{Z}$  be the Pontryagin number obtained by evaluating the square of the Pontryagin class  $p_m(M) \in H^m(M; \mathbb{Z})$  (of the tangent bundle of  $M$ ) on the fundamental class  $[M] \in H_{2m}(M; \mathbb{Z})$ . Eells and Kuiper show that the Pontryagin number  $p_m^2(M)[M] \in \mathbb{Z}$  determines the diffeomorphism type *up to connected sum with a homotopy sphere*; in other words, if  $M'$  is another such manifold of the same dimension and the same Pontryagin number, then  $M'$  is diffeomorphic to the connected sum  $M \# \Sigma$  of  $M$  with a  $2m$ -dimensional homotopy sphere  $\Sigma$  (see 1.3 and [4, §9]; we note that the Pontryagin number determines the Eells-Kuiper integer  $h$  and vice versa via their formulas (2) resp. (5) in §9).

A complete homeomorphism classification of topological manifolds which look like projective planes was obtained by the first author in [9]. The main result of this paper is the following.

**Theorem A.** *Let  $M$  be a smooth simply connected manifold of dimension  $2m \neq 4$ , with integral homology  $H_*(M) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ . Then for any homotopy sphere  $\Sigma$  of dimension  $2m$  the connected sum  $M \# \Sigma$  is diffeomorphic to  $M$ .*

In view of the results of Eells-Kuiper discussed above, this implies the following diffeomorphism classification of projective plane like manifolds.

**Corollary B.** *Let  $M$  be a smooth simply connected  $2m$ -manifold with integral homology  $H_*(M) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ . Then the diffeomorphism type of  $M$  is determined by the Pontryagin number  $p_m^2(M)[M] \in \mathbb{Z}$ .*

Results of Eells-Kuiper combined with a result of Wall [22] allow a characterization of those integers which occur as the Pontryagin numbers of such manifolds. We will give a precise statement as Theorem 1.3 in the next section; for now we remark that the above result provides us with an infinite family of manifolds  $M$  which have a unique differentiable structure in the sense that any manifold homeomorphic to  $M$  is in fact diffeomorphic to  $M$  [10].

Another motivation for this paper came from the first author's attempt to classify the underlying spaces of topological projective planes

in the sense of Salzman [18]. In [9] he obtained a homeomorphism classification for the point sets of smooth topological projective planes, showing that only the four classical spaces  $\mathbb{F}P^2$ ,  $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$  appear. However, the diffeomorphism classification remained open. Except for the case of  $\mathbb{C}P^2$ , the results of the present paper settle this question. Combining our results with McKay’s diffeomorphism classification of 2-dimensional smooth topological projective planes [14], we obtain the following result.

**Corollary C.** *The point space of a smooth topological projective plane (in the sense of [18]) is diffeomorphic to its classical counterpart, i.e., to  $\mathbb{R}P^2$ ,  $\mathbb{C}P^2$ ,  $\mathbb{H}P^2$ , or  $\mathbb{O}P^2$ .*

**Outline of the paper.** In Section 1 we state in more detail the Eells-Kuiper results concerning the diffeomorphism classification of projective plane like manifolds up to connected sum with homotopy spheres. For the convenience of the reader, we also outline the proofs. The other sections are devoted to proving our main Theorem A.

In Section 2 we use Kreck’s modified surgery approach [11] to show that for a closed simply connected manifold  $M$  of dimension  $2m \neq 4$  the connected sum  $M \# \Sigma$  with a homotopy sphere  $\Sigma$  is diffeomorphic to  $M$  provided  $\Sigma$  represents zero in a suitable bordism group  $\Omega_{2m}^B$  (cf. Corollary 2.5).

The bordism groups  $\Omega_*^B$  depend on a fibration  $B \rightarrow BO$  which in turn depends on the manifold  $M$ . In Section 3 we determine the relevant fibration in the case that  $M$  is a projective plane like  $2m$ -manifold (cf. Proposition 3.4). In Section 4 we prove that any homotopy sphere  $\Sigma$  of dimension  $2m = 8, 16$  represents zero in  $\Omega_{2m}^B$  for  $B$  as above, thus completing the proof of Theorem A.

### 1. Classification up to connected sums with homotopy spheres

As mentioned in the introduction, the diffeomorphism classification of projective plane like manifolds up to connected sum with homotopy spheres was obtained by Eells-Kuiper [4] (plus one result of Wall’s [22, Thm. 4, p. 178]) or by specializing Wall’s much more general classification of ‘almost closed’  $(m - 1)$ -connected  $2m$  manifolds [22] to this case. Still, we feel that it is worthwhile to outline in this section how this classification follows from the classification of  $m$ -dimensional vector bundles over  $S^m$  and the  $h$ -cobordism theorem.

We recall that a smooth manifold  $N$  is ‘almost closed’ if it is a compact manifold whose boundary is a homotopy sphere. Such a manifold is obtained for example by removing an open  $n$ -disk from a closed manifold  $M$  of dimension  $n$ . The boundary  $\partial N$  of an almost closed  $n$ -manifold  $N$  is homeomorphic to the standard sphere  $S^{n-1}$ , and we denote by

$N(\alpha) = N \cup_{\alpha} D^n$  the closed topological manifold obtained by gluing  $N$  and the disk  $D^n$  along their common boundary via a homeomorphism  $\alpha: \partial D^n \rightarrow \partial N$ . We note that  $N(\alpha)$  is again a smooth manifold if  $\alpha$  is a *diffeomorphism*; moreover, if  $\beta: \partial N \rightarrow \partial D^n$  is a second diffeomorphism, then  $N(\beta)$  is diffeomorphic to the connected sum  $N(\alpha) \# \Sigma$  of  $N$  with the homotopy sphere  $\Sigma = D^n \cup_{\alpha^{-1}\beta} D^n$  obtained by gluing two discs along their boundaries via the diffeomorphism  $\alpha^{-1}\beta: \partial D^n \rightarrow \partial D^n$ .

An almost closed manifold  $N$  is called *projective plane like* if the integral homology  $H_*(N(\alpha))$  (which is independent of the choice of the homeomorphism  $\alpha$ ) is isomorphic to  $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ . This implies that  $N$  is a manifold of dimension  $2m$  with  $m = 2, 4, 8$ .

**Theorem 1.1** (Eells-Kuiper). *The diffeomorphism classes of simply connected almost closed projective plane like manifolds of dimension  $2m$  for  $m = 4, 8$  are in one-to-one correspondence with the non-negative integers. The manifold  $N_t$  corresponding to  $t \in \mathbb{Z}$  is the disk bundle of the vector bundle  $\xi_t$  over  $S^m$  with Euler class  $e(\xi_t) = x$ , and Pontryagin class  $p_m(\xi_t) = 2(1 + 2t)x$  (for  $m = 4$ ) resp.  $p_m(\xi_t) = 6(1 + 2t)x$  (for  $m = 8$ ), where  $x$  is the generator of  $H^m(S^m)$ .*

*Proof.* It is an easy homology calculation to show that the disk bundle  $N_t = D(\xi_t)$  is an almost closed projective space like manifold (the condition  $e(\xi_t) = x$  guarantees that the boundary  $\partial D(\xi_t)$  is a homotopy sphere). We note that pulling  $\xi_t$  back via a map  $S^m \rightarrow S^m$  of degree  $-1$  we obtain a bundle isomorphic to  $\xi_{-t-1}$ ; it follows that the manifolds  $N_t$  and  $N_{-t-1}$  are diffeomorphic and hence it suffices to consider only  $t \geq 0$ .

Conversely, if  $N$  is any simply connected almost closed projective space like manifold of dimension  $2m$ , consider the normal bundle  $\xi$  of an embedding  $S^m \hookrightarrow N$  which represents a generator for  $H_m(N; \mathbb{Z}) \cong \mathbb{Z}$ .

Then  $\xi$  is an  $m$ -dimensional oriented vector bundle over  $S^m$ , whose disc bundle  $D(\xi)$  can be identified with a tubular neighborhood of  $S^m \subset N$ . Up to isomorphism  $\xi$  is determined by its Euler class  $e(\xi)$  and its Pontryagin class  $p_m(\xi)$ . The assumption that  $N$  is projective space like implies that the integral cohomology ring of  $N/\partial N$  is isomorphic to  $\mathbb{Z}[x]/(x^3)$ , which in turn implies  $e(\xi) = x$ . By the classification of  $m$ -dimensional vector bundles over  $S^m$ , this implies that  $\xi$  is isomorphic to  $\xi_t$  for some  $t \in \mathbb{Z}$ . Now removing the interior of the disc bundle  $D(\xi) \subset N$  from  $N$ , we obtain a bordism  $W$  between  $\partial D(\xi)$  and  $\partial N$ . A homology calculation shows that this is in fact an  $h$ -cobordism (i.e., the inclusion of either boundary component into  $W$  is a homotopy equivalence). By Smale's  $h$ -cobordism theorem,  $W$  is diffeomorphic to  $\partial D(\xi) \times [0, 1]$ ; in particular,  $N = D(\xi) \cup_{\partial D(\xi)} W$  is diffeomorphic to  $D(\xi) = D(\xi_t)$ , which proves the theorem. q.e.d.

The theorem above begs the question for which  $t \in \mathbb{Z}$  is the boundary of  $N_t$  diffeomorphic to the standard sphere  $S^{2m-1}$ . The answer is given by the next result:

**Proposition 1.2** (Eells-Kuiper, Wall). *The boundary  $\partial N_t$  is diffeomorphic to  $S^{2m-1}$  if and only if  $t \equiv 0, 7, 48, 55 \pmod{56}$  (for  $m = 4$ ) resp.  $t \equiv 0, 127, 16128, 16255 \pmod{16256}$  (for  $m = 8$ ).*

*Proof.* Choose a homeomorphism  $\alpha: S^{2m-1} \rightarrow \partial N_t$  and consider the closed topological manifold  $N_t(\alpha) = N_t \cup_{\alpha} D^{2m}$ . Its  $\widehat{A}$ -genus  $\widehat{A}(N_t(\alpha))$ , a certain rational linear combination of the Pontryagin numbers  $p_m^2(M)[M]$  and  $p_{2m}(M)[M]$ , turns out to be independent of  $\alpha$ , and can be expressed in terms of  $t$  by the following formula [4, §9, Thms. on p. 216, resp. p. 218], [9, §7.2]:

$$\widehat{A}(N_t(\alpha)) = \begin{cases} -\frac{t(t+1)}{7 \cdot 8} & m = 4 \\ -\frac{t(t+1)}{127 \cdot 128} & m = 8 \end{cases}$$

If  $\partial N_t$  is diffeomorphic to  $S^{2m-1}$ , we may choose  $\alpha$  to be a diffeomorphism, and then  $N_t(\alpha)$  is a smooth manifold. This manifold can be equipped with a spin structure, since  $H^i(M; \mathbb{Z}/2) = 0$  for  $i = 1, 2$  and hence the Stiefel-Whitney classes  $w_i(M) \in H^i(M; \mathbb{Z}/2)$ ,  $i = 1, 2$  (the potential obstructions against a spin structure) vanish. This implies that  $\widehat{A}(N_t(\alpha))$  is an integer, namely the index of the ‘Dirac operator’ which can only be constructed for smooth spin manifolds. The formula above then implies that  $t$  satisfies the congruence of the proposition.

Conversely, according to a result of Wall [22, Thm. 4, p. 178], the integrality of  $\widehat{A}(N_t(\alpha))$  implies that  $\partial N_t$  is diffeomorphic to the standard sphere. q.e.d.

We note that the Pontryagin number  $p_m^2(M)[M]$  of the projective plane like manifold  $M = N_t(\alpha)$  is equal to  $2^2(1 + 2t)^2$  (for  $m = 4$  — unfortunately, the formula stated in [9] p. 2 is off by a factor 2; the correct number given here appears in *loc.cit.* Thm. 7.1) resp.  $6^2(1 + 2t)^2$  (for  $m = 8$ ). Hence the theorem and the proposition above imply the following result.

**Theorem 1.3.** *Let  $M$  be a smooth projective plane like manifold of dimension  $2m$ ,  $m = 4, 8$ . Then up to connected sum with a homotopy sphere, the diffeomorphism type of  $M$  is determined by the Pontryagin number  $p_m^2(M)[M] \in \mathbb{Z}$ . Moreover, an integer  $k$  is equal to the Pontryagin number  $p_m^2(M)[M]$  of such a manifold if and only if  $k$  is of the form  $k = 2^2(1 + 2t)^2$  with  $t \equiv 0, 7, 48, 55 \pmod{56}$  (for  $m = 4$ ) resp.  $k = 6^2(1 + 2t)^2$  with  $t \equiv 0, 127, 16128, 16255 \pmod{16256}$  (for  $m = 8$ ).*

## 2. Bordism groups and surgery

In this section we briefly describe a main result of Kreck’s ‘modified surgery theory’ [11] (Theorem 2.2 below). A direct consequence of this result (see Corollary 2.5) is that the connected sum  $M\#\Sigma$  of a closed simply connected manifold  $M$  of dimension  $2m \neq 4$  with a homotopy sphere  $\Sigma$  is diffeomorphic to  $M$  provided  $\Sigma$  represents zero in a suitable bordism group  $\Omega_{2m}^B$ .

We begin by defining the bordism groups  $\Omega_n^B$ .

**2.1.** Fix a fibration  $B \rightarrow BO$  over the classifying space  $BO$  of the stable orthogonal group. We recall that  $BO$  is the union of the classifying spaces  $BO_k$  of the orthogonal groups and that  $BO_k$  is the union of the Grassmann manifolds  $Gr_k(\mathbb{R}^{n+k})$  of  $k$ -planes in  $\mathbb{R}^{n+k}$  via natural inclusion maps  $Gr_k(\mathbb{R}^{n+k}) \subseteq Gr_k(\mathbb{R}^{n+k+1})$ . Let  $(N, \partial N)$  be a compact  $n$ -manifold, and let  $\iota : (N, \partial N) \hookrightarrow (\mathbb{R}_+^{n+k}, \partial\mathbb{R}_+^{n+k})$  be a smooth embedding into euclidean half-space. Recall that the normal Gauss map  $\nu : N \rightarrow Gr_k(\mathbb{R}^{n+k})$  assigns to any point  $x \in N$  its normal space in  $\mathbb{R}^{n+k}$ . A  $B$ -structure on  $N$  is an equivalence class of pairs  $(\iota, \bar{\nu})$ , where  $\bar{\nu}$  is a map making the following diagram commutative

$$\begin{array}{ccc}
 & & B \\
 & \nearrow \bar{\nu} & \downarrow \\
 N & \xrightarrow{\nu} & Gr_k(\mathbb{R}^{n+k}) \subset BO_k \subset BO
 \end{array}$$

The equivalence relation is generated by simultaneous deformations of  $\iota$  and  $\bar{\nu}$ , and by the stabilization map  $\mathbb{R}^{n+k} \hookrightarrow \mathbb{R}^{n+k+1}$ .

A  $B$ -manifold is a manifold equipped with a  $B$ -structure; a  $B$ -bordism between  $B$ -manifolds  $M_1 \xrightarrow{\bar{\nu}_1} B$  and  $M_2 \xrightarrow{\bar{\nu}_2} B$  is a bordism  $W$  between  $M_1$  and  $M_2$  equipped with a  $B$ -structure  $\bar{\nu} : W \rightarrow B$  which restricts to  $\bar{\nu}_1$  resp.  $\bar{\nu}_2$  on the boundary  $\partial W = M_1 \cup M_2$ .

A  $B$ -structure  $\beta : M \rightarrow B$  is called a *normal  $k$ -smoothing* if  $\bar{\nu}$  is a  $(k + 1)$ -equivalence, i.e., if the induced homomorphism  $\bar{\nu}_* : \pi_i(M) \rightarrow \pi_i(B)$  is an isomorphism for  $i \leq k$  and surjective for  $i = k + 1$ . We remark that if there exists a  $k$ -smoothing  $\bar{\nu} : M \rightarrow BO$ , the fibration  $B \rightarrow BO$  is determined by the manifold  $M$  up to fiber homotopy equivalence, if we assume that  $\pi_i B \rightarrow \pi_i BO$  is an isomorphism for  $i > k + 1$  and injective for  $i = k + 1$ . In this case, Kreck refers to  $B \rightarrow BO$  as the *normal  $k$ -type* of  $M$ .

**Theorem 2.2** (Kreck [11, Theorem B]). *Let  $M_1, M_2$  be closed manifolds of dimension  $n \geq 5$  with the same Euler characteristic which are equipped with  $B$ -structures that are normal  $k$ -smoothings. For  $k \geq [n/2] - 1$  a  $B$ -bordism  $W$  between  $M_1$  and  $M_2$  is bordant to an  $s$ -cobordism if and only if a certain obstruction  $\theta(W)$  is elementary.*

We recall that a bordism  $W$  between  $M_1$  and  $M_2$  is an  $s$ -cobordism if the inclusions  $M_1 \rightarrow W$  and  $M_2 \rightarrow W$  are simple homotopy equivalences. The  $s$ -cobordism Theorem implies that then  $M_1$  and  $M_2$  are diffeomorphic (assuming that  $\dim M_1 = \dim M_2 \geq 5$ ).

**2.3.** The obstruction  $\theta(W)$  is an element of an abelian monoid  $\ell_{n+1}(\pi, w)$  which depends on the fundamental group  $\pi = \pi_1(B)$  and the induced map  $w: \pi_1(B) = \pi \rightarrow \pi_1(BO) = \mathbb{Z}/2$ . Even if  $\pi$  is the trivial group (this is the case we care about in this paper), the obstruction  $\theta(W)$  is difficult to handle for  $k = [n/2] - 1$  (see [11, §7]).

The situation greatly simplifies for  $k \geq [n/2]$ :

- The normal  $k$ -smoothings induce isomorphisms  $H_i(M_1) \cong H_i(B) \cong H_i(M_2)$  for  $i \leq [n/2]$ . By Poincaré duality, we also have isomorphisms  $H_i(M_1) \cong H_i(M_2)$  for  $[n/2] + 1 \leq i \leq n$  and hence in particular the Euler characteristics of  $M_1$  and  $M_2$  agree.
- By [11] p. 734 the obstruction  $\theta(W)$  is contained in an abelian subgroup  $L_{n+1}(\pi, w)$  of the monoid  $\ell_{n+1}(\pi, w)$ . Moreover, this group projects to the Whitehead group  $Wh(\pi)$ , and the kernel

$$L_{n+1}^s(\pi, w) = \ker(L_{n+1}(\pi, w) \rightarrow Wh(\pi))$$

is Wall's classical surgery group [23].

If  $B$  is simply connected, then  $Wh(\pi) = 0$ , and so  $L_{n+1}^s(\pi, w) = L_{n+1}(\pi, w)$ ; moreover, these groups are zero if  $n$  is even [23]. As the zero-element in  $L_{n+1}(\pi, w)$  is certainly elementary in Kreck's sense, the obstruction  $\theta(W)$  is elementary in this case, and we conclude:

**Corollary 2.4.** *Let  $M_1, M_2$  be closed simply connected  $2m$ -dimensional manifolds which are equipped with  $B$ -structures that are normal  $m$ -smoothings,  $m \geq 3$ . If  $M_1$  and  $M_2$  represent the same element in the bordism group  $\Omega_{2m}^B$ , then  $M_1$  is diffeomorphic to  $M_2$ .*

**Corollary 2.5.** *Let  $\bar{\nu}: M \rightarrow B$  be a normal  $m$ -smoothing of a simply connected  $2m$ -manifold,  $m \geq 3$ . Let  $\Sigma$  be a homotopy sphere equipped with a  $B$ -structure such that  $[\Sigma] = 0 \in \Omega_{2m}^B$ . Then  $M \# \Sigma$  is diffeomorphic to  $M$ .*

*Proof.* It is well-known that the connected sum  $M \# N$  of two  $B$ -manifolds admits a  $B$ -structure such that it represents the same element in  $\Omega_*^B$  as the disjoint union of  $M$  and  $N$ ; the desired  $B$ -bordism  $W$  is constructed by taking the disjoint union of  $M \times [0, 1]$  and  $N \times [0, 1]$  and attaching a 1-handle  $D^n \times [0, 1]$  to it, connecting these two parts. The boundary of the resulting  $n+1$ -manifold  $W$  consists of the disjoint union of  $M, N$  and  $M \# N$ ; obstruction theory shows that the  $B$ -structure can be extended over the 1-handle to give a  $B$ -structure on  $W$ .

We note that the  $B$ -structure constructed on  $M \# \Sigma$  in this way is again an  $m$ -smoothing; hence the previous corollary implies that  $M \# \Sigma$  is diffeomorphic to  $M$ . q.e.d.

### 3. The normal $m$ -type of projective space like $2m$ -manifolds

In order to apply this result, we need to identify for a given projective plane like  $2m$ -manifold  $M$  a suitable fibration  $B \rightarrow BO$  such that  $M$  admits a normal  $m$ -smoothing  $\bar{\nu}: M \rightarrow B$  (i.e.,  $\bar{\nu}_*: \pi_i M \rightarrow \pi_i B$  is an isomorphism for  $i \leq m$  and surjective for  $i = m + 1$ ). To find  $B$ , we will need the following information about  $M$ .

**Lemma 3.1.** *Let  $M$  be a projective plane like  $2m$ -manifold,  $m = 4, 8$ , such that the almost closed manifold  $\mathring{M}$  obtained by removing an open disk from  $M$  is diffeomorphic to  $N_t$ . Let  $\nu: M \rightarrow BO_k \subset BO$  be the normal Gauss map induced by an embedding  $M \subset \mathbb{R}^{2m+k}$ . Then the induced map  $\nu_*: \pi_m M \cong \mathbb{Z} \rightarrow \pi_m BO \cong \mathbb{Z}$  is multiplication by  $\pm(2t + 1)$ .*

*Proof.* Let  $i: S^m \hookrightarrow D(\xi_t) = N_t \subset M$  be the inclusion of the zero-section.

The normal bundle of this embedding is  $\xi_t$ . The normal bundle of the embedding  $N_t \subset M \hookrightarrow \mathbb{R}^{n+k}$  is the pull back  $\nu^*\gamma^k$  of the universal bundle  $\gamma^k \rightarrow BO_k$  via the normal Gauss map  $\nu: M \rightarrow BO_k$ . This implies that the vector bundle

$$TS^m \oplus \xi_t \oplus i^*\nu^*\gamma^k \cong i^*TM \oplus i^*\nu^*\gamma^k = i^*(TM \oplus \nu^*\gamma^k)$$

is the restriction of the tangent bundle of  $\mathbb{R}^{n+k}$  to  $S^m \subset M \subset \mathbb{R}^{n+k}$  and hence trivial. Identifying stable vector bundles over  $S^m$  with their classifying map  $[S^m \rightarrow BO] \in \pi_m BO$ , we conclude  $i^*\nu^*\gamma = -\xi_t \in \pi_m BO$ . Comparing the Pontryagin classes  $p_m(\xi_t), p_m(\xi_1) \in H^m(S^m; \mathbb{Z})$ , we see that  $\xi_t = (2t + 1)\xi_1 \in \pi_m BO$ . Combining these facts, we have  $i^*\nu^*\gamma = -(2t + 1)\xi_1 \in \pi_m BO$ .

Reinterpreting this equation, it tells us that the map  $\nu_*: \pi_m M \rightarrow \pi_m BO$  maps the generator  $[i: S^m \rightarrow M] \in \pi_m M$  to  $-(2t + 1)\xi_1 \in \pi_m BO$ , which implies the lemma, since  $\xi_1$  is a generator of  $\pi_m BO \cong \mathbb{Z}$ .  
q.e.d.

**3.2.** We note that a projective plane like  $2m$ -manifold  $M$  is  $(m - 1)$ -connected, i.e.,  $\pi_i M = 0$  for  $i < m$ . This implies by standard obstruction theory that the normal Gauss map  $\nu: M \rightarrow BO_k \subset BO$  of an embedding  $M \subset \mathbb{R}^{2m+k}$  can be factored through the  $(m - 1)$ -connected cover  $q: BO\langle m \rangle \rightarrow BO$ , a fibration determined up to fiber homotopy equivalence by the requirement that  $\pi_i BO\langle m \rangle = 0$  for  $i < m$  and that  $q_*: \pi_i BO\langle m \rangle \rightarrow \pi_i BO$  induces an isomorphism for  $i \geq m$  (we note that the 1-connected cover  $X\langle 2 \rangle \rightarrow X$  of a space  $X$  is just the universal covering of  $X$ ). The lift  $\bar{\nu}: M \rightarrow BO\langle m \rangle$  of  $\nu$  constructed this way is *not* a normal  $m$ -smoothing of  $M$ , since by the above lemma, the induced map  $\pi_m M \rightarrow \pi_m BO\langle m \rangle = \pi_m BO$  is not an isomorphism unless  $t = 0$ .



In particular,  $BO\langle m \rangle \rightarrow BO$  is not the normal  $m$ -type of  $M$  unless  $t = 0$ .

**3.3.** Now we proceed to construct the fibration  $B \rightarrow BO$  which will turn out to be the normal  $m$ -type of  $M$ . Let  $K(\mathbb{Z}, m)$  be the Eilenberg-MacLane space characterized up to homotopy equivalence by the requirement that the homotopy group  $\pi_i K(\mathbb{Z}, m)$  is zero for  $i \neq m$  and equal to  $\mathbb{Z}$  for  $i = m$ . The long exact homotopy sequence of the path fibration

$$\Omega K(\mathbb{Z}, m) \rightarrow PK(\mathbb{Z}, m) \rightarrow K(\mathbb{Z}, m)$$

together with the fact that the path space  $PK(\mathbb{Z}, m)$  is contractible shows that the loop space  $\Omega K(\mathbb{Z}, m)$  is the Eilenberg-MacLane space  $K(\mathbb{Z}, m - 1)$ . For  $m \equiv 0 \pmod 4$ , let us denote by  $B_{d,m} \rightarrow BO\langle m \rangle$  the pull-back of the above path fibration via a map  $\phi: BO\langle m \rangle \rightarrow K(\mathbb{Z}, m)$  such that the induced map  $\pi_*: \pi_m BO\langle m \rangle = \mathbb{Z} \rightarrow \pi_m K(\mathbb{Z}, m) = \mathbb{Z}$  is multiplication by  $d$  (this requirement determines  $\phi$  up to homotopy). We note that the long exact homotopy sequence of this fibration shows that the induced map  $\mathbb{Z} \cong \pi_m B_{d,m} \rightarrow \pi_m BO\langle m \rangle \cong \mathbb{Z}$  is multiplication by  $\pm d$ .

**Proposition 3.4.** *Let  $M$  be as in Lemma 3.1. Then the normal  $m$ -type of  $M$  is the composite fibration  $B_{2t+1,m} \rightarrow BO\langle m \rangle \rightarrow BO$ .*

*Proof.* Let  $\bar{\nu}': M \rightarrow BO\langle m \rangle$  be the lift of the normal Gauss map  $M \rightarrow BO$  associated to an embedding  $M \hookrightarrow \mathbb{R}^{n+k}$ . Again obstruction theory shows that  $\bar{\nu}'$  can be lifted to a map  $\bar{\nu}: M \rightarrow B_{2t+1,m}$ . Lemma 3.1 implies that the induced map  $\bar{\nu}_*: \pi_i M \rightarrow \pi_i B_{d,m}$  is an isomorphism for  $i = m$ . Moreover, it is surjective for  $i = m + 1$ : for  $m = 4$  this is obvious, since  $\pi_5 BO = 0$ ; for  $m = 8$ , it follows from the fact that the Hopf map  $\eta: S^9 \rightarrow S^8$  induces a surjection  $\pi_8 BO = \mathbb{Z} \rightarrow \pi_9 BO = \mathbb{Z}/2$ . q.e.d.

Applying now Corollary 2.5 to projective plane like manifolds, we conclude:

**Corollary 3.5.** *If  $M$  is as in Lemma 3.1, and  $\Sigma$  is a homotopy sphere of dimension  $2m$  with  $[\Sigma] = 0 \in \Omega_{2m}^{B_{2t+1,m}}$ , then  $M \# \Sigma$  is diffeomorphic to  $M$ .*

#### 4. The bordism class of homotopy spheres

In view of the last corollary our main result follows from the following statement whose proof is the goal of this section.

**Proposition 4.1.** *Let  $\Sigma$  be a homotopy sphere of dimension  $2m = 8, 16$ . Then  $[\Sigma] = 0 \in \Omega_{2m}^{B_{2t+1,m}}$  for any  $t$ .*

To prove this result we note that the map  $B_{2t+1,m} \rightarrow BO\langle m \rangle$  is a map of fiber bundles over  $BO$  and hence it induces a homomorphism of bordism groups

$$(1) \quad \Omega_*^{B_{2t+1,m}} \longrightarrow \Omega_*^{BO\langle m \rangle}.$$

The groups  $\Omega_{2m}^{BO\langle m \rangle}$  are known for  $m = 4, 8$ , see Milnor [15] and Giambalvo [6]:

$$\Omega_8^{BO\langle 4 \rangle} = \Omega_8^{BSpin} \cong \mathbb{Z} \oplus \mathbb{Z} \quad \text{and} \quad \Omega_{16}^{BO\langle 8 \rangle} \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Since  $\Theta_{2m} \cong \mathbb{Z}/2$  for  $m = 4, 8$ , it follows that for any homotopy  $2m$ -sphere  $\Sigma$  the connected sum  $\Sigma \# \Sigma$  is diffeomorphic to  $S^{2m}$ . In particular,  $\Sigma$  represents an element of order at most 2 in  $\Omega_*^{B_{2t+1,m}}$ . Hence the next result implies the proposition above.

**Lemma 4.2.** *The homomorphism (1) is a 2-local isomorphism (i.e., its kernel and cokernel belong to the class of torsion groups without elements of order 2).*

Before proving this lemma we recall some relevant facts.

**4.3. The Pontryagin-Thom construction.** Let  $f_k: B_k \rightarrow BO_k$  be the restriction of the fibration  $f: B \rightarrow BO$  to  $BO_k \subset BO$ . Let  $\gamma_k \rightarrow BO_k$  be the universal  $k$ -dimensional vector bundle, let  $\hat{\gamma}^k \rightarrow B_k$  be its pull-back via  $f_k$ , and let  $T(\hat{\gamma}^k)$  be the Thom space of  $\hat{\gamma}^k$  (the quotient space of its total space obtained by collapsing all vectors of length  $\geq 1$  to a point). Then the Pontryagin-Thom construction (see [21, Thm., p. 18]) produces an isomorphism

$$(2) \quad \Omega_n^B \cong \lim_{k \rightarrow \infty} \pi_{n+k} T(\hat{\gamma}^k).$$

**4.4. Thom spectra.** It is usual and convenient to express the right hand side of the Pontryagin-Thom isomorphism (2) in terms of Thom spectra. We recall that a *spectrum* is a sequence  $E_k$  of pointed spaces together with pointed maps  $\Sigma E_k \rightarrow E_{k+1}$  from the suspension of  $E_k$  to  $E_{k+1}$ . For example, if  $B \rightarrow BO$  is a fibration, there is an associated *Thom spectrum*  $MB$ , whose  $k$ -th space is the Thom space  $T(\hat{\gamma}^k)$ .

Many constructions with spaces can be generalized to spectra; e.g., the homotopy (resp. homology) groups of a spectrum  $E = \{E_k\}$  are defined as

$$\pi_n E \stackrel{\text{def}}{=} \lim_{k \rightarrow \infty} \pi_{n+k} E_k \quad H_n(E) \stackrel{\text{def}}{=} \lim_{k \rightarrow \infty} \tilde{H}_{n+k}(E_k).$$

With these definitions, the Pontryagin-Thom isomorphism takes the pleasant form

$$(3) \quad \Omega_n^B \cong \pi_n MB.$$

Assuming that  $B$  is 1-connected, the vector bundles  $\hat{\gamma}^k \rightarrow B_k$  are all oriented and hence we have Thom-isomorphisms  $H_i(B_k; \mathbb{Z}) \cong$

$\tilde{H}_{i+k}(T\hat{\gamma}^k; \mathbb{Z})$ . It turns out that these isomorphisms are all compatible and so passing to the  $k \rightarrow \infty$  limit, one obtains a Thom-isomorphism

$$(4) \quad H_i(B; \mathbb{Z}) \cong H_i(MB; \mathbb{Z}).$$

*Proof of lemma.* By construction, the induced homomorphism

$$\pi_i(B_{2t+1,4k}) \longrightarrow \pi_i(BO\langle 4k \rangle)$$

is an isomorphism for  $i \neq 4k$ ; for  $i = 4k$ , it is injective with cokernel isomorphic to  $\mathbb{Z}/(2t+1)$ . In particular, for all  $i$  it is a 2-local isomorphism. Then the generalized Whitehead Theorem ([19, Thm. 22 of Chap. 9, §6]) implies that

$$(5) \quad p_* : H_i(B_{2t+1,4k}; \mathbb{Z}) \longrightarrow H_i(BO\langle 4k \rangle; \mathbb{Z})$$

is also a 2-local isomorphism. Now we consider the map of Thom spectra

$$Mp : MB_{2t+1,4k} \longrightarrow MBO\langle 4k \rangle$$

induced by  $p$ . Via the Thom isomorphism (4) the induced map in *homology* may be identified with the homomorphism (5), while the induced map on *homotopy groups* via the Pontryagin-Thom isomorphism (3) corresponds to the homomorphism (1) of bordism groups. Again by the generalized Whitehead Theorem, the latter is a 2-local isomorphism since the former is. q.e.d.

### References

- [1] J.F. Adams, *Stable homotopy and generalised homology*, Reprint of the 1974 original, Univ. Chicago Press, Chicago, IL, 1995, MR 1324104, Zbl 0309.55016.
- [2] J.F. Adams & M.F. Atiyah, *K-theory and the Hopf invariant*, Quart. J. Math. Oxford Ser. (2) **17** (1966) 31–38, MR 0198460, Zbl 0136.43903.
- [3] R. Bott, *The stable homotopy of the classical groups*, Ann. of Math. (2) **70** (1959) 313–337, MR 0110104, Zbl 0129.15601.
- [4] J. Eells, Jr. & N.H. Kuiper, *Manifolds which are like projective planes*, Inst. Hautes Études Sci. Publ. Math. **14** (1962) 5–46, MR 0145544, Zbl 0109.15701.
- [5] M.H. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geom. **17**(3) (1982) 357–453, MR 0679066, Zbl 0528.57011.
- [6] V. Giambalvo, *On  $\langle 8 \rangle$ -cobordism*, Illinois J. Math. **15** (1971) 533–541, MR 0287553, Zbl 0221.57019.  
V. Giambalvo, *Correction to my paper: “On  $\langle 8 \rangle$ -cobordism”*, Illinois J. Math. **16** (1972) 704, MR 0309132, Zbl 0238.57020.
- [7] M. A. Kervaire & J.W. Milnor, *Groups of homotopy spheres, I*, Ann. of Math. (2) **77** (1963) 504–537, MR 0148075, Zbl 0115.40505.
- [8] L. Kramer, *The topology of smooth projective planes*, Arch. Math. (Basel) **63**(1) (1994) 85–91, MR 1277915, Zbl 0831.51011.
- [9] ———, *Projective planes and their look-alikes*, J. Differential Geom. **64**(1) (2003) 1–55, MR 2015043, Zbl 1068.57019.

- [10] M. Kreck, *Manifolds with unique differentiable structure*, Topology **23**(2) (1984) 219–232, MR 0744852, Zbl 0547.57025.
- [11] ———, *Surgery and duality*, Ann. of Math. (2) **149**(3) (1999) 707–754, MR 1709301, Zbl 0935.57039.
- [12] ———, *A guide to the classification of manifolds*, in ‘Surveys on surgery theory’, Vol. 1, 121–134, Ann. of Math. Stud., **145**, Princeton Univ. Press, Princeton, NJ, MR 1747533, Zbl 0947.57001.
- [13] I. Madsen & R.J. Milgram, *The classifying spaces for surgery and cobordism of manifolds*, Ann. of Math. Stud., **92**, Princeton Univ. Press, Princeton, NJ, 1979, MR 0548575, Zbl 0446.57002.
- [14] B. McKay, *Smooth projective planes*, Geom. Ded. **116** (2005) 157–202, MR 2195446, Zbl 05021438.
- [15] J. Milnor, *Spin structures on manifolds*, Enseignement Math. (2) **9** (1963) 198–203, MR 0157388, Zbl 0116.40403.
- [16] J.W. Milnor & J.D. Stasheff, *Characteristic classes*, Ann. of Math. Stud., **76**, Princeton Univ. Press, Princeton, NJ, 1974, MR 0440554, Zbl 0298.57008.
- [17] Y.B. Rudyak, *On Thom spectra, orientability, and cobordism*, Springer, Berlin, 1998, MR 1627486, Zbl 0906.55001.
- [18] H. Salzmann et al., *Compact projective planes*, de Gruyter, Berlin, 1995, MR 1384300, Zbl 0851.51003.
- [19] E.H. Spanier, *Algebraic topology*, Corrected reprint, Springer, New York, 1981, MR 0666554, Zbl 0477.55001.
- [20] S. Stolz, *Hochzusammenhängende Mannigfaltigkeiten und ihre Ränder*, Springer, Berlin, 1985, MR 0871476, Zbl 0561.57021.
- [21] R.E. Stong, *Notes on cobordism theory*, Mathematical notes Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1968, MR 0248858, Zbl 0181.26604.
- [22] C.T.C. Wall, *Classification of  $(n-1)$ -connected  $2n$ -manifolds*, Ann. of Math. (2) **75** (1962) 163–189, MR 0145540, Zbl 0218.57022.
- [23] ———, *Surgery on compact manifolds*, Second edition, Amer. Math. Soc., Providence, RI, 1999, MR 1687388, Zbl 0935.57003.

MATHEMATISCHES INSTITUT  
UNIVERSITÄT MÜNSTER  
EINSTEINSTR. 62, 48149 MÜNSTER  
GERMANY

*E-mail address:* linus.kramer@math.uni-muenster.de

DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF NOTRE DAME  
NOTRE DAME, IN 46556  
*E-mail address:* stolz.1@math.nd.edu